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STABILITY OF QUADRATIC INTERPOLATION POLYNOMIALS IN VERTICES OF TRIANGLES WITHOUT OBTUSE ANGLES

Josef Dalík

Abstract. An explicit description of the basic Lagrange polynomials in two variables related to a six-tuple $a^1, \dots, a^6$ of nodes is presented. Stability of the related Lagrange interpolation is proved under the following assumption: $a^1, \dots, a^6$ are the vertices of triangles $T_1, \dots, T_4$ without obtuse inner angles such that T_1 has one side common with T_j for $j = 2, 3, 4$.

1. Introduction and notations

We say that the nodes $a^1, \dots, a^6$ are *regular* if the following conditions (a), (b) are satisfied.

(a) $T_1 = \overline{a^1a^3a^5}$ is a central triangle and $T_2 = \overline{a^1a^2a^3}$, $T_3 = \overline{a^3a^4a^5}$, $T_4 = \overline{a^5a^6a^1}$ are its neighbours.

(b) $\alpha \le \pi/2$ for all inner angles α of $T_1, \dots, T_4$.

We denote by h the maximal length of sides of $T_1, \dots, T_4$, by α_0 the minimal inner angle of $T_1, \dots, T_4$ and by T the (closed) convex hull of $T_1 \cup \dots \cup T_4$ in R^2. We adopt the convention that a^{i+j} denotes the node a^{i+j-6} for any $i, j \in \{1, \dots, 6\}$ such that $i + j > 6$.

For $i = 1, \dots, 6$, we denote by l_i a certain quadratic polynomial such that $l_i(a^j) = 0$ whenever $j \ne i$ and describe a positive lower bound of $|l_i(a^i)|$ explicitly.

The fact that $l_i(a^i) \ne 0$ for $i = 1, \dots, 6$ guaranties existence and unicity of the solution of Lagrange interpolation problem in the nodes $a^1, \dots, a^6$ by second–degree polynomials in two variables. In Sauer, Xu [2], such nodes are called poised and are studied in a general setting. The value of lower estimate of $|l_i(a^i)|$ from our Main Theorem assures stability of the interpolation polynomial in the following sense: In Dalík [1], it is derived from this estimate that interpolation errors of values of functions as well as of their first and second partial derivatives are of optimal order.

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We denote by x_1, x_2 the coordinates of any point $x \in R^2$. Let a, b, c, $d \in R^2$. We define the halflines

$$\overleftharpoon{ab} = \{a + t\cdot \overrightarrow{ab}; 0 \le t\}, \quad \overrightharpoon{ab} = \{b + t\cdot \overrightarrow{ba}; 0 \le t\}$$

and put $\overset{\rightleftharpoons}{ab} = \overleftharpoon{ab} \cap \overrightharpoon{ab}$, $\overline{ab} = \overleftharpoon{ab} \cup \overrightharpoon{ab}$. We denote by $|ab|$ the length of the segment $\overset{\rightleftharpoons}{ab}$ and by $d(a, \overline{bc})$ the distance of a from the line $\overline{bc}$. Further, we put

$$D(abc) = \frac{1}{2}\begin{vmatrix} a_1 - c_1 & a_2 - c_2 \\ b_1 - c_1 & b_2 - c_2 \end{vmatrix}, \qquad P(abc) = |D(abc)|.$$

Then $P(abc)$ is the area of the triangle $\overline{abc}$ and $D(abc) > 0$, $D(abc) < 0$ whenever the orientation of a, b, c is positive, negative respectively. We denote by $P(abcd)$ the area of the tetragon $\overline{abcd}$ and abbreviate $P(T_i)$ by P_i for $i = 1, \dots, 4$.

2. Main theorem

Definition. We relate the polynomial

$$\begin{aligned} l_i(x) = {} & D(xa^{4+i}a^{5+i})D(xa^{1+i}a^{2+i})D(a^{3+i}a^{4+i}a^{2+i})D(a^{3+i}a^{5+i}a^{1+i}) \\ & - D(xa^{2+i}a^{4+i})D(xa^{1+i}a^{5+i})D(a^{3+i}a^{4+i}a^{5+i})D(a^{3+i}a^{1+i}a^{2+i}) \end{aligned}$$

to $i = 1, \dots, 6$.

Lemma 1. *If $i, j \in \{1, \dots, 6\}$ then we have*

$$l_i(a^j) = \begin{cases} 0 & j \ne i, \\ (-1)^{i+1}[D(a^1a^5a^6)D(a^1a^2a^3)D(a^4a^5a^3)D(a^4a^6a^2) & \\ \qquad -D(a^1a^3a^5)D(a^1a^2a^6)D(a^4a^5a^6)D(a^4a^2a^3)] & j = i. \end{cases}$$

Proof. See [1]. □

Notations. We denote by $\alpha_i, \beta_i, \gamma_i, \delta_i$ the inner angle at a^i in the triangle T_1, T_2, T_3, T_4 respectively and put $\varphi_1 = \alpha_1 + \beta_1 + \delta_1$, $\varphi_2 = \beta_2$, $\varphi_3 = \alpha_3 + \beta_3 + \gamma_3$, $\varphi_4 = \gamma_4$, $\varphi_5 = \alpha_5 + \gamma_5 + \delta_5$, $\varphi_6 = \delta_6$. See Fig.1. By definition,

$$\begin{aligned} l_1(a^1) = {} & D(a^1a^5a^6)D(a^1a^2a^3)D(a^4a^5a^3)D(a^4a^6a^2) \\ & - D(a^1a^3a^5)D(a^1a^2a^6)D(a^4a^5a^6)D(a^4a^2a^3). \end{aligned} \tag{1}$$

We denote this expression more exactly by $l(a^1, a^2, a^3, a^4, a^5, a^6)$ and more briefly by l.

