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A NOTE TO FRIEDRICHS' INEQUALITY

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Abstract. The main aim of this paper is to derive continuous and discrete forms of inequalities which are similar to Friedrichs' inequality and to show that for h sufficiently small the constant C appearing in discrete inequalities written for functions from finite element spaces X_h is independent of h. The discrete forms of Friedrichs' inequality are restricted to two-dimensional domains in this paper. These inequalities have applications in the theory of two-dimensional electromagnetic field and in the analysis of the approximate solution of Maxwell's equations.

1. Introduction

Let Ω, Ω_E, Ω_P be two-dimensional bounded domains with continuous boundaries in the sense of Nečas (see [6, p. 14]) such that

$$\bar{\Omega} = \bar{\Omega}_E \cup \bar{\Omega}_P, \quad \Omega_E \cap \Omega_P = \emptyset, \quad \operatorname{mes}_2 \Omega_P > 0\,.$$

Further, in Theorem 3.3 we shall assume that boundaries $\partial\Omega$, $\partial\Omega_E$, $\partial\Omega_P$ are Lipschitz continuous and piecewise of class C^2 and we restrict our considerations to the case of a simply connected domain Ω divided into subdomains Ω_E and Ω_P like this:

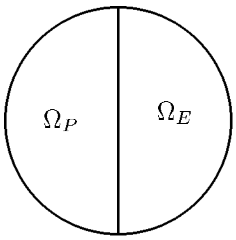

Fig. 1

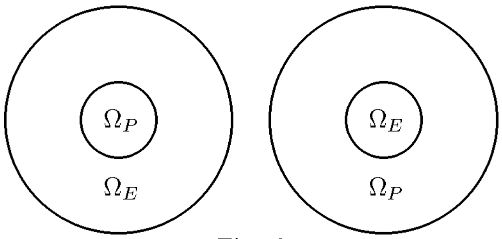

Fig. 2

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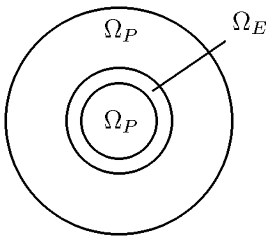

Fig. 3

Let us note that some of such cases of Ω can occur in problems of two-dimensional nonlinear quasistationary electromagnetic fields in electrical machines.

We shall use the Lebesgue space $L_2(\Omega)$ and the Sobolev space $H^1(\Omega)$ equipped with their usual norms $\|.\|_0$, $\|.\|_1$, respectively (see [4]). The seminorm in the space $H^1(\Omega)$ will be denoted by $|.|_1$. If G is a domain different from Ω then the norms in $L_2(G)$ and $H^1(G)$ will be denoted by $\|.\|_{0,G}$, $\|.\|_{1,G}$, respectively. In order to simplify the notation we shall write $\|.\|_{k,M}$ and $\|.\|_{k,M_h}$ instead of $\|.\|_{k,\Omega_M}$ and $\|.\|_{k,\Omega_{hM}}$, respectively, $(k = 0, 1)$ $(M = E, P)$, where Ω_{hM} is an approximation of Ω_M.

2. Continuous form

2.1 Theorem. *Let Ω be a bounded domain with a continuous boundary. Then we have*

$$\|v\|_1^2 \le K(\Omega)\left(\|v\|_{0,P}^2 + |v|_1^2\right) \qquad \forall v \in H^1(\Omega)\,. \tag{1}$$

Proof. The method is analogical to the proof of Friedrichs' inequality (see [6, Theorems 1.1.8, 1.1.9]); however, on the contrary to [6] we go to details in the proof.

A) First we shall prove that $H^1(\Omega)$ is a Banach (and even a Hilbert) space for the norm given by the right-hand side of (1). Let us denote $B_1 = H^1(\Omega)$; the symbol ${}^1\|.\|$ will denote a norm in B_1, i.e. ${}^1\|v\| = \|v\|_1$.

