ARCHIVUM MATHEMATICUM (BRNO)
Tomus 35 (1999), 337 – 345

# ASYMPTOTIC ESTIMATION FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH SEVERAL DELAYS

Jan Čermák

Abstract. We discuss the asymptotic behaviour of all solutions of the functional differential equation

$$y'(x) = \sum_{i=1}^{m} a_i(x)y(\tau_i(x)) + b(x)y(x)\,,$$

where $b(x) < 0$. The asymptotic bounds are given in terms of a solution of the functional nondifferential equation

$$\sum_{i=1}^{m} |a_i(x)|\omega(\tau_i(x)) + b(x)\omega(x) = 0.$$

## 1. Introduction and preliminaries

The linear functional differential equation

$$y'(x) = \sum_{i=1}^{m} a_i(x)y(\tau_i(x)) + b(x)y(x)\,, \qquad x \in I = [x_0, \infty) \tag{1.1}$$

has been discussed, under special hypotheses, in many papers (for references see [4]). Equations (1.1) with bounded $r_i(x) = x - \tau_i(x)$ are usually studied preferably, whereas the theory for equations (1.1) with unbounded $r_i(x)$ is less developed. In this paper we use the transformation approach described by F. Neuman in [8] and [9] to obtain asymptotic formulas valid especially for equations (1.1) with unbounded $r_i(x)$.

This approach consists in introducing a change of variables converting every solution $y(x)$ of (1.1) into a solution of an equation with constant or bounded delays.

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1991 *Mathematics Subject Classification*: 34K15, 34K25; Secondary 39B99.

*Key words and phrases*: functional differential equation, functional nondifferential equation, asymptotic behaviour, transformation.

Research supported by the grant # A101/99/02 of the Academy of Sciences of the Czech Republic.

Received December 16, 1998.

Under certain assumptions we relate asymptotic properties of solutions of (1.1) to the behaviour of a solution $\omega(x)$ of an auxiliary linear functional nondifferential equation

$$\sum_{i=1}^{m} |a_i(x)|\omega(\tau_i(x)) + b(x)\omega(x) = 0, \qquad x \in I \tag{1.2}$$

and thus we extend or generalize some parts of [3], [5], [6], [1] and [2].

Throughout this paper we assume that $\tau_i : I \to \mathbb{R}$ are increasing continuous functions such that $\tau_i(x) < x$ in $I$ and $\lim_{x\to\infty} \tau_i(x) = \infty$, $i = 1, 2, \ldots, m$. By the symbol $\tau^n$ we mean the $n$-th iterate of $\tau$ (for $n > 0$) or the -$n$-th iterate of the inverse function $\tau^{-1}$ (for $n < 0$) and put $\tau^0 = \mathrm{id}$. Further, we denote by $(x_i)_{i=1}^{\infty}$ the increasing consequence of reals formed by all numbers $\tau_i^{-n}(x_0)$, $i = 1, 2, \ldots, m$, $n = 1, 2, \ldots$.

Set $x_{-1} = \min\,\{\tau_i(x_0),\ i = 1, 2, \ldots, m\}$ and let $I_{-1} = [x_{-1}, \infty)$. By a solution of (1.1) we understand a function $y(x) \in C^0(I_{-1}) \cap C^1(I)$ such that $y(x)$ satisfies (1.1) in $I$.

We start off with the study of equation (1.2), where $a_i(x)$, $\tau_i(x)$, $b(x)$ are known, $i = 1, 2, \ldots, m$, $b(x) < 0$ and $\omega(x)$ is unknown.