We find a positive lower estimate of $l_1(a^1)$ in each of the four cases characterized by the number of angles from the set $\{\varphi_1, \varphi_3, \varphi_5\}$ which are less than π separately.

Lemma 2. *Let $a^1, \dots, a^6$ be regular nodes such that $\varphi_1 < \pi$, $\varphi_3 < \pi$, $\varphi_5 < \pi$. Then*

$$l_1(a^1) \geq \frac{7}{8} \cdot P_1 P_2 P_3 P_4.$$

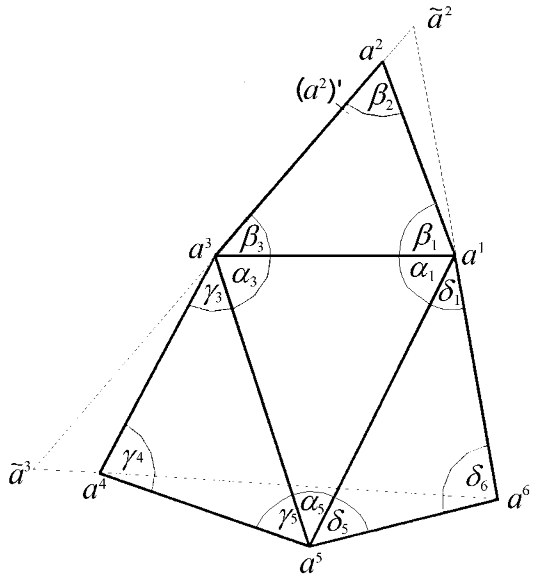

Fig. 1

Proof. In this case we have

$$D(a^1a^5a^6)D(a^1a^2a^3)D(a^4a^5a^3)D(a^4a^6a^2) > 0 \quad \text{and}$$

$$D(a^1a^3a^5)D(a^1a^2a^6)D(a^4a^5a^6)D(a^4a^2a^3) > 0,$$

so that l is a difference of two positive values. We observe

$$\text{(2)} \qquad \overleftarrow{a^4a^5} \cap \overleftarrow{a^2a^1} \neq \emptyset, \quad \overleftarrow{a^6a^1} \cap \overleftarrow{a^4a^3} \neq \emptyset, \quad \overleftarrow{a^2a^3} \cap \overleftarrow{a^6a^5} \neq \emptyset :$$

It follows by $\varphi_3 < \pi$, $\varphi_2 \leq \frac{\pi}{2}$ and $\varphi_4 \leq \frac{\pi}{2}$ that $\overleftarrow{a^4a^5} \cap \overleftarrow{a^2a^1} \neq \emptyset$. The rest of (2) can be justified analogically.

$$\text{(3)} \quad (a^2)' \in \overline{\overline{a^2a^3}} \;\Rightarrow\; l(a^1, (a^2)', a^3, a^4, a^5, a^6) \leq l(a^1, a^2, a^3, a^4, a^5, a^6)\frac{|(a^2)'a^3|}{|a^2a^3|} :$$

With respect to (2) we put $\overleftarrow{a^3a^2} \cap \overleftarrow{a^6a^1} = \{\tilde{a}^2\}$ and $\overleftarrow{a^2a^3} \cap \overleftarrow{a^6a^4} = \{\tilde{a}^3\}$; see Fig.1. Then

$$D(a^1(a^2)'a^3) = D(a^1a^2a^3)\frac{|(a^2)'a^3|}{|a^2a^3|}, \quad D(a^4a^6(a^2)') = D(a^4a^6a^2)\frac{|(a^2)'\tilde{a}^3|}{|a^2\tilde{a}^3|},$$

$$D(a^4(a^2)'a^3) = D(a^4a^2a^3)\frac{|(a^2)'a^3|}{|a^2a^3|}, \quad D(a^1(a^2)'a^6) = D(a^1a^2a^6)\frac{|(a^2)'\tilde{a}^2|}{|a^2\tilde{a}^2|}.$$

These identities and the fact that $\frac{|(a^2)'\tilde{a}^3|}{|a^2\tilde{a}^3|} \le 1 \le \frac{|(a^2)'\tilde{a}^2|}{|a^2\tilde{a}^2|}$ lead to

$$l(a^1, (a^2)', a^3, a^4, a^5, a^6) =$$

$$D(a^1a^5a^6)D(a^1a^2a^3)D(a^4a^5a^3)D(a^4a^6a^2)\frac{|(a^2)'a^3||(a^2)'\tilde{a}^3|}{|a^2a^3||a^2\tilde{a}^3|} -$$

$$D(a^1a^3a^5)D(a^1a^2a^6)D(a^4a^5a^6)D(a^4a^2a^3)\frac{|(a^2)'\tilde{a}^2||(a^2)'a^3|}{|a^2\tilde{a}^2||a^2a^3|} \le$$

$$l(a^1, a^2, a^3, a^4, a^5, a^6)\frac{|(a^2)'a^3||(a^2)'\tilde{a}^3|}{|a^2a^3||a^2\tilde{a}^3|} \le l(a^1, a^2, a^3, a^4, a^5, a^6)\frac{|(a^2)'a^3|}{|a^2a^3|}.$$