Let B_2 be a normed linear space which consists of the same elements as B_1 provided with the norm

$${}^2\|v\| = \left\{ \int_{\Omega_P} v^2\,dx + \int_{\Omega} \sum_{i=1}^{N} \left(\frac{\partial v}{\partial x_i}\right)^2 dx \right\}^{\frac{1}{2}}.$$

We show that the expression ${}^2\|.\|$ is a norm. Let ${}^2\|v\| = 0$. Then $D^i v = 0, |i| = 1$ and $\int_{\Omega_P} v^2\,dx = 0$. As the derivatives are equal to zero we have $v = \text{const}$, thus the integral over Ω_P ($\mathrm{mes}_2\,\Omega_P > 0$) which is equal to zero gives $v \equiv 0$. The other properties of the norm are evident.

From the definition of the norms ${}^1||.||$, ${}^2||.||$ it follows that

$$ {}^2||v|| \le {}^1||v|| \,. \tag{2}$$

Let $A = I$ be the identity operator which maps every element $v \in B_1$ onto the same element $v \in B_2$. Then relation (2) implies

$$ {}^2||Av|| = {}^2||v|| \le {}^1||v|| \,. \tag{3}$$

Thus the operator A is bounded. As it is linear (which is evident) it is continuous. In the part B) of the proof we shall show that the space B_2 is complete. Therefore we can use Banach's theorem on isomorphism (see [5, Theorem 2.20.1]). According to it, the inverse operator $A^{-1} = I$ from B_2 to B_1 is linear and bounded:

$$ {}^1||A^{-1}v|| = {}^1||v|| \le \mathrm{const}\,\left({}^2||v||\right)\,, \tag{4}$$

which is inequality (1).

B) Now we prove the completeness of the space B_2. Let $\{v_s\} \subset B_2$ be a Cauchy sequence in the norm ${}^2||.||$. Then we have

$$ \int\limits_{\Omega_P} |v_m - v_n|^2\,dx \to 0 \quad \text{for} \quad m,n \to \infty\,, \tag{5}$$

$$ \int\limits_{\Omega} \sum_{i=1}^{N} \left|\frac{\partial v_m}{\partial x_i} - \frac{\partial v_n}{\partial x_i}\right|^2 dx \to 0 \quad \text{for} \quad m,n \to \infty. \tag{6}$$

According to [6, Theorem 1.1.6] and relation (6), we have (the symbol P_0 denotes the set of polynomials of degree zero)

$$ ||\tilde v_m - \tilde v_n||_{H^1(\Omega)/P_0} \to 0 \quad \text{for} \quad m,n \to \infty\,,$$

i.e. a sequence of classes $\{\tilde v_s\} \subset H^1(\Omega)/P_0$ corresponding to $\{v_s\}$ is a Cauchy sequence. Let us note that the symbol $H^1(\Omega)/P_0$ denotes a factorspace of classes $\tilde v$ of functions v from $H^1(\Omega)$ such that

$$ v, u \in \tilde v \quad \Leftrightarrow \quad v - u \in P_0\,,$$

in which a norm is defined by

$$ ||\tilde v||_{H^1(\Omega)/P_0} = \inf_{v \in \tilde v} ||v||_1\,.$$

As the space $H^1(\Omega)/P_0$ is complete the sequence $\{\tilde v_s\}$ converges, i.e. there exists a class $\tilde v \in H^1(\Omega)/P_0$ such that

$$ \lim_{s\to\infty} ||\tilde v_s - \tilde v||_{H^1(\Omega)/P_0} = 0\,. \tag{7}$$

As $\tilde{v}_s - \tilde{v} = (v_s - v)^{\sim}$ we have

$$\|\tilde{v}_s - \tilde{v}\|_{H^1(\Omega)/P_0} = \inf_{u \in (v_s - v)^{\sim}} \|u\|_1 = \inf_{q \in P_0} \|v_s - w + q\|_1 \,, \tag{8}$$

where $w \in \tilde{v}$ is an arbitrary element (independent on v_s), but fixed. We show the existence of a sequence $\{p_s\} \subset P_0$ depending on w and such that $v_s + p_s \to w$ in the first norm ${}^1\|.\|$.