**Proposition 1.** *Assume that $|a_i(x)|$, $b(x)$, $\tau_i(x) \in C^r(I)$, $r \geq 0$, $\sum_{i=1}^{m} |a_i(x)| \neq 0$ in $I$, $b(x) < 0$ in $I$, $i = 1, 2, \ldots, m$. Let $\omega_0(x) \in C^r([x_{-1}, x_0])$ be a positive function such that*

$$(\omega_0(x_0))^{(s)} = \Big(\sum_{i=1}^{m} \frac{|a_i(x_0)|}{-b(x_0)}\omega_0(\tau_i(x_0))\Big)^{(s)}, \qquad s = 0, 1, \ldots, r.$$

*Then there exists a unique positive solution $\omega(x) \in C^r(I_{-1})$ of (1.2) such that*

$$\omega(x) = \omega_0(x)\,, \qquad x \in [x_{-1}, x_0]\,.$$

*This solution is given inductively by*

$$\omega(x) = \omega_1(x) = \sum_{i=1}^{m} \frac{|a_i(x)|}{-b(x)}\omega_0(\tau_i(x))\,, \qquad x_0 \leq x \leq x_1\,, \tag{1.3}$$

$$\omega(x) = \omega_n(x) = \sum_{i=1}^{m} \frac{|a_i(x)|}{-b(x)}\omega_j(\tau_i(x))\,, \qquad x_{n-1} \leq x \leq x_n\,,$$

*where $n = 2, 3, \ldots$ and $j$ is an integer ($0 \leq j \leq n-1$) depending on $\tau_i(x)$ such that $x_{j-1} \leq \tau_i(x) \leq x_j$.*

**Proof.** The existence and uniqueness of the solution $\omega(x)$ can be proved by the step method (see, e.g., [7]). We show that $\omega(x)$ is positive in $I$. Suppose not and denote $x^* = \min\,\{x \in I, \omega(x) = 0\}$. Then

$$0 = \omega(x^*) = \sum_{i=1}^{m} \frac{|a_i(x^*)|}{-b(x^*)}\omega(\tau_i(x^*))\,.$$

However, $\sum_{i=1}^{m} \frac{|a_i(x)|}{-b(x)} \neq 0$ in $I$, which means that $\omega(\tau_i(x^*)) = 0$ for $i = 1, 2, \ldots, m$. This is a contradiction with the definition of $x^*$. □

Now we consider the corresponding linear autonomous functional equation

$$\sum_{i=1}^{m} |a_i|\omega(\tau_i(x)) + b\omega(x) = 0\,, \qquad x \in I\,, \tag{1.4}$$

where $b < 0$. We are going to discuss conditions under which the required solution $\omega(x)$ of (1.4) can be exhibited in the form $\omega(x) = \exp(\alpha\psi(x))$, where $\alpha \in \mathbb{R}$ is a constant and $\psi(x)$ is a solution of Abel equation

$$\psi(\tau_1(x)) = \psi(x) - 1\,, \qquad x \in I\,. \tag{1.5}$$

We recall some basic facts about differentiable solutions of (1.5). Let $\tau_1 \in C^r(I)$, $r \geq 1$ and $\tau_1'(x) > 0$ in $I$. Then there exists a solution $\psi(x) \in C^r([\tau_1(x_0), \infty))$ of (1.5) such that $\psi'(x) > 0$ in $[\tau_1(x_0), \infty)$ (see, e.g., [9]). This solution is given by the formula

$$\psi(x) = \psi_0(\tau_1^n(x)) + n, \qquad \tau_1^{-n+1}(x_0) \leq x \leq \tau_1^{-n}(x_0)\,, \tag{1.6}$$

where $\psi_0(x) \in C^r([\tau_1(x_0), x_0])$, $\psi_0'(x) > 0$ in $[\tau_1(x_0), x_0]$ and

$$(\psi_0(x_0))^{(s)} = (\psi_0(\tau_1(x_0)) + 1)^{(s)}\,, \qquad s = 0, 1, \ldots, r\,.$$