Hence, if we choose a point $(a^2)'$, $(a^4)'$ and $(a^6)'$ in the segment $\overset{\rightleftharpoons}{a^2a^3}$, $\overset{\rightleftharpoons}{a^4a^5}$ and $\overset{\rightleftharpoons}{a^6a^1}$ in such a way that the triangle $T_2' = \overline{a^1(a^2)'a^3}$, $T_3' = \overline{a^3(a^4)'a^5}$ and $T_4' = \overline{a^5(a^6)'a^1}$ has a right angle at $(a^2)'$, $(a^4)'$ and $(a^6)'$ respectively then we obtain

$$l(a^1, (a^2)', a^3, (a^4)', a^5, (a^6)') \le l(a^1, a^2, a^3, a^4, a^5, a^6)\frac{P_2'P_3'P_4'}{P_2P_3P_4}; \tag{4}$$

here P_i' denotes the area of T_i' for $i = 2, 3, 4$.

Obviously, we have

$$\varphi_2 = \varphi_4 = \varphi_6 = \frac{\pi}{2} \quad \Rightarrow \quad \varphi_1 + \varphi_3 + \varphi_5 = \frac{5}{2}\pi \quad \text{and} \tag{5}$$

$$\varphi_2 = \varphi_4 = \varphi_6 = \frac{\pi}{2} \quad \Rightarrow \tag{6}$$

$$l = P_2P_3P_4\big[P(a^4a^6a^2) - P(a^1a^3a^5)\sin\varphi_1\sin\varphi_3\sin\varphi_5\big],$$

since in this case $l = \frac{1}{8}|a^1a^2||a^2a^3||a^3a^4||a^4a^5||a^5a^6||a^6a^1|\big[P(a^4a^6a^2) - P(a^1a^3a^5)\sin\varphi_1\sin\varphi_3\sin\varphi_5\big]$.

$$\varphi_2 = \varphi_4 = \varphi_6 = \frac{\pi}{2} \quad \Rightarrow \quad \sin\varphi_1\sin\varphi_3\sin\varphi_5 \le \frac{1}{8}: \tag{7}$$

According to (5), it is sufficient to find a maximum of the expression

$$V(\varphi_1, \varphi_3) = \sin\varphi_1\sin\varphi_3\sin(\frac{5}{2}\pi - \varphi_1 - \varphi_3)$$

on the domain

$$\Omega = \{[\varphi_1, \varphi_3]; \frac{\pi}{2} < \varphi_1 < \pi, \frac{3\pi}{2} - \varphi_1 < \varphi_3 < \pi\}.$$

One can easily see that V attains its maximum $\frac{1}{8}$ at the point $[\frac{5\pi}{6}, \frac{5\pi}{6}]$ from Ω.

$$\varphi_2 = \varphi_4 = \varphi_6 = \frac{\pi}{2} \quad \Rightarrow \quad P(a^1a^3a^5) \le P(a^4a^6a^2) : \tag{8}$$

In the proof of (8), we denote by $\overset{\frown}{a^3a^5}$ that half of the circle with diameter $\overline{\overline{a^3a^5}}$ whose endpoints are a^3, a^5 which satisfies $a^4 \in \overset{\frown}{a^3a^5}$. We assume that $a^3 \notin \overset{\frown}{a^3a^5}$ and $a^5 \notin \overset{\frown}{a^3a^5}$. In the same sense we will use the arc $\overset{\frown}{a^5a^1}$.

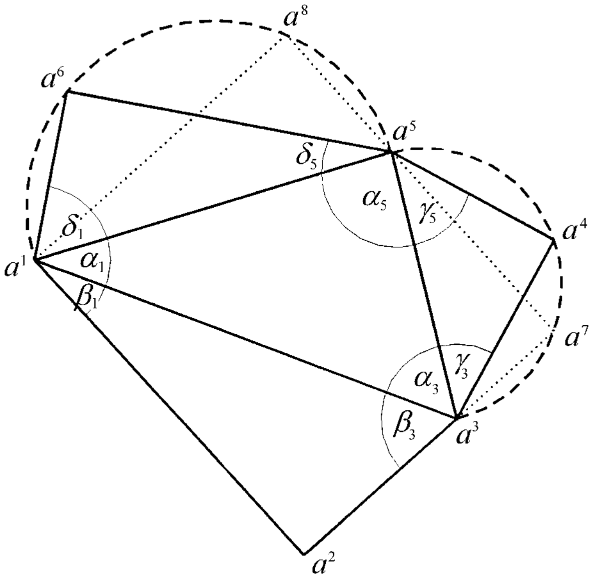

Fig. 2

We first prove (8) under the following condition:

(*) There is a sharp angle between $\delta_1+\alpha_1, \alpha_1+\beta_1, \beta_3+\alpha_3, \alpha_3+\gamma_3, \gamma_5+\alpha_5, \alpha_5+\delta_5$.