Let us choose $\varepsilon > 0$ arbitrarily. Relation (7) implies that there exists $N(\varepsilon)$ such that

$$\|\tilde{v}_s - \tilde{v}\|_{H^1(\Omega)/P_0} < \frac{\varepsilon}{2} \quad \text{for} \quad s \geq N(\varepsilon)\,. \tag{9}$$

Further, according to the definition of infimum, there exists $q_s^{(\varepsilon,w)} \in P_0$ dependent on v_s, w, ε such that

$$\|v_s - w + q_s^{(\varepsilon,w)}\|_1 < \inf_{q \in P_0} \|v_s - w + q\|_1 + \frac{\varepsilon}{2}\,. \tag{10}$$

We restrict ourselves to $s \geq N(\varepsilon)$ in (10). Then using (8) and (9) we can rewrite this relation in the form

$$\|v_s - w + q_s^{(\varepsilon,w)}\|_1 < \varepsilon \quad \text{for} \quad s \geq N(\varepsilon)\,. \tag{11}$$

Let us set $\varepsilon_j = \frac{1}{j}\,(j = 1, 2, \ldots)$. The preceding considerations imply the existence of numbers $N(\varepsilon_1) \leq N(\varepsilon_2) \leq \ldots \leq N(\varepsilon_n) \leq \ldots$ and $q_s^{(\varepsilon_1,w)} \in P_0\,(s \geq N(\varepsilon_1))$, $q_s^{(\varepsilon_2,w)} \in P_0\,(s \geq N(\varepsilon_2))$, $\ldots$, $q_s^{(\varepsilon_n,w)} \in P_0\,(s \geq N(\varepsilon_n))$, $\ldots$ such that

$$\|v_s - w + q_s^{(\varepsilon_j,w)}\|_1 < \varepsilon_j \quad \text{for} \quad s \geq N(\varepsilon_j)\,.$$

Let us define a sequence $\{p_s\} \subset P_0$ as follows:

$$p_s = \text{an arbitrary element of } P_0 \quad \text{for} \quad s < N(\varepsilon_1)\,,$$

$$p_s = q_s^{(\varepsilon_j,w)} \quad \text{for} \quad N(\varepsilon_j) \leq s < N(\varepsilon_{j+1})\,.$$

As for every $\varepsilon > 0$ we can find ε_j such that $\varepsilon_j < \varepsilon$, we see that the sequence $\{p_s\}$ defined by this way has the following property: for every $\varepsilon > 0$ we can find $N = N(\varepsilon_j)$ such that

$$\|v_s + p_s - w\|_1 < \varepsilon \quad \text{for} \quad s \geq N\,, \tag{12}$$

i.e.

$$v_s + p_s \to w \quad \text{in} \quad H^1(\Omega)$$

in the first norm ${}^1\|.\|$. (Let us stress that $\{p_s\}$ depends on $w \in \tilde{v}$.)

Now we prove that $\{p_s\}$ is a Cauchy sequence in $H^1(\Omega)$. We have

$$\|p_m - p_n\|_{0,P} \leq \|(v_m - v_n) + (p_m - p_n)\|_{0,P} + \|v_m - v_n\|_{0,P} \leq$$

$$\leq \|(v_m + p_m) - (v_n + p_n)\|_0 + \|v_m - v_n\|_{0,P}\,. \tag{13}$$

According to (5), $\{v_s\}$ is a Cauchy sequence in $L_2(\Omega_P)$. Taking into account (12) $\{v_s + p_s\}$ is convergent in $H^1(\Omega)$, therefore it is a Cauchy sequence in $H^1(\Omega)$ and consequently a Cauchy sequence in $L_2(\Omega)$. Hence with respect to (13) $\{p_s\}$ is a Cauchy sequence in $L_2(\Omega_P)$. As P_0 is a finite dimensional space all norms in P_0 are equivalent, i.e.

