ARCHIVUM MATHEMATICUM (BRNO)
Tomus 35 (1999), 329 – 336

ON (1,1)-TENSOR FIELDS ON SYMPLECTIC MANIFOLDS

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Abstract. Two symplectic structures on a manifold M determine a (1,1)-tensor field on M. In this paper we study some properties of this field. Conversely, if A is (1,1)-tensor field on a symplectic manifold (M, ω) then using the natural lift theory we find conditions under which $\omega^A, \omega^A(X,Y) = \omega(AX,Y)$, is symplectic.

INTRODUCTION

Let M be a manifold with two symplectic structures $\omega, \overline{\omega}$. Then the vector bundle morphisms $I_\omega, I_{\overline{\omega}} : TM \to T^*M, I_\omega(X) = i_X\omega, I_{\overline{\omega}}(X) = i_X\overline{\omega}$ determine a (1,1)-tensor field $A = I_\omega^1 \cdot I_\omega$. In Proposition 1 we conclude some properties of A from the point of view of both symplectic structures.

Let A be a (1,1)-tensor field and ω be a symplectic structure on M. Using natural lifts on TM and T^*M we find conditions under which the (0,2)-tensor field $\omega^A, \omega^A(X,Y) = \omega(AX,Y)$, is symplectic in both cases when ω is closed only (Proposition 2) and when ω is exact (Proposition 3). Proposition 4 deals with the same problem in the case when $\omega = dd_vL$ is the basic symplectic structure on TM of a Lagrangian L on TM.

Finally we show (Proposition 5) that if C_*A is the complete lift of A on T^*M, ε is the Liouville 1-form on $T^*M, \omega = d\varepsilon, a = \varepsilon \cdot C_*A$, then $\omega^{C_*A} = da$.

All manifolds and maps in this paper are assumed to be infinitely differentiable.

Two symplectic structures on a manifold M

Let A be a (1,1)-tensor field on a manifold M. Denote by $A : TM \to TM$ and by $A^* : T^*M \to T^*M$ the corresponding vector bundle isomorphisms over Id_M. Let ω be a (0,2)-tensor field on M. We will use the following notations:

$$I_\omega : TM \to T^*M, \quad I_\omega(X) = i_X\omega = \omega(X, -)$$
$$\omega^A : M \to \otimes^2 T^*M, \ \omega^A(X,Y) = \omega(AX,Y),$$
$$\omega_A : M \to \otimes^2 T^*M, \ \omega_A(X,Y) = \omega(X,AY).$$

1991 *Mathematics Subject Classification*: 53C05, 58A20.
Key words and phrases: symplectic structure, natural lifts on tangent and cotangent bundles.
Supported by the VEGA SR No. 1/5011/98.
Received December 10, 1998.

Evidently $I_\omega \cdot A : TM \to T^*M, I_\omega A(X) = i_{AX}\omega, [I_\omega A(X)](Y) = \omega(AX, Y) = \omega^A(X, Y)$.

If ω is symmetric or skew-symmetric, then $\omega_A(X, Y) = (\omega^A)^t(X, Y)$ or $\omega_A(X, Y) = -(\omega^A)^t(X, Y)$, respectively, where $(\omega^A)^t$ is transposed to ω^A. Therefore, if ω is a 2-form, then ω^A is symmetric or skew-symmetric if and only if $\omega_A = -\omega^A$ or $\omega_A = \omega^A$ respectively.

Definition 1. We will say that a (1,1)-tensor field A on M is ω-symmetric if $I_\omega A = A^* I_\omega$.