Let us assume that $\alpha_1 + \beta_1 < \frac{\pi}{2}$ like in Fig.2. Then, necessarily, $\beta_3 + \alpha_3 > \frac{\pi}{2}$ and we put $\overleftarrow{a^2a^3} \cap \overset{\frown}{a^3a^5} = \{a^7\}$. It follows by $\alpha_1 + \beta_1 < \frac{\pi}{2}$ and $\overline{a^1a^2} \parallel \overline{a^5a^7}$ that $\sphericalangle a^7a^5a^1 > \frac{\pi}{2}$. Then we denote by a^8 the point of intersection of $\overleftarrow{a^7a^5}$ and $\overset{\frown}{a^5a^1}$ and we see that $\overline{a^1a^2a^7a^8}$ is a rectangle such that

$$P(a^1a^3a^5) \le \frac{1}{2} P(a^1a^2a^7a^8).$$

At the same time,

$$\frac{1}{2}P(a^1a^2a^7a^8) = P(a^1a^2a^5)$$

and, as $\overleftarrow{a^4a^5} \cap \overleftarrow{a^2a^1} \neq \emptyset$ by (2), we have

$$P(a^1a^2a^5) \leq P(a^1a^2a^4).$$

Similarly, $\overleftarrow{a^6a^1} \cap \overleftarrow{a^4a^3} \neq \emptyset$ implies

$$P(a^1a^2a^4) \leq P(a^4a^6a^2)$$

and the last four relations give us (8).

Let us now assume that (*) is not true. We put $\overleftarrow{a^2a^3} \cap \overset{\frown}{a^3a^5} = \{a^7\}$, $\overleftarrow{a^6a^5} \cap \overset{\frown}{a^3a^5} = \{(a^7)'\}$ and prove (8) in the first of the two symmetric cases

$$d(a^7, \overline{a^6a^2}) \leq d((a^7)', \overline{a^6a^2}), \quad d(a^7, \overline{a^6a^2}) > d((a^7)', \overline{a^6a^2})$$

only. See Fig.3. If we denote $\overleftarrow{a^2a^1} \cap \overset{\frown}{a^5a^1} = \{a^8\}$ then $\overline{a^2a^7a^5a^8}$ is a rectangle and

$$P(a^1a^3a^5) \leq \frac{1}{2}P(a^2a^7a^5a^8) = P(a^2a^7a^5).$$

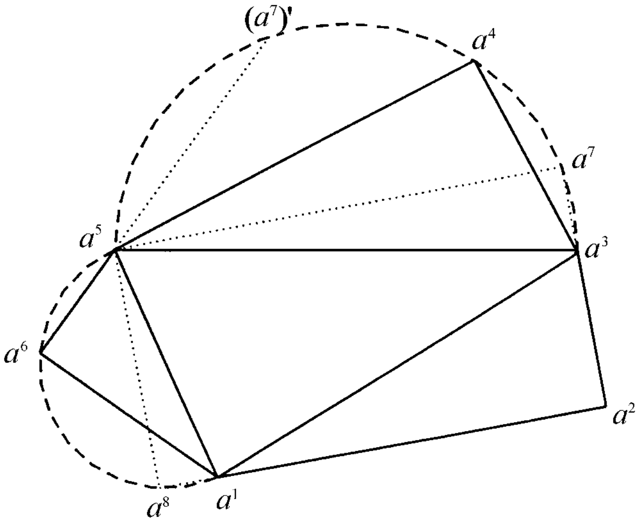

Fig. 3

As $\overleftarrow{a^2a^3} \cap \overleftarrow{a^6a^5} \neq \emptyset$ by (2), we have

$$P(a^2a^7a^5) \leq P(a^2a^7a^6)$$

and the condition $d(a^7, \overline{a^6a^2}) \leq d((a^7)', \overline{a^6a^2})$ leads to $d(a^7, \overline{a^6a^2}) \leq d(a^4, \overline{a^6a^2})$. Hence

$$P(a^2a^7a^6) \leq P(a^2a^4a^6)$$

and the last three relations imply (8).

Now, the statement is a consequence of (6), (7), (8) and (4). □

Lemma 3. *Let $a^1, \ldots, a^6$ be regular nodes such that $\varphi_1 < \pi$, $\varphi_3 < \pi$ and $\varphi_5 \geq \pi$. Then*

$$l_1(a^1) > P_2(P_3)^2 P_4 \quad \text{or} \quad l_1(a^1) > P_2 P_3 (P_4)^2.$$

Proof. We have

$$D(a^1a^5a^6)D(a^1a^2a^3)D(a^4a^5a^3)D(a^4a^6a^2) > 0 \quad \text{and}$$

$$D(a^1a^3a^5)D(a^1a^2a^6)D(a^4a^5a^6)D(a^4a^2a^3) \leq 0,$$

so that

$$l \geq D(a^1a^5a^6)D(a^1a^2a^3)D(a^4a^5a^3)D(a^4a^6a^2). \tag{9}$$

We can see that either $d(a^2, \overline{a^4a^6}) > d(a^1, \overline{a^4a^6})$ or $d(a^2, \overline{a^4a^6}) > d(a^3, \overline{a^4a^6})$ and, at the same time,