$$c_1\|p\|_{0,P} \le \|p\|_1 \le c_2\|p\|_{0,P} \quad \forall p \in P_0\,.$$

Let us note that the norm $\|.\|_{0,P}$ is really the norm for polynomials of degree zero (even of an arbitrary degree): the equality

$$\|p\|^2_{0,P} = \int\limits_{\Omega_P} p^2\,dx = 0\,,$$

where $p \in P_0$, implies $p \equiv 0$ because $\operatorname{mes}_2 \Omega_P > 0$. Thus $\{p_s\}$ is a Cauchy sequence in the norm of the space $H^1(\Omega)$ and there exists an element $p \in P_0$ such that $p_s \to p$ in $H^1(\Omega)$.

Both of the relations $v_s + p_s \to w$ in the norm of $H^1(\Omega)$ and $p_s \to p$ in the norm of $H^1(\Omega)$ give

$$\|v_m - v_n\|_1 \le \|(v_m + p_m) - (v_n + p_n)\|_1 + \|p_m - p_n\|_1 \to 0 \quad \text{for} \quad m, n \to \infty\,.$$

Thus $\{v_s\}$ is a Cauchy sequence in $H^1(\Omega)$ and there exists an element $v \in H^1(\Omega)$ such that $v_s \to v$ in the norm of the space $H^1(\Omega)$, i.e. in the norm ${}^1\|.\|$. Hence according to (3), $v_s \to v$ in the norm ${}^2\|.\|$ so that B_2 is a complete space. $\square$

2.2 Remark. Let us note that the proof of the preceding inequality (1) does not depend on the dimension of the domain. Further, the constant $K(\Omega) > 0$ occurring in this inequality is independent of v.

2.3 Remark. The inequality

$$\|v\|^2_k \le K(\Omega)\left(\|v\|^2_{0,P} + |v|^2_k\right) \qquad \forall v \in H^k(\Omega)$$

can be proved in the same way. The only change is that we substitute $H^1(\Omega)$ and P_0 by $H^k(\Omega)$ and P_{k-1}, respectively, where P_{k-1} is the space of polynomials the degree of which is not greater than $k-1$.

3. Discrete form

So called discrete forms of Friedrichs' inequality are studied with connections of solving various variational problems by the finite element method (see, e.g., [8], [1], [3]).

In this section we restrict ourselves to the two-dimensional case of discrete inequalities corresponding to inequality (1). Let us approximate a bounded two-dimensional domain Ω by a domain Ω_h with a polygonal boundary $\partial\Omega_h$ the vertices of which lie on $\partial\Omega$. Let $\mathcal{T}_h$ be a triangulation of Ω_h, i.e. a set $\mathcal{T}_h = \{\overline{T}_1, \overline{T}_2, \dots, \overline{T}_m\}$ consisting of a finite number of closed triangles which have the following properties:

(1) $\bar{\Omega}_h = \bigcup_{i=1}^m \overline{T}_i$

(2) two arbitrary triangles are either disjoint or possess a common vertex or a common side

We assume that the points of $\partial\Omega$ where the condition of C^2 -smoothness is not satisfied are vertices of the triangles in $\mathcal{T}_h$.

We further assume that every triangulation $\mathcal{T}_h$ consists of two subtriangulations $\mathcal{T}_{hE}$ and $\mathcal{T}_{hP}$ such that

$$\mathcal{T}_h = \mathcal{T}_{hE} \cup \mathcal{T}_{hP}, \quad \mathcal{T}_{hE} \cap \mathcal{T}_{hP} = \emptyset .$$

The subtriangulation $\mathcal{T}_{hM}$ is a triangulation of the polygonal domain Ω_{hM} approximating Ω_M $(M = E, P)$ and has all properties described above.