Lemma 1. *Let ω be a 2-form on M. Then a (1,1)-tensor field A is ω-symmetric if and only if ω^A is skew-symmetric.*

Proof. We have the equalities:

$$I_\omega A(X)(Y) = \omega^A(X, Y),$$
$$[A^* I_\omega(X)](Y) = I_\omega(X)(AY) = \omega(X, AY) = \omega_A(X, Y).$$

Then $I_\omega A = A^* I_\omega$ iff $\omega^A = \omega_A$, i.e. iff ω^A is skew-symmetric. □

Lemma 2. *If both (0,2)-tensor fields ω and ω^A are 2-forms, then $i_A\omega = 2\omega^A$.*

Proof. Recall that $i_A\omega(X, Y) = \omega(AX, Y) + \omega(X, AY)$. By our assumption $\omega_A = \omega^A$. It completes our proof. □

Definition 2. Let $\omega, \overline{\omega}$ be (0,2)-tensor fields on M. We will say that $\overline{\omega}$ is A-related with ω if $I_{\overline{\omega}} = I_\omega \cdot A$, i.e. if $\overline{\omega} = \omega^A$.

Let a (0,2)-tensor field be regular. Then $A := I_\omega^{-1} \cdot I_{\overline{\omega}}$ is a (1,1)-tensor field on M and $\overline{\omega}$ is A-related with ω.

Lemma 3. *If two (0,2)-tensor fields $\omega, \overline{\omega}$ are symmetric or skew-symmetric and ω is regular, then $(I_\omega^{-1} \cdot I_{\overline{\omega}})^* = I_{\overline{\omega}} \cdot I_\omega^{-1}$.*

The proof is evident when using the coordinate expressions.

Corollary of Lemma 1. *If ω and $\overline{\omega}$ are 2-forms and ω is regular then $A = I_\omega^{-1} \cdot I_{\overline{\omega}}$ is ω-symmetric.*

Let both forms ω and $\overline{\omega}$ be symplectic. Then $A = I_\omega^{-1} \cdot I_{\overline{\omega}}$ is regular, $\omega^A = \omega_A = \overline{\omega}$, $I_{\overline{\omega}} = I_\omega A = A^* I_\omega$, $A^* = I_{\overline{\omega}} \cdot I_\omega^{-1}$. As $0 = \omega^A(X, X) = \omega(AX, X)$ therefore the vector fields X and AX are ω-orthogonal for every vector field X on M.

Let $(\alpha, \beta)_\omega$ denote the Poisson bracket of 1-forms α and β in the symplectic manifold (M, ω). Recall that if we denote $I_\omega(X_\gamma) = \gamma$ then two forms α, β are in ω-involution if $\omega(X_\alpha, X_\beta) = 0$. Further, it is said that a vector field X is a local ω-Hamiltonian or an ω-Hamiltonian if $I_\omega(X)$ is closed or exact, respectively, see [4].

Proposition 1. *Let ω and $\overline{\omega}$ be symplectic 2-forms on M. Let $A = I_\omega^{-1} \cdot I_{\overline{\omega}}$. Then*

a) *1-forms $I_{\overline{\omega}}(X), I_{\overline{\omega}}(Y)$ are in $\overline{\omega}$-involution if and only if the 1-forms $A^* I_\omega(X), I_\omega(Y)$ are in ω-involution.*
b) *The forms $I_\omega(X)$ and $A^* I_\omega(X)$ are in ω-involution.*
c) *A vector field X is a local $\overline{\omega}$-Hamiltonian if and only if AX is a local ω-Hamiltonian.*
d) *We have the identities*

$$I_\omega(A[X,Y]) = A^*(I_\omega X, I_\omega(Y))_\omega = (I_{\overline{\omega}}(X), I_{\overline{\omega}}(Y))_{\overline{\omega}} .$$

Proof.

a) $\overline{\omega}(X,Y) = \omega^A(X,Y) = \omega(AX,Y)$. Then the equality $A^* I_\omega(X) = I_\omega(AX)$ yields the proof.
b) The proof is evident from $\omega(AX,X) = 0$.
c) The assertion is the consequence of the identity $I_{\overline{\omega}}(X) = I_\omega(AX)$.
d) By the definition of the Poisson bracket we get

$$(I_{\overline{\omega}}(X), I_{\overline{\omega}}(Y))_{\overline{\omega}} = I_{\overline{\omega}}[X,Y] = A^* I_\omega([X,Y]) = A^*(I_\omega X, I_\omega Y),$$
$$I_\omega A([X,Y]) = A^* I_\omega([X,Y]) = A^*(I_\omega(X), I_\omega(Y)) . \qquad \square$$