$$d(a^6, \overline{a^3a^4}) > d(a^5, \overline{a^3a^4}). \tag{10}$$

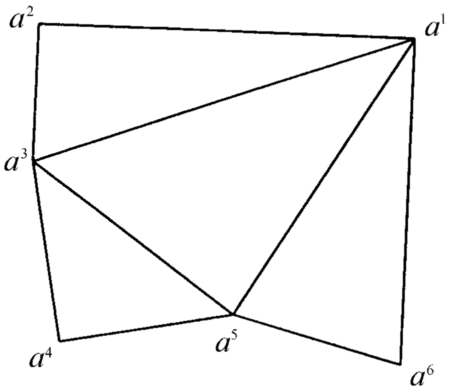

Fig. 4

Let us consider the case $d(a^2, \overline{a^4a^6}) > d(a^3, \overline{a^4a^6})$ from Fig.4. This relation and (10) imply $P(a^4a^6a^2) > P(a^4a^5a^3)$. This and (9) give $l > P_2(P_3)^2P_4$. The case $d(a^2, \overline{a^4a^6}) > d(a^1, \overline{a^4a^6})$ leads to $l > P_2P_3(P_4)^2$. □

Lemma 4. *Let $a^1, \dots, a^6$ be regular nodes such that $\varphi_1 < \pi$, $\varphi_3 \geq \pi$ and $\varphi_5 \geq \pi$. Then*

$$l_1(a^1) > \frac{2}{3} \sin^2 \alpha_0 P_2 (P_3)^2 P_4.$$

Proof. In this case

$$D(a^1 a^5 a^6) D(a^1 a^2 a^3) D(a^4 a^5 a^3) D(a^4 a^6 a^2) > 0 \quad \text{and}$$

$$D(a^1 a^3 a^5) D(a^1 a^2 a^6) D(a^4 a^5 a^6) D(a^4 a^2 a^3) \geq 0,$$

so that l is a difference of two non-negative values. If we use the points $(a^3)'$, $(a^3)''$, $(a^5)'$, $(a^5)''$ from Fig.5 then we have

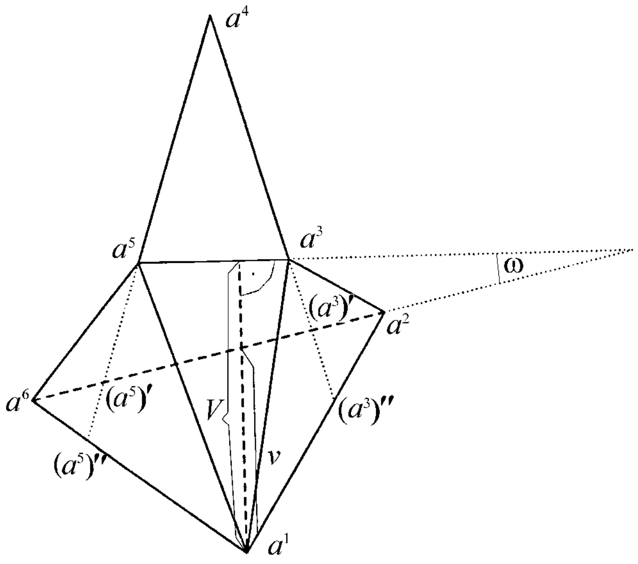

Fig. 5

$$(11) \qquad \begin{aligned} &D(a^1 (a^5)' a^6) D(a^1 a^2 (a^3)') D(a^4 (a^5)' (a^3)') D(a^4 a^6 a^2) = \\ &D(a^1 (a^3)' (a^5)') D(a^1 a^2 a^6) D(a^4 (a^5)' a^6) D(a^4 a^2 (a^3)'), \end{aligned}$$

since the value of both sides of (11) is equal to

$$\frac{1}{16} d(a^1, \overline{a^6 a^2})^2 d(a^4, \overline{a^6 a^2})^2 |(a^5)' a^6| |a^2 (a^3)'| |(a^5)' (a^3)'| |a^2 a^6|.$$

$$(12) \qquad l > P_2 (P_3)^2 P_4 \left(1 - \frac{P(a^1 a^3 a^5) |(a^3)' (a^3)''| |(a^5)' (a^5)''|}{P(a^1 (a^3)' (a^5)') |a^3 (a^3)''| |a^5 (a^5)''|}\right) :$$

If we insert

$$D(a^1a^5a^6)\frac{|(a^5)'(a^5)''|}{|a^5(a^5)''|},\ D(a^1a^2a^3)\frac{|(a^3)'(a^3)''|}{|a^3(a^3)''|},\ D(a^4a^5a^3)\frac{|(a^3)'a^4||(a^5)'a^4|}{|a^3a^4||a^5a^4|},$$

$$D(a^4a^5a^6)\frac{|a^4(a^5)'|}{|a^4a^5|},\ D(a^4a^2a^3)\frac{|a^4(a^3)'|}{|a^4a^3|}$$

instead of $D(a^1(a^5)'a^6)$, $D(a^1a^2(a^3)')$, $D(a^4(a^5)'(a^3)')$, $D(a^4(a^5)'a^6)$, $D(a^4a^2(a^3)')$ respectively into (11) then we get