With every triangulation $\mathcal{T}_h$ we associate three parameters h, $\bar{h}$ and ϑ defined

$$h = \max_{T \in \mathcal{T}_h} h_T , \quad \bar{h} = \min_{T \in \mathcal{T}_h} h_T , \quad \vartheta = \min_{T \in \mathcal{T}_h} \vartheta_T$$

where h_T and ϑ_T are the length of the greatest side and the smallest angle, respectively, of the triangle $\overline{T} \in \mathcal{T}_h$. We restrict ourselves to triangulations $\{\mathcal{T}_h\}$ $(h \in (0, h_0), h_0 > 0)$ satisfying the conditions

(14) $$\vartheta_h \geq \vartheta_0 > 0 \quad \forall h \in (0, h_0) \quad \vartheta_0 = \text{ const}$$

(15) $$\bar{h}/h \geq C_0 > 0 \quad \forall h \in (0, h_0) \quad C_0 = \text{ const.}$$

Let us define a finite dimensional subspace of $H^1(\Omega_h) \cap C(\bar{\Omega}_h)$ by the relation

$$X_h = \{v \in C(\bar{\Omega}_h) : v|_T \text{ is linear for all } T \in \mathcal{T}_h\} .$$

The space X_h is a finite element approximation of the space $H^1(\Omega)$ defined on the triangulation $\mathcal{T}_h$.

For the purpose of the proof of Theorem 3.3 let us set

(16) $$\omega_h = \Omega - \bar{\Omega}_h , \quad \tau_h = \Omega_h - \bar{\Omega} ,$$

(17) $$\omega_{hM} = \Omega_M - \bar{\Omega}_{hM}, \quad \tau_{hM} = \Omega_{hM} - \bar{\Omega}_M \quad (M = E, P) .$$

In the proof we shall also need the following notions.

Let $\overline{T} \in \mathcal{T}_h$ be a boundary triangle lying along the curved part of $\partial\Omega$. (It has two vertices on $\partial\Omega$.) The closed curved triangle $\overline{T}^{\text{id}}$ with two straight sides and one curved side, which is the part of $\partial\Omega$, is called the ideal triangle associated with the triangle $\overline{T} \in \mathcal{T}_h$. The triangle $\overline{T}$ is an approximation of $\overline{T}^{\text{id}}$.

3.1 Definition. Let $w \in X_h$. The function

$$\bar{w} : \bar{\Omega}_h \cup \bar{\Omega} \to R^1$$

is called the natural extension of w if

$$\bar{w} = w \quad \text{on} \quad \bar{\Omega}_h$$

and

$$\bar{w}|_{\overline{T}^{\text{id}}} = p|_{\overline{T}^{\{id}} \quad \text{on} \quad \overline{T}^{\text{id}} \supset \overline{T}$$

where $p \in P_1$ is the linear polynomial satisfying

$$p|_{\overline{T}} = w|_{\overline{T}} .$$

(The symbol P_1 denotes the space of all polynomials with the degree less than or equal to one and $\overline{T}^{\mathrm{id}}$ denotes the ideal triangle which is approximated by $\overline{T}$.)

In the proof of Theorem 3.3 the following estimates will be useful.

3.2 Lemma. *Let $\bar{w}$ be the natural extension of $w \in X_h$. Then we have*

$$|\bar{w}|_{1,\varepsilon_h} \le Ch^{\frac{1}{2}} |\bar{w}|_1 \quad (\varepsilon = \tau, \omega), \tag{18}$$

$$\|\bar{w}\|_{0,\varepsilon_h} \le Ch^{\frac{1}{2}} \|\bar{w}\|_0 \quad (\varepsilon = \tau, \omega) \tag{19}$$

where the constant C does not depend on h and w.

See the proof of [10, Lemma 28.8].

The inequality appearing in Theorem 3.3 is the discrete form of inequality (1) because it is written only for the functions from the finite dimensional space X_h.

3.3 Theorem. *Let Ω be a two-dimensional bounded domain with boundary $\partial\Omega$ piecewise of class C^2. Then we have*

$$\|v\|^2_{1,\Omega_h} \le C \left(\|v\|^2_{0,P_h} + |v|^2_{1,\Omega_h} \right) \quad \forall v \in X_h \tag{20}$$

where the constant $C > 0$ does not depend on $h \in (0, h_0)$ and v.