Remark. Denote by H_ω or $H_{\overline{\omega}}$ the Lie algebras of all local ω- or $\overline{\omega}$-Hamiltonians, respectively. By Proposition 1, $X \in H_{\overline{\omega}}$ if and only if $AX \in H_\omega$. It is clear that $A|_{H_{\overline{\omega}}} : H_{\overline{\omega}} \to H_\omega$ is an isomorphism of linear spaces which is not the Lie algebras isomorphism in general.

(1,1)-tensor fields on symplectic manifolds

We will deal with a question: Let (M,ω) be a symplectic manifold and A be a (1,1)-tensor field on M. Under what conditions the (0,2)-tensor field ω^A is symplectic?

First of all we recall some lifts of geometrical fields on M to the tangent bundle $p_M : TM \to M$, see [2], [3], [5].

Let (x^i) be a local chart on M. It induces the chart (x^i, x_1^i) on TM. If f or F is a function on M or on TM then we will use the following shortened notations

$$f_i := \frac{\partial f}{\partial x^i} , \quad F_i := \frac{\partial F}{\partial x^i} , \quad F_{i_1} := \frac{\partial F}{\partial x_1^i} .$$

The complete lift of a function $f : M \to R$ is a function $Cf : TM \to R$ such that $Cf(X) = Xf$, $X \in TM$, or equivalently $Cf = S(p_M^* f)$, where S is an arbitrary semispray (a second order differential equation) on TM and $p_M^* f$ is the p_M-pullback of f. In coordinates: $Cf = f_i x_1^i$.

The complete lift of a vector field X on M is the vector field CX on TM the flow of which is the tangent prolongation of the flow of X, $CX = \xi^i \partial/\partial x^i + \xi_k^i x_1^k \partial/\partial x_1^i$, where $X = \xi^i \partial/\partial x^i$.

The complete lift of a p-form ε on M is the p-form $C\varepsilon$ on TM which satisfies the equality

$$C\varepsilon(CX_1, \dots, CX_p) = C(\varepsilon(X_1, \dots, X_p)) \tag{1}$$

for any vector fields $X_1, \dots, X_p$ on M.

In coordinates, if $\varepsilon = \frac{1}{p!}\varepsilon_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$, then

$$C\varepsilon = \frac{1}{p!}\varepsilon_{i_1 \dots i_p, k} x_1^k dx^{i_1} \wedge \dots \wedge dx^{i_p} + \frac{1}{(p-1)!}\varepsilon_{i_1 \dots i_p} dx_1^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} . \tag{2}$$

The C_T-lift of a p-form ε on M is the $p-$form $C_T\varepsilon$ on TM defined by

$$C_T\varepsilon = di_S(p_M^*\omega) \ ,$$

where S is again a semispray on TM and $p_M^*\omega$ is the pull-back of ω. Equivalently, this form can be constructed by the following procedure: Let $X \in TM$. Then the map $\varepsilon_T : X \to i_X\varepsilon$ is a $(p-1)$-form on TM such that

$$C_T\varepsilon = d\varepsilon_T \ .$$

In coordinates,

$$C_T\varepsilon = \frac{1}{(p-1)!}(\varepsilon_{i_1 \dots i_{p-1}k, i_p} x_1^k dx^{i_1} \wedge \dots \wedge dx^{i_p} + \varepsilon_{i_1 \dots i_p} dx_1^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p}) . \tag{3}$$

Finally we recall that the complete lift of a tensor (1,1)-field A on M is a tensor field CA on TM such that $CA(CX) = C(AX)$ for every vector field X on M. In coordinates, if $A = a_j^i dx^j \otimes \partial/\partial x^i$, then

$$CA = a_j^i dx^j \otimes \partial/\partial x^i + (a_{jk}^i x_1^k dx^j + a_j^i dx_1^j) \otimes \partial/\partial x_1^i \ .$$

There are well known the following properties of complete lifts, see [2], [3].