$$D(a^1a^5a^6)D(a^1a^2a^3)D(a^4a^5a^3)D(a^4a^6a^2)\frac{|(a^3)'(a^3)''||(a^5)'(a^5)''|}{|a^3(a^3)''||a^5(a^5)''|} =$$

$$D(a^1(a^3)'(a^5)')D(a^1a^2a^6)D(a^4a^5a^6)D(a^4a^2a^3).$$

Now, we multiply this equality by $P(a^1a^3a^5)/P(a^1(a^3)'(a^5)')$ and write the resulting left-hand side instead of $D(a^1a^3a^5)D(a^1a^2a^6)D(a^4a^5a^6)D(a^4a^2a^3)$ into l. We arrive at $l =$

$$D(a^1a^5a^6)D(a^1a^2a^3)D(a^4a^5a^3)D(a^4a^6a^2)(1-\frac{P(a^1a^3a^5)|(a^3)'(a^3)''||(a^5)'(a^5)''|}{P(a^1(a^3)'(a^5)')|a^3(a^3)''||a^5(a^5)''|}).$$

This and the fact that $P(a^4a^6a^2) > P(a^4a^5a^3)$ give (12).

Let us denote by ω the angle between the lines $\overline{a^3a^5}$, $\overline{(a^3)'(a^5)'}$ and put

$$V = d(a^1, \overline{a^3a^5}),\ v = \frac{d(a^1, \overline{(a^3)'(a^5)'})}{\cos\omega}.$$

See Fig.5. We can easily see that $|a^3a^5| < |(a^3)'(a^5)'|\cos\omega$. By means of this fact, we verify

$$\frac{P(a^1a^3a^5)}{P(a^1(a^3)'(a^5)')} < \frac{V}{v}: \tag{13}$$

$$P(a^1a^3a^5) = \frac{1}{2}V|a^3a^5| < \frac{1}{2}V|(a^3)'(a^5)'|\cos\omega =$$

$$\frac{1}{2}v\cos\omega|(a^3)'(a^5)'|\frac{V}{v} = P(a^1(a^3)'(a^5)')\frac{V}{v}.$$

It follows by (13) that

$$1-\frac{P(a^1a^3a^5)|(a^3)'(a^3)''||(a^5)'(a^5)''|}{P(a^1(a^3)'(a^5)')|a^3(a^3)''||a^5(a^5)''|} > 1-\frac{V|(a^3)'(a^3)''||(a^5)'(a^5)''|}{v|a^3(a^3)''||a^5(a^5)''|}. \tag{14}$$

Because $\frac{V}{v} < \frac{|a^3(a^3)''|}{|(a^3)'(a^3)''|}$ and $\frac{V}{v} < \frac{|a^5(a^5)''|}{|(a^5)'(a^5)''|}$ hold obviously, we obtain

$$1-\frac{V|(a^3)'(a^3)''||(a^5)'(a^5)''|}{v|a^3(a^3)''||a^5(a^5)''|} > \max\{\frac{|a^3(a^3)'|}{|a^3(a^3)''|}, \frac{|a^5(a^5)'|}{|a^5(a^5)''|}\}. \tag{15}$$

We put

$$s \equiv s(a^1, a^2, a^3, a^4, a^5, a^6) \equiv \max\Big\{\frac{|a^3(a^3)'|}{|a^3(a^3)''|}, \frac{|a^5(a^5)'|}{|a^5(a^5)''|}\Big\}.$$

Since the following implications

$$\begin{gathered} b \in \overleftarrow{a^1 a^6},\ |a^1 b| > |a^1 a^6| \quad \Rightarrow \\ s(a^1, a^2, a^3, a^4, a^5, b) < s(a^1, a^2, a^3, a^4, a^5, a^6), \end{gathered} \tag{16}$$

$$\overline{a^3 a^4 a^5} \subset \overline{a^3 b a^5} \ \Rightarrow\ s(a^1, a^2, a^3, b, a^5, a^6) < s(a^1, a^2, a^3, a^4, a^5, a^6) \tag{17}$$

are true obviously, we will find the lower bound of s under the assumptions

$$\beta_3 = \frac{\pi}{2} = \delta_5,\ \gamma_4 = \alpha_0. \tag{18}$$

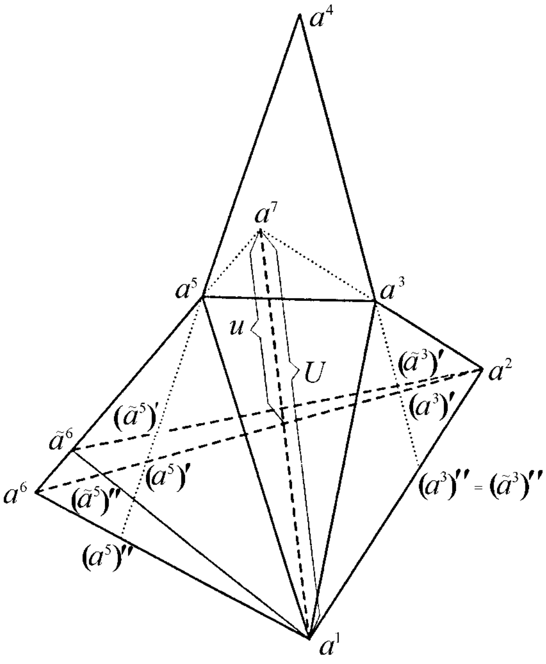

Fig. 6

Then we have

$$\overleftarrow{a^5 a^4} \cap \overleftarrow{a^1 a^2} = \emptyset, \tag{19}$$

weil $\gamma_4 = \alpha_0 \leq \beta_1 \leq \sphericalangle a^2(a^3)''a^3$. By means of the symbols from Fig.6, we verify the assertion

$$\tilde{a}^6 \in a^5 \overset{=}{} a^6 \;\Rightarrow\; s(a^1,a^2,a^3,a^4,a^5,\tilde{a}^6) \leq s(a^1,a^2,a^3,a^4,a^5,a^6) : \tag{20}$$