Proof. According to (16), let us write for $v \in X_h$

$$\|v\|^2_{1,\Omega_h} = \|\bar{v}\|^2_1 + \|\bar{v}\|^2_{1,\tau_h} - \|\bar{v}\|^2_{1,\omega_h} = \|\bar{v}\|^2_1 \left(1 + \delta_\tau - \delta_\omega\right), \tag{21}$$

where

$$\delta_\tau = \frac{\|\bar{v}\|^2_{1,\tau_h}}{\|\bar{v}\|^2_1}, \quad \delta_\omega = \frac{\|\bar{v}\|^2_{1,\omega_h}}{\|\bar{v}\|^2_1} . \tag{22}$$

Similarly, we obtain

$$\|v\|^2_{0,P_h} + |v|^2_{1,\Omega_h} = \|v\|^2_{0,P_h} + \|\bar{v}\|^2_{0,P} - \|\bar{v}\|^2_{0,P} + |\bar{v}|^2_1 + |\bar{v}|^2_{1,\tau_h} - |\bar{v}|^2_{1,\omega_h} =$$

$$= \left(\|\bar{v}\|^2_{0,P} + |\bar{v}|^2_1\right) \left(1 + \varepsilon_\Delta + \varepsilon_\tau - \varepsilon_\omega\right), \tag{23}$$

where

$$\varepsilon_\Delta = \frac{\|v\|^2_{0,P_h} - \|\bar{v}\|^2_{0,P}}{\|\bar{v}\|^2_{0,P} + |\bar{v}|^2_1} , \tag{24}$$

$$\varepsilon_\chi = \frac{|\bar v|^2_{1,\chi_h}}{\|\bar v\|^2_{0,P} + |\bar v|^2_1} \quad (\chi = \tau, \omega)\,. \tag{25}$$

It should be noted that the function $\bar v$ appearing in $\|\bar v\|_{0,P}$ is the restriction $(\bar v)_P$ of $\bar v$ $(v \in X_h)$ to the domain Ω_P and $(\bar v)_P \neq \bar v_P = \bar{(v_P)}$. (We use the rule: first indices, then bars.) Relations (21) and (23) imply

$$\frac{\|v\|^2_{1,\Omega_h}}{\|v\|^2_{0,P_h} + |v|^2_{1,\Omega_h}} = \frac{\|\bar v\|^2_1}{\|\bar v\|^2_{0,P} + |\bar v|^2_1}\,\frac{1+\delta_\tau-\delta_\omega}{1+\varepsilon_\Delta+\varepsilon_\tau-\varepsilon_\omega}\,. \tag{26}$$

Taking into account (1) we see that it suffices to prove

$$\delta_\chi = O(h), \quad \varepsilon_\chi = O(h) \quad (\chi=\tau,\omega), \quad \varepsilon_\Delta = O(h)\,. \tag{27}$$

As we have

$$\delta_\chi = \frac{\|\bar v\|^2_{1,\chi_h}}{\|\bar v\|^2_1} \le \frac{|\bar v|^2_{1,\chi_h}}{|\bar v|^2_1} + \frac{\|\bar v\|^2_{0,\chi_h}}{\|\bar v\|^2_0}$$

the estimate $\delta_\chi \le Ch\,(\chi=\tau,\omega)$ follows from (18) and (19). Similarly, as

$$\varepsilon_\chi \le \frac{|\bar v|^2_{1,\chi_h}}{|\bar v|^2_1}$$

the second estimate (27) for ε_χ follows from (18).