Lemma 4. *Let ε be a p-form and A be a (1,1)-tensor field on M. Then*

a) $dC\varepsilon = Cd\varepsilon$

aa) $C(A \otimes^S \varepsilon) = CA \otimes^S C\varepsilon$,

where $\otimes^S$ denotes a contraction of tensor products.

Corollaries.

1. *If ε is closed, then $C\varepsilon$ is also closed.*
2. *A 2-form $\overline{\omega}$ is A-related with ω if and only if $C\overline{\omega}$ is CA-related with $C\omega$.*

Lemma 5. *Let ω be p-form on M and let $C\omega$ or $C_T\omega$ be its complete or C_T-lifts, respectively. Then ω is closed if and only if $C\omega = C_T\omega$.*

Proof in coordinates. If $\omega = \frac{1}{p!}\omega_{i_1\dots i_p} dx^{i_1}\wedge\dots\wedge dx^{i_p}$ then $d\omega = \frac{1}{p!}\omega_{i_1\dots i_p,k} dx^k \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p}$ and so ω is closed iff

$$(4)\qquad \omega_{i_1\dots i_p,k} - \omega_{i_1\dots i_{p-1}k,i_p} + \omega_{i_1\dots i_{p-2}i_pk,i_{p-1}} + \dots + (-1)^p\omega_{i_2\dots i_pk,i_1} = 0\,.$$

By (2) and (3) we get that $C\omega = C_T\omega$ if and only if

$$(5)\qquad \frac{1}{p!}\,\omega_{i_1\dots i_p,k}x_1^k dx^{i_1}\wedge\dots\wedge dx^{i_p} = \frac{1}{(p-1)!}\,\omega_{i_1\dots i_{p-1}k,i_p}x_1^k dx^{i_1}\wedge\dots\wedge dx^{i_p}\,.$$

For arbitrary vector fields $X_1,\dots,X_p$ on TM we get for the left side L or for the right side R of the equality (5), respectively:

$$\begin{aligned}
L =&\omega_{i_1\dots i_p,k}x_1^k\xi_1^{i_1}\dots\xi_p^{i_p}\\
R =&\frac{1}{(p-1)!}\,\omega_{i_1\dots i_{p-1}k,i_p}x_1^k[(p-1)!\xi_1^{i_1}\dots\xi_p^{i_p} - (p-1)!\xi_1^{i_1}\dots\xi_{p-2}^{i_{p-2}}\xi_p^{i_{p-1}}\xi_{p-1}^{i_p}\\
&+(p-1)!\xi_1^{i_1}\dots\xi_{p-3}^{i_{p-3}}\xi_{p-1}^{i_{p-2}}\xi_p^{i_{p-1}}\xi_{p-2}^{i_p} + \dots + (-1)^{p-1}(p-1)!\xi_2^{i_1}\xi_3^{i_2}\dots\xi_p^{i_{p-1}}\xi_1^{i_p}]\\
=&\omega_{i_1\dots i_{p-1}k,i_p}x_1^k\xi_1^{i_1}\dots\xi_p^{i_p} - \omega_{i_1\dots i_pk,i_{p-1}}x_1^k\xi_1^{i_1}\dots\xi_{p-2}^{i_{p-2}}\xi_p^{i_p}\xi_{p-1}^{i_{p-1}}\\
&+\omega_{i_1\dots i_{p-3}i_{p-1}i_pk,i_{p-2}}\xi_1^{i_1}\dots\xi_{p-3}^{i_{p-3}}\xi_p^{i_p}\xi_{p-1}^{i_{p-1}}\xi_{p-2}^{i_{p-2}} + \dots + (-1)^{p-1}\omega_{i_2\dots i_pk,i_1}\xi_2^{i_2}\dots\\
&\dots\xi_p^{i_p}\xi_1^{i_1} = (\omega_{i_1\dots i_{p-1}k,i_p} - \omega_{i_1\dots i_{p-2}i_pk,i_{p-1}} + \omega_{i_1\dots i_{p-3}i_{p-1}i_pk,i_{p-2}} + \dots\\
&+(-1)^{p-1}\omega_{i_2\dots i_pk,i_1})\xi_1^{i_1}\dots\xi_p^{i_p}\,.
\end{aligned}$$