In the sequel we denote by $\{\tau_1^u(x), u \in \mathbb{R}\}$ a continuous iteration group defined in $I$ and generated by $\psi(x)$. Hence,

$$\tau_1^u(x) = \psi^{-1}(\psi(x) - u)\,, \qquad x \in I, u \in \mathbb{R}\,.$$

Now assume that functions $\tau_i(x) \in C^r(I)$, $\tau_i'(x) > 0$ in $I$, $i = 1, 2, \ldots, m$ can be embedded into $\{\tau_1^u(x), u \in \mathbb{R}\}$. Then there exists a simultaneous solution $\psi(x) \in C^r(I_{-1})$, $\psi'(x) > 0$ in $I_{-1}$ of a system of Abel equations

$$\psi(\tau_i(x)) = \psi(x) - c_i\,, \qquad x \in I\,, i = 1, 2, \ldots, m, \tag{1.7}$$

where $c_i > 0$ are suitable constants.

Set $\omega(x) = \exp(\alpha\psi(x))$ in (1.4) to obtain

$$\sum_{i=1}^{m} |a_i| \exp(\alpha\psi(\tau_i(x))) + b\exp(\alpha\psi(x)) = 0\,, \qquad x \in I\,,$$

i.e.,

$$\sum_{i=1}^{m} |a_i|\lambda_i^{\alpha} + b = 0\,, \tag{1.8}$$

where $\lambda_i = \exp(-c_i)$, $i = 1, 2, \ldots, m$. It is clear that (1.8) has a unique real root $\alpha^*$. Consequently, the function $\omega(x) = \exp(\alpha^*\psi(x))$ is a solution of (1.4) such that $\omega(x) \in C^r(I_{-1})$ and $\omega'(x) \neq 0$ in $I_{-1}$.

Our previous ideas yield

**Proposition 2.** *Let $a_i \neq 0$, $b < 0$ be scalars and let functions $\tau_i(x) \in C^r(I)$, $r \geq 1$, $\tau_i'(x) > 0$ in $I$ can be embedded into a continuous iteration group $\{\tau_1^u(x), u \in \mathbb{R}\}$, $i = 1, 2, \ldots, m$. Further, let $\psi(x) \in C^r(I_{-1})$ be a solution of (1.5) given by (1.6) and let $\alpha^*$ be a real root of (1.8). Then function $\omega(x) = \exp(\alpha^* \psi(x))$, $x \in I_{-1}$ defines a solution of (1.4) such that $\omega(x) \in C^r(I_{-1})$ and $\omega'(x) \neq 0$ in $I_{-1}$.*

**Remark 1.** The problem of embeddability of given functions $\tau_i(x)$ into a continuous iteration group has been dealt with by F. Neuman [8] and M. Zdun [10]. We note that the most important necessary condition is commutativity of any pair $\tau_i(x)$, $\tau_j(x)$, $i, j = 1, 2, \ldots, m$.

Other results concerning the theory of functional nondifferential equations can be found in [7].

## 2. Main results

We consider the case $b(x) < 0$ in $I$ and put

$$\omega'_-(x) = \max\left(0, -\omega'(x)\right), \qquad x \in I\,.$$

**Theorem 1.** *Let $|a_i(x)|$, $b(x)$, $\tau_i(x) \in C^2(I)$, $\sum_{i=1}^m |a_i(x)| \neq 0$ in $I$ and $b(x) < 0$ in $I$, $i = 1, 2, \ldots, m$. Assume that $\tau_1(x) \geq \tau_i(x)$ for each $x \in I$, $i = 2, \ldots, m$ and $\tau_1'(x) > 0$ in $I$. Further, let $\omega(x) \in C^2(I_{-1})$ be a positive solution of (1.2) given by (1.3) and let $\psi(x) \in C^2(I_{-1})$ be a solution of (1.5) given by (1.6) such that $\psi'(x) > 0$ in $I_{-1}$. If*

(i) $\omega'(x) - b(x)\omega(x) > 0$ *in $I$,*

(ii) $\frac{\omega'_-(x)}{\omega'(x) - b(x)\omega(x)}$ *is nonincreasing in $I$,*