Now we estimate ε_Δ. For this purpose we denote by the symbols τ^Λ_{hP}, τ^Λ_{hE} the parts of τ_{hP}, τ_{hE} along the common boundary $\Lambda = \partial\Omega_P \cap \partial\Omega_E$ and by τ^P_{hP}, ω^P_{hP} the parts of τ_{hP}, ω_{hP} along the boundary $\partial\Omega_P - \Lambda$. ($\tau_{hP}$, τ_{hE}, ω_{hP} were defined by (17).) Let us consider only such division of the domain Ω into subdomains Ω_E and Ω_P demonstrated on Fig. 1, Fig. 2 and Fig. 3. Thus we can write

$$\|v\|^2_{0,P_h} - \|\bar v\|^2_{0,P} = \|v\|^2_{0,\tau^P_{hP}} - \|\bar v\|^2_{0,\omega^P_{hP}} + \|v\|^2_{0,\tau^\Lambda_{hP}} - \|\bar v\|^2_{0,\tau^\Lambda_{hE}}$$

and (24) gives

$$|\varepsilon_\Delta| \le \frac{\|v\|^2_{0,\tau^P_{hP}} + \|\bar v\|^2_{0,\omega^P_{hP}} + \|v\|^2_{0,\tau^\Lambda_{hP}} + \|\bar v\|^2_{0,\tau^\Lambda_{hE}}}{\|\bar v\|^2_{0,P} + |\bar v|^2_1}\,. \tag{28}$$

We have

$$\|v\|^2_{0,\tau^P_{hP}} \le \|v\|^2_{0,\tau_h} = \|\bar v\|^2_{0,\tau_h}\,.$$

Using (19) we obtain

$$\|\bar v\|^2_{0,\tau_h} \le Ch\|\bar v\|^2_0\,.$$

Thus the preceding relations imply

$$\|v\|^2_{0,\tau^P_{hP}} \le Ch\|\bar v\|^2_0 \tag{29}$$

and similarly

$$\|\bar v\|^2_{0,\omega^P_{hP}} \le \|\bar v\|^2_{0,\omega_h} \le Ch\|\bar v\|^2_0\,. \tag{30}$$

To estimate the remaining terms on the right-hand side of (28) we shall need the following relation

$$\max_{\overline{T}} |p| \le C h_T^{-1} \, \|p\|_{0,T} \tag{31}$$

where $p \in P_1$ and the constant C does not depend on T and p.

Relation (31) follows from [2, Lemma 2.2.6], [10, (9.4)] and the fact that $|J| = 2 \ \mathrm{meas}\,(T)$.

It remains to estimate the terms in (28). It holds

$$\|v\|^2_{0,\tau^\Lambda_{hP}} = \|\bar{v}\|^2_{0,\tau^\Lambda_{hP}} \,. \tag{32}$$

Let us denote $\pi_T = \overline{T} - \overline{T}^{\mathrm{id}}$. As $meas(T^{\mathrm{id}} - T) \le C h_T^3$ (see [10, (28.5)]) we find using (31) and the inclusion $\overline{T}^{\mathrm{id}} \subset \overline{T}$

$$\|\bar{v}\|^2_{0,\pi_T} \le \max_{\overline{T}} |\bar{v}|^2 \, meas\, \pi_T \le C h_T^3 \max_{\overline{T}} |\bar{v}|^2 \le C h_T \|\bar{v}\|^2_{0,T} \,.$$

Summing over all π_T from τ^Λ_{hP} and considering (15), (32) and the fact that all triangles $\overline{T}$ containing π_T are interior triangles of the domain Ω we get

$$\|v\|^2_{0,\tau^\Lambda_{hP}} \le C h \|\bar{v}\|^2_0 \,. \tag{33}$$

The proof of the estimate

$$\|\bar{v}\|^2_{0,\tau^\Lambda_{hE}} \le C h \|\bar{v}\|^2_0 \tag{34}$$

follows the same lines.