We prove that $\frac{|a^3(\tilde{a}^3)'|}{|a^3(\tilde{a}^3)''|} \leq \frac{|a^3(a^3)'|}{|a^3(a^3)''|}$ and $\frac{|a^5(\tilde{a}^5)'|}{|a^5(\tilde{a}^5)''|} \leq \frac{|a^5(a^5)'|}{|a^5(a^5)''|}$. Clearly, it is sufficient to verify the second inequality. Under the assumption $|(a^5)''(\tilde{a}^5)''| \leq |(a^5)''(a^5)'|$, we can see that

$$\frac{|a^5(\tilde{a}^5)'|}{|a^5(\tilde{a}^5)''|} \leq \frac{|a^5(a^5)'|}{|a^5(a^5)''|} \quad\Leftrightarrow\quad \frac{|(a^5)'(a^5)''|}{|a^5(a^5)''|} \leq \frac{|(\tilde{a}^5)'(\tilde{a}^5)''|}{|a^5(\tilde{a}^5)''|} \quad\Leftrightarrow$$

$$\frac{|(a^5)''(\tilde{a}^5)''| + |(\tilde{a}^5)''(a^5)'|}{|(a^5)''(\tilde{a}^5)''| + |(\tilde{a}^5)''(a^5)'| + |(a^5)'a^5|} \leq \frac{|(\tilde{a}^5)'(a^5)'| + |(a^5)'(\tilde{a}^5)''|}{|a^5(a^5)'| + |(a^5)'(\tilde{a}^5)''|} \quad\Leftrightarrow$$

$$\frac{|a^5(a^5)'|}{|(\tilde{a}^5)'(a^5)'|} \leq \frac{|a^5(a^5)''|}{|(\tilde{a}^5)''(a^5)''|} \quad\Leftrightarrow\quad \frac{|a^5a^6||a^2\tilde{a}^6|}{|\tilde{a}^6a^6||a^2(\tilde{a}^5)'|} \leq \frac{|a^5a^6||a^1\tilde{a}^6|}{|\tilde{a}^6a^6||a^1(\tilde{a}^5)''|} \quad\Leftrightarrow$$

$$\frac{|a^2\tilde{a}^6|}{|a^2(\tilde{a}^5)'|} \leq \frac{|a^1\tilde{a}^6|}{|a^1(\tilde{a}^5)''|}$$

and the last inequality is true by (19). The case $|(a^5)''(\tilde{a}^5)''| > |(a^5)''(a^5)'|$ leads to the same conclusion.

Because of (20), we can extend the assumptions (18) by

$$\beta_1 = \alpha_0 = \delta_1. \tag{21}$$

The lengths u and U from Fig.6 satisfy

$$s > \frac{u}{U}, \tag{22}$$

because it holds either $\frac{u}{U} < \frac{|a^3(a^3)'|}{|a^3(a^3)''|}$ or $\frac{u}{U} < \frac{|a^5(a^5)'|}{|a^5(a^5)''|}$. Finally, under the assumptions (18), (21) we prove that

$$\frac{u}{U} > \frac{2}{3}\sin^2\alpha_0 : \tag{23}$$

Let us put $a = |a^1a^6|$, $b = |a^1a^2|$ and choose the cartesian coordinate system in such a way that

$$a^1 = [0,0],\; a^6 = [0,a],\; a^2 = [b\sin(\alpha_1 + 2\alpha_0), b\cos(\alpha_1 + 2\alpha_0)].$$