(iii) $\displaystyle\int_{x_0}^{\infty} \frac{\omega'_-(s)\psi'(s)}{\omega'(s) - b(s)\omega(s)} ds < \infty$,

*then*

$$y(x) = O(\omega(x)) \qquad \text{as } x \to \infty$$

*for every solution $y(x)$ of (1.1).*

**Proof.** The change of variables

$$t = \psi(x), \quad z(t) = \frac{y(x)}{\omega(x)}$$

converts equation (1.1) into

$$\dot z(t) = \sum_{i=1}^m \Big(a_i(h(t))\dot h(t) \frac{\omega(\tau_i(h(t)))}{\omega(h(t))} z(\mu_i(t))\Big) + \Big(b(h(t))\dot h(t) - \frac{\dot\omega(h(t))\dot h(t)}{\omega(h(t))}\Big) z(t)\,,$$

where $t \in J = [t_0, \infty)$, $h(t) = \psi^{-1}(t)$ and $\mu_i(t) = t - r_i(h(t))$, $r_i(h(t)) \geq 1$ for every $t \in J$, $i = 1, 2, \ldots, m$. This form can be rewritten as

$$\begin{aligned} &\frac{d}{dt}\Big[z(t)\omega(h(t))\exp\Big\{-\int_{x_0}^{h(t)} b(u)du\Big\}\Big] \\ (2.1)\qquad &= \sum_{i=1}^{m}\Big(a_i(h(t))\dot h(t)\omega(\tau_i(h(t)))\exp\Big\{-\int_{x_0}^{h(t)} b(u)du\Big\}z(\mu_i(t))\Big)\,. \end{aligned}$$

Now we denote by $I_k$ the interval $[t_0+k-1, t_0+k]$ and put $M_k = \sup\ \{|z(t)|, t \in \cup_{j=1}^{k} I_j\}$, $k = 1, 2, \ldots$. We consider $t \in I_{k+1}$ and integrate (2.1) over $[t_0+k, t]$ to obtain

$$\begin{aligned} z(t) =&\frac{\omega(h(t_0+k))}{\omega(h(t))}\exp\Big\{\int_{h(t_0+k)}^{h(t)} b(u)du\Big\}z(t_0+k) \\ &+\int_{t_0+k}^{t}\Big(\sum_{i=1}^{m}\Big(a_i(h(s))\dot h(s)\frac{\omega(\tau_i(h(s)))}{\omega(h(t))}\exp\Big\{\int_{h(s)}^{h(t)} b(u)du\Big\}z(\mu_i(s))\Big)\Big)ds\,. \end{aligned}$$

Then

$$\begin{aligned} |z(t)| \leq& M_k\frac{\omega(h(t_0+k))}{\omega(h(t))}\exp\Big\{\int_{h(t_0+k)}^{h(t)} b(u)du\Big\} \\ &+ M_k\int_{t_0+k}^{t}\Big(\sum_{i=1}^{m}|a_i(h(s))|\dot h(s)\frac{\omega(\tau_i(h(s)))}{\omega(h(t))}\exp\Big\{\int_{h(s)}^{h(t)} b(u)du\Big\}\Big)ds \\ \leq& M_k\Big\{\frac{\omega(h(t_0+k))}{\omega(h(t))}\exp\Big\{\int_{h(t_0+k)}^{h(t)} b(u)du\Big\} \\ &+\int_{t_0+k}^{t}\Big(-b(h(s))\dot h(s)\frac{\omega(h(s))}{\omega(h(t))}\exp\Big\{\int_{h(s)}^{h(t)} b(u)du\Big\}\Big)ds\Big\} \end{aligned}$$

by use of (1.2). Further,

(2.2)

$$\begin{aligned} |z(t)| \leq& M_k\Big\{\frac{\omega(h(t_0+k))}{\omega(h(t))}\exp\Big\{\int_{h(t_0+k)}^{h(t)} b(u)du\Big\} \\ &+\frac{\exp\Big\{\int_{x_0}^{h(t)} b(u)du\Big\}}{\omega(h(t))}\int_{t_0+k}^{t}\Big(\omega(h(s))\frac{d}{ds}\Big[\exp\Big\{-\int_{x_0}^{h(s)} b(u))du\Big\}\Big]\Big)ds\Big\}\,. \end{aligned}$$