So $L = R$ if and only if

$$\begin{aligned}
\omega_{i_1\dots i_p,k} = \omega_{i_1\dots i_{p-1}k,ip} - \omega_{i_1\dots i_{p-2}i_pki_{p-1}} + \omega_{i_1\dots i_{p-3}i_{p-1}i_pk,i_{p-2}} + \dots\\
+(-1)^{p-1}\omega_{i_2\dots i_pk,i_1}\,.
\end{aligned}$$

Comparing it with (4) we complete our proof. □

Now we get

Proposition 2. *Let ω be a symplectic 2-form. Let ω^A be skew-symmetric. Then ω^A is symplectic if and only if A is regular and $C\omega^A = C_T\omega^A$.*

Proof. $I_\omega A$ is regular iff A is regular. Then Lemma 5 completes our proof. □

Remark. Let a 2-form $\overline{\omega}$ is A-related to ω. Let X be a vector field on M. Then $A^*I_\omega(X)$ is closed if and only if $C\alpha_X = C_T\alpha_X,\ \alpha_X = I_{\overline{\omega}}(X)$.

(1,1)-TENSOR FIELD ON A MANIFOLD $(\mathbf{M},\omega)$ WITH AN EXACT 2-FORM ω

Let $\varepsilon=\varepsilon_i dx^i$ be a 1-form on M and A be a given (1,1)-tensor field on M. Then we have the forms:

$$\begin{aligned}\overline{\varepsilon}=A^*\varepsilon=\varepsilon_t a_i^t dx^i,\ \omega=d\varepsilon=\varepsilon_{ij}dx^j\wedge dx^i\ ,\\ \overline{\omega}=d(A^*\varepsilon)=(\varepsilon_{tj}a_i^t+\varepsilon_t a_{ij}^t)dx^j\wedge dx^i,\\ C\varepsilon=\varepsilon_{ik}x_1^k dx^i+\varepsilon_i dx_1^i,\ C_T\varepsilon=\varepsilon_{ti}x_1^t dx^i+\varepsilon_i dx_1^i\ .\end{aligned}$$

Let $X=\xi^i\partial/\partial x^i$ is a vector field on M. Then we get in coordinates:

$$\begin{aligned}I_\omega(AX)=(\varepsilon_{it}-\varepsilon_{ti})a_j^t\xi^j dx^i\ ,\\ I_{\overline{\omega}}(X)=(\varepsilon_{tj}a_i^t+\varepsilon_t a_{ij}^t-\varepsilon_{ti}a_j^t-\varepsilon_t a_{ji}^t)\xi^j dx^i\ .\end{aligned}$$

So the form ω^A is skew-symmetric if and only if

$$(\varepsilon_{it}-\varepsilon_{ti})a_j^t=-(\varepsilon_{jt}-\varepsilon_{tj})a_i^t\ . \tag{6}$$

Proposition 3. *The 2-form $\overline{\omega}=d(A^*\varepsilon)$ is A-related to the 2-form $\omega=d\varepsilon$ if and only if ω^A is skew-symmetric and the 2-form $di_{CA}C_T\varepsilon$ is semibasic.*

Proof. As $\overline{\omega}$ is a 2-form, then it is A-related to ω iff ω^A is skew-symmetric and $I_\omega(AX)=I_{\overline{\omega}}(X)$, i.e. iff the equalities (6) and

$$\varepsilon_{tj}a_i^t+\varepsilon_t a_{ij}^t-\varepsilon_t a_{ji}^t=\varepsilon_{it}a_j^t \tag{7}$$

are satisfied.