Now we prove the third estimate in (27). Combining (28), (29), (30) with (33) and (34) we obtain

$$|\varepsilon_\Delta| \le \frac{C h \|\bar{v}\|^2_0}{\|\bar{v}\|^2_{0,P} + |\bar{v}|^2_1} . \tag{35}$$

Thus, as $\bar{v} \in H^1(\Omega)$ we can write by (1) and (35)

$$|\varepsilon_\Delta| \le \frac{C h \|\bar{v}\|^2_1}{\|\bar{v}\|^2_{0,P} + |\bar{v}|^2_1} \le K(\Omega) h$$

which proves $(27)_3$.

Because of validity (27) we have for $h \in (0, h_0)$

$$\frac{1}{2} \le 1 + \varepsilon_\Delta + \varepsilon_\tau - \varepsilon_\omega \,, \quad \frac{1}{2} \le 1 + \delta_\tau - \delta_\omega \le 2 \,.$$

Hence, the preceding estimates, relations (26), (1) and the fact that $\bar{v} \in H^1(\Omega)$ imply

$$\frac{\|v\|^2_{1,\Omega_h}}{\|v\|^2_{0,P_h} + |v|^2_{1,\Omega_h}} \le 2K(\Omega)\,(1 + \delta_\tau - \delta_\omega) \le 4K(\Omega) \quad \forall v \in X_h \,. \tag{36}$$

Inequality (36) is inequality (20) with the constant $C = 4K(\Omega)$ where $K(\Omega)$ is the constant appearing in (1). $\square$

4. Applications

For two media the computation of nonlinear quasistationary two-dimensional electromagnetic field leads to the nonlinear second order parabolic-elliptic initial-boundary value problem of the following type.

There is given a two-dimensional bounded domain Ω and an open nonempty set $\Omega_P \subset \Omega$. We are looking for a function $u = u(x_1, x_2, t)$ (magnetic vector potential) such that

$$\begin{aligned}
\sigma \frac{\partial u_P}{\partial t} &= \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\nu_P \frac{\partial u_P}{\partial x_i}\right) + f_P \quad \text{in} \quad \Omega_P \times (0,T)\,, \\
0 &= \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\nu_E \frac{\partial u_E}{\partial x_i}\right) + f_E \quad \text{in} \quad \Omega_E \times (0,T), \quad \Omega_E = \Omega - \bar{\Omega}_P\,, \\
u_P(x_1, x_2, 0) &= u_0^P(x_1, x_2) \quad \text{in} \quad \Omega_P\,,
\end{aligned}$$

u satisfies a boundary condition of Dirichlet type at least on a part of $\partial\Omega \times (0,T)$ and the transition conditions

$$[u]_E^P = \left[\nu \frac{\partial u}{\partial n^*}\right]_E^P = 0 \quad \text{on} \quad \partial\Omega_E \cap \partial\Omega_P\,.$$

Here the conductivity $\sigma = \sigma(x_1, x_2)$ is a positive function on $\bar{\Omega}$, the reluctivity $\nu_M = \nu_M(x_1, x_2, |\operatorname{grad} u_M|)$ is a positive function on $\Omega_M \times [0, \infty)$ $(M = E, P)$, $f_M = f_M(x_1, x_2, t)$ is a given current density, $u_0^P = u_0^P(x_1, x_2)$ is a given function defined on Ω_P and n^* denotes the normal to $\partial\Omega_E \cap \partial\Omega_P$ oriented in a unique way.

The numerical solution by the finite element method of the above problem has been studied, e.g., in [11], [12], [13], [9], [10]. Let us note that papers [11], [12], [13] have been restricted to domains which can be covered by finite elements exactly; only the domains Ω, Ω_E and Ω_P with polygonal boundaries have been considered.

Taking into account the introduced problem with a nonhomogeneous Dirichlet boundary condition on a part Γ_1 of the boundary we cannot use "classical" Friedrichs' inequality. When we formulate the discrete problem corresponding to this one by using the finite element method with linear functions on triangular elements (the discretization in space) and for example by the implicit Euler method (the discretization in time) then inequality (20) is used in the proof of the existence, and the uniqueness of the approximate solution. Inequality (1) is used in the proofs of both the existence of the solution of the variational formulation of this problem and the convergence of the method (see [7]).

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