Integrating by parts we have

$$\begin{aligned} &\int_{t_0+k}^{t}\Big(\omega(h(s))\frac{d}{ds}\Big[\exp\Big\{-\int_{x_0}^{h(s))} b(u)du\Big\}\Big]\Big)ds \\ &\quad=\Big[\omega(h(s))\exp\Big\{-\int_{x_0}^{h(s)} b(u)du\Big\}\Big]_{t_0+k}^{t} \\ &\qquad+\int_{t_0+k}^{t}\Big(-\dot\omega(h(s))\dot h(s)\exp\Big\{-\int_{x_0}^{h(s)} b(u)du\Big\}\Big)ds\,. \end{aligned}$$

Repeated integration by parts yields

$$
\begin{aligned}
&\int_{t_0+k}^{t}\Big(-\dot\omega(h(s))\dot h(s)\exp\Big\{-\int_{x_0}^{h(s)}b(u)du\Big\}\Big)ds\\
\le&\int_{t_0+k}^{t}\Big(\dot\omega_-(h(s))\dot h(s)\exp\Big\{-\int_{x_0}^{h(s)}b(u)du\Big\}\Big)ds\\
=&\int_{t_0+k}^{t}\Big(\frac{\dot\omega_-(h(s))}{\dot\omega(h(s))-b(h(s))\omega(h(s))}\frac{d}{ds}\Big[\omega(h(s))\exp\Big\{-\int_{x_0}^{h(s)}b(u)du\Big\}\Big]\Big)ds\\
=&\Big[\omega(h(s))\exp\Big\{-\int_{x_0}^{h(s)}b(u)du\Big\}\frac{\dot\omega_-(h(s))}{\dot\omega(h(s))-b(h(s))\omega(h(s))}\Big]_{t_0+k}^{t}\\
&+\int_{t_0+k}^{t}\Big(\omega(h(s))\exp\Big\{-\int_{x_0}^{h(s)}b(u)du\Big\}\frac{d}{ds}\Big[\frac{-\dot\omega_-(h(s))}{\dot\omega(h(s))-b(h(s))\omega(h(s))}\Big]\Big)ds\\
\le&\Big[\omega(h(s))\exp\Big\{-\int_{x_0}^{h(s)}b(u)du\Big\}\frac{\dot\omega_-(h(s))}{\dot\omega(h(s))-b(h(s))\omega(h(s))}\Big]_{t_0+k}^{t}\\
&+\omega(h(t))\exp\Big\{-\int_{x_0}^{h(t)}b(u)du\Big\}\Big[\frac{-\dot\omega_-(h(s))}{\dot\omega(h(s))-b(h(s))\omega(h(s))}\Big]_{t_0+k}^{t}\\
=&\Big[\omega(h(s))\exp\Big\{-\int_{x_0}^{h(s)}b(u)du\Big\}\Big]_{t_0+k}^{t}\frac{\dot\omega_-(h(t_0+k))}{\dot\omega(h(t_0+k))-b(h(t_0+k))\omega(h(t_0+k))}\,.
\end{aligned}
$$

Substituting this into (2.2) we obtain

$$
|z(t)|\le M_k\Big\{1+\frac{\dot\omega_-(h(t_0+k))}{\dot\omega(h(t_0+k))-b(h(t_0+k))\omega(h(t_0+k))}\Big\}
$$

for every $t\in I_{k+1}$, i.e.,

$$
\begin{aligned}
M_{k+1}\le&M_k\Big\{1+\frac{\dot\omega_-(h(t_0+k))}{\dot\omega(h(t_0+k))-b(h(t_0+k))\omega(h(t_0+k))}\Big\}\\
\le&M_1\prod_{j=1}^{k}\Big\{1+\frac{\dot\omega_-(h(t_0+j))}{\dot\omega(h(t_0+j))-b(h(t_0+j))\omega(h(t_0+j))}\Big\}
\end{aligned}
$$

for every $k=1,2,\dots$.