We get

$$\begin{aligned}CA^*C_T\varepsilon=&i_{CA}C_T\varepsilon=(\varepsilon_{ku}x_1^k a_i^u+\varepsilon_t a_{ik}^t x_1^k)dx^i+\varepsilon_t a_i^t dx_1^i\ ,\\ d(CA^*C_T\varepsilon)=&(\varepsilon_{kuj}x_1^k a_i^u+\varepsilon_{ku}x_1^k a_{ij}^u+\varepsilon_{tj}a_{ik}^t x_1^k+\varepsilon_t a_{ikj}^t x_1^k)dx^j\wedge dx^i\\ &+(\varepsilon_{tj}a_i^t-\varepsilon_{it}a_j^t+\varepsilon_t(a_{ij}^t-a_{ji}^t))dx^i\wedge dx_1^j\ .\end{aligned}$$

Comparing this with (7) we finish our proof. □

REMARK ON A LAGRANGIAN L OF FIRST ORDER ON M WITH A (1,1)-TENSOR FIELD A

Let $L:TM\to R$ be a Lagrangian on M and $v=dx^i\otimes\partial/\partial x_1^i$ be the canonical endomorphism (almost tangent structure). Then $\varepsilon=d_vL=L_{i_1}dx^i$, $\omega=d\varepsilon=L_{i_1j}dx^j\wedge dx^i+L_{i_1j_1}dx_1^j\wedge dx^i$ are the Lagrange forms on TM which are the fundamental objects of the Lagrange formalism of classical mechanics. If A is a (1,1)-tensor field on M we put $\overline{\varepsilon}=i_{CA}\varepsilon=L_{t_1}a_i^t dx^i$ and $\overline{\omega}=d\overline{\varepsilon}=(L_{t_1j}a_i^t+L_{t_1}a_{ij}^t)dx^j\wedge dx^i+L_{t_1j_1}a_i^t dx_1^j\wedge dx^i$. It is easy to prove the following assertion.

Proposition 4. *The 2-form $\overline{\omega}=d(i_{CA}d_vL)$ is CA-related to the Lagrange 2-form $\omega=dd_vL$ if and only if ω^{CA} is skew-symmetric and the 2-form $di_{CA}dL$ is semibasic.*

An example of the canonical 2-form $\overline{\omega}$ which is A-related to the Liouville 2-form on the cotangent bundle T^*M

Let (x^i, z_i) be the induced chart on the cotangent bundle $\pi_M : T^*M \to M$. Then $\varepsilon = z_i dx^i$ is the Liouville 1-form on T^*M and $\omega = d\varepsilon = dz_i \wedge dx^i$ is the canonical symplectic form. A tensor field $A = a^i_j dx^j \otimes \partial/\partial x^i$ on M determines some geometrical objects on T^*M which are closely connected with the natural lifts of A to T^*M. We recall some of them, ([1], [5]):

1. Let $A^* : T^*M \to T^*M$ be the dual vector bundle morphism to $A : TM \to TM$. Then $a = a^j_i z_j dx^i : T^*M \to T^*T^*M$ is a 1-form on T^*M such that $a(z, X) = A^*z(T\pi_M X)$ for every $X \in T_z T^*M$. Put

$$\overline{\omega} = da = a^k_{ij} z_k dx^j \wedge dx^i + a^j_i dz_j \wedge dx^i \,.$$

This immediately gives

Lemma 6. *The 2-form da is symplectic if and only if the (1,1)-tensor field A is regular.*

2. The complete lift C_*A of A to T^*M is a (1,1)-tensor field on T^*M such that

$$da(Y, X) = < i_Y d\varepsilon,\ C_*A(X) > \,, \tag{8}$$

where the symbol $< >$ means the evaluation mapping.