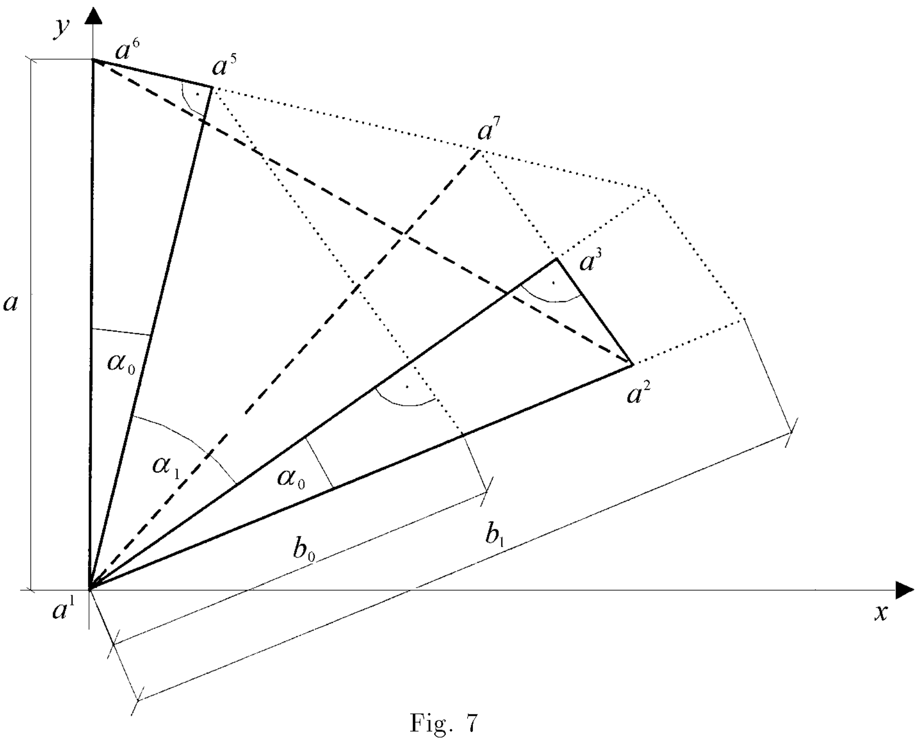

Fig. 7

See Fig.7. One can easily compute that

$$a^7 = \Big[\frac{\cos\alpha_0}{\sin\alpha_1}(b\cos\alpha_0 - a\cos(\alpha_1+\alpha_0)), a - \frac{\sin\alpha_0}{\sin\alpha_1}(b\cos\alpha_0 - a\cos(\alpha_1+\alpha_0))\Big]$$

and, further, by means of the parametrization

$$x_1 = t\frac{\cos\alpha_0}{\sin\alpha_1}(b\cos\alpha_0 - a\cos(\alpha_1+\alpha_0)),$$

$$x_2 = t\Big[a - \frac{\sin\alpha_0}{\sin\alpha_1}(b\cos\alpha_0 - a\cos(\alpha_1+\alpha_0))\Big],$$

$t \in <0,1>$, of the segment $\overline{\overline{a^1a^7}}$, one can see that the value of the parameter t in the cross point $\overline{\overline{a^1a^7}} \cap \overline{\overline{a^6a^2}}$ is

$$t(Q) = \frac{Q\sin\alpha_1\sin(\alpha_1+2\alpha_0)}{\cos\alpha_0[2Q\cos\alpha_0 - (1+Q^2)\cos(\alpha_1+\alpha_0)]}.$$

Here $Q = \frac{a}{b}$ and the fact that $Q \in <\frac{a}{b_1}, \frac{a}{b_0}> = <\cos\alpha_1, \frac{1}{\cos\alpha_1}>$ is apparent from Fig.7. Because of $t(Q) = 1 - \frac{u}{U}$, an optimal lower bound of $\frac{u}{U}$ corresponds to

$$\max\{t(Q); \cos\alpha_1 \le Q \le \frac{1}{\cos\alpha_1}\}.$$

We can find this maximum by a standard procedure. Then we conclude that

$$\frac{u}{U} \geq \frac{\sin\alpha_0 \sin\alpha_1 \sin(\alpha_1+\alpha_0)}{\cos\alpha_0[\cos\alpha_1 \sin(\alpha_1+\alpha_0)+\sin\alpha_0]}.$$

As the expression on the right-hand side attains its minimum for $\alpha_1 = \alpha_0$, we find

$$\frac{u}{U} \geq \frac{\sin^2\alpha_0 \sin 2\alpha_0}{\cos\alpha_0(\cos\alpha_0 \sin 2\alpha_0 + \sin\alpha_0)} = \frac{2\sin^2\alpha_0}{1+2\cos^2\alpha_0} > \frac{2}{3}\sin^2\alpha_0.$$

The statement is a consequence of (12), (14), (15), (22) and (23). □

Lemma 5. *Let $a^1,\ldots,a^6$ be regular nodes such that $\varphi_1 \geq \pi$, $\varphi_3 \geq \pi$ and $\varphi_5 \geq \pi$. Then*

$$l_1(a^1) > P_1P_2P_3P_4.$$

Proof. We have

$$D(a^1a^5a^6)D(a^1a^2a^3)D(a^4a^5a^3)D(a^4a^6a^2) > 0 \quad \text{and}$$

$$D(a^1a^3a^5)D(a^1a^2a^6)D(a^4a^5a^6)D(a^4a^2a^3) \leq 0.$$

Since, at the same time, $\overline{a^4a^6a^2} \supset \overline{a^1a^3a^5}$, we conclude

$$l \geq D(a^1a^5a^6)D(a^1a^2a^3)D(a^4a^5a^3)D(a^4a^6a^2) > P_1P_2P_3P_4\,. \qquad \square$$

Main Theorem. *Let the nodes $a^1,\ldots,a^6$ be regular. Then there exist an index $k \in \{1,2,3,4\}$ and a positive constant C independent of h such that*

$$|l_i(a^i)| \geq C\cdot P_kP_2P_3P_4 \quad \textit{for} \quad i=1,\ldots,6.$$

Proof. This statement is a consequence of the lemmas 1, 2, 3, 4, 5. □

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Department of Mathematics, Technical University of Brno
Žižkova 17, 602 00 Brno, CZECH REPUBLIC
E-mail: `mddal@fce.vutbr.cz`