Applying Cauchy's integral criterion we can see that the infinite product converges as $k\to\infty$, hence $z(t)$ is bounded as $t\to\infty$. □

**Remark 2.** If we replace $x_0\in I$ in conditions (i), (ii) and (iii) by $x^*\in I$ large enough, then the conclusion of Theorem 1 remains valid.

**Remark 3.** The assumption $\tau_1(x)\in C^2(I)$, $\tau_1(x)\ge\tau_i(x)$ for each $x\in I$, $i=2,\dots,m$, $\tau_1'(x)>0$ in $I$ can be replaced by the assumption that there exists $\tau(x)\in C^2(I)$, $\tau(x)\ge\tau_i(x)$ for each $x\in I$, $i=1,2,\dots,m$ and $\tau'(x)>0$ in $I$. Of course, $\psi(x)$ is then a solution of Abel equation (1.5) with $\tau(x)$ instead of $\tau_1(x)$.

**Remark 4.** If $\displaystyle\sum_{i=1}^{m} \frac{|a_i(x_0)|}{-b(x_0)} \geq 1$ and functions $\dfrac{|a_i(x)|}{-b(x)}$ are nondecreasing in $I$ for $i = 1, 2, \ldots, m$, then there exists a positive nondecreasing solution $\omega(x)$ of (1.2). Hence, we can simplify the assumptions of Theorem 1.

Now we consider autonomous equation (1.1), i.e., the equation

$$y'(x) = \sum_{i=1}^{m} a_i y(\tau_i(x)) + by(x)\,, \qquad x \in I\,. \tag{2.3}$$

Using Proposition 2 we get

**Theorem 2.** *Let $a_i \neq 0$, $b < 0$ be scalars, $i = 1, 2, \ldots, m$. Assume that functions $\tau_i(x) \in C^2(I)$ fulfil $0 < \tau_i'(x) \leq (\frac{\tau_i(x)}{x})^\gamma$ in $I$ for a suitable real constant $\gamma > \frac{1}{2}$, $\tau_1'(x)$ is nonincreasing in $I$ and let $\tau_i(x)$ can be embedded into a continuous iteration group $\{\tau_1^u(x), u \in \mathbb{R}\}$, $i = 1, 2, \ldots, m$. Further, let $\psi(x) \in C^2(I_{-1})$, $\psi'(x) > 0$ in $I_{-1}$ be a solution of (1.5) given by (1.6) and let $\alpha^*$ be a real root of (1.8). Then every solution $y(x)$ of (2.3) satisfies*

$$y(x) = O(\exp(\alpha^* \psi(x)) \qquad \text{as } x \to \infty.$$

**Proof.** The function $\psi'(x)$ is a solution of the functional equation

$$\psi'(x) = \psi'(\tau_1(x))\tau_1'(x)\,.$$

If $\psi'(x)$ is nonincreasing for $\tau_1(x_0) \leq x \leq x_0$, then $\psi'(x)$ is nonincreasing in $I_{-1}$. Moreover, if $\psi'(x) \leq \frac{M}{x^\gamma}$ for a suitable $M > 0$ and every $\tau_1(x_0) \leq x \leq x_0$, then

$$\psi'(x) \leq \frac{M}{(\tau_1(x))^\gamma}\tau_1'(x) \leq \frac{M}{x^\gamma}$$

for every $x_0 \leq x \leq \tau_1^{-1}(x_0)$. By induction on $n$ we can similarly show that $\psi'(x) \leq \frac{M}{x^\gamma}$ for every $\tau_1^{-n+1}(x_0) \leq x \leq \tau_1^{-n}(x_0)$, $n = 1, 2, \ldots$, i.e., $\displaystyle\int_{x_0}^{\infty} (\psi'(s))^2 ds < \infty$.