In coordinates

$$C_*A = a^i_j dx^j \otimes \partial/\partial x^i + [(a^k_{ji} - a^k_{ij}) z_k dx^j + a^j_i dz_j] \otimes \partial/\partial z_i \,.$$

It is evident that $a = i_{C_*A}\varepsilon$.

Proposition 5. *The 2-form $\overline{\omega} = da$ is C_*A-related to the 2-form $d\varepsilon$.*

Proof. As $< i_Y d\varepsilon, C_*A(X) >= d\varepsilon(Y, C_*A(X)) = -d\varepsilon(C_*A(X), Y)$ therefore the equality (8) can be rewritten in the form $i_X da = i_{C_*AX} d\varepsilon$. It means that da is C_*A-related with $d\varepsilon$, i.e. $da = (d\varepsilon)^{C_*A}$. □

Corollary. *As da is a 2-form therefore $d\varepsilon^{C_*A}$ is skew-symmetric and therefore C_*A is $d\varepsilon$-symmetric, i.e. $I_{d\varepsilon}(C_*A) = (C_*A)^* I_{d\varepsilon}$.*

Let $\alpha = \alpha_i dx^i$, $\alpha : M \to T^*M$, be a 1-form on M. Then $\alpha^v = \pi^*_M \alpha = \alpha_i dx^i$ is the socalled vertical lift and $X^v_\alpha = \alpha_i \partial/\partial x^i$ is the vertical vector field on T^*M, induced by the section α and by the identification $VT^*M = T^*M \times_M T^*M$. Since $I_{d\varepsilon}(X^v_\alpha) = \alpha^v$ therefore X^v_α is a Hamiltonian of the symplectic manifold $(T^*M, d\varepsilon)$ iff α is closed. It implies that the 1-form $A^*\alpha$ is closed iff the vertical field $X^v_{A^*\alpha}$ is a dε-Hamiltonian.

Recall that the complete lift $C_*X = \xi^i \partial/\partial x^i - \xi^k_i z_k \partial/\partial z_i$ of a vector field $X = \xi^i \partial/\partial x^i$ on M to the cotangent bundle is a $d\varepsilon$-Hamiltonian and $I_{d\varepsilon}(C_*X) = -df_X$, where $f_X(z) =< z, X >= z_i \xi^i$ is a function on T^*M determined by X. If A is regular, then AX is a vector field on M and $I_{d\varepsilon}(C_*AX) = -df_{AX}$. We have

$$C_*A(C_*X) = a^i_t \xi^t \partial/\partial x^i + [(a^k_{it} - a^k_{ti}) z_k \xi^t - a^j_i \xi^k_j z_k] \partial/\partial z_i \ ,$$
$$C_*(AX) = a^i_t \xi^t \partial/\partial x^i - (a^k_{ti} \xi^t + a^k_t \xi^t_i) z_k \partial/\partial z_i \,.$$

Proposition 6. *Let A be a regular (1,1)-tensor field. Then the vector fields $C_*(AX)$ and $C_*A(C_*X)$ are equal if and only if the 1-forms df_{AX} and $i_{C_*A}df_X$ are also equal.*

Proof. The field C_*A is $d\varepsilon$-symmetric, therefore $I_{d\varepsilon}(C_*A(C_*X)) = (C_*A)^* I_{d\varepsilon}(C_*X) = -(C_*A)^*(df_X)$. Then the equality $I_{d\varepsilon}(C_*A(C_*X) - C_*(AX)) = -(C_*A)^*(df_X) + df_{AX}$ completes our proof. □

Corollary. *The vector field $C_*A(C_*X)$ is a $d\varepsilon$-Hamiltonian iff $i_{C_*A}df_X = df_{AX}$.*

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