By Proposition 2 $\omega(x) = \exp(\alpha^* \psi(x))$ is a solution of (1.4). Due to the above given properties of $\psi'(x)$ it is easy to verify that all the assumptions of Theorem 1 are fulfilled with the respect to Remark 3. □

**Corollary.** *In addition to assumptions of Theorem 2 suppose that $\displaystyle\sum_{i=1}^{m} |a_i| < -b$. Then every solution $y(x)$ of (2.3) tends to zero as $x \to \infty$.*

## 3. Applications

**Example 1.** We consider the equation

$$y'(x) = \sum_{i=1}^{m} b_i(x)[y(x) - y(\tau_i(x))]\,, \qquad x \in I\,, \tag{3.1}$$

where $b_i(x) \in C^0(I)$, $b_i(x) < 0$ in $I$, $i = 1, 2, \dots, m$. Auxiliary functional equation (1.2) then becomes

$$\sum_{i=1}^{m} b_i(x)[\omega(x) - \omega(\tau_i(x))] = 0, \qquad x \in I$$

and admits the solution $\omega(x) = const$. We apply conclusions of Theorem 1 with the respect to Remark 4.

Assume that $b_i(x) < 0$ in $I$, $i = 1, 2, \dots, m$. Let $\tau_1(x) \in C^1(I)$, $\tau_1'(x) > 0$ in $I$ and $\tau_1(x) \geq \tau_i(x)$ for each $x \in I$, $i = 2, \dots, m$. Then every solution $y(x)$ of (3.1) is bounded.

We note that equation (3.1) with $b_i(x) > 0$ and constant delays $\tau_i(x)$ has been studied by J. Diblík [3]. Our previous result extend some parts of [3].

**Example 2.** Now we investigate the asymptotic behaviour of all solutions of the equation

$$y'(x) = a_1 x y(\lambda_1 x) + a_2 x y(\lambda_2 x) + b y(x)\,, \qquad x \in [1, \infty)\,, \tag{3.2}$$

where $a_1$, $a_2 \neq 0$, $0 < \lambda_1, \lambda_2 < 1$ and $b < 0$. The corresponding functional equation (1.2) is

$$|a_1| x \omega(\lambda_1 x) + |a_2| x \omega(\lambda_2 x) + b\omega(x) = 0\,, \qquad x \in [1, \infty)\,. \tag{3.3}$$

This equation has a positive increasing solution given by (1.3). Hence, by Theorem 1 and Remark 4, every solution $y(x)$ of (3.2) fulfil $y(x) = O\{\omega(x)\}$, where $\omega(x)$ is a positive and increasing solution of (3.3) given by (1.3).

To obtain a more applicable form of the estimate we can simplify equation (3.3) in the following way. We consider the equation

$$a x \varphi(\lambda x) + b\varphi(x) = 0\,, \qquad x \in [1, \infty)\,,$$

where $a = \max\,(|a_1|, |a_2|)$, $\lambda = \max\,(\lambda_1, \lambda_2)$. It can be easily verified that this equation has a solution

$$\varphi^*(x) = \exp\Big\{\frac{\log^2 x}{2\log\lambda^{-1}} + \frac{\log x}{2} + \frac{\log\frac{a}{-b}}{\log\lambda^{-1}}\log x\Big\}\,, \qquad x \in [1, \infty)\,.$$

Since obviously every positive and increasing solution $\omega(x)$ of (3.3) is of order not exceeding $\varphi^*(x)$ we get that

$$y(x) = O\Big(\exp\Big\{\frac{\log^2 x}{2\log\lambda^{-1}} + \frac{\log x}{2} + \frac{\log\frac{a}{-b}}{\log\lambda^{-1}}\log x\Big\}\Big) \qquad \text{as } x \to \infty$$

for every solution $y(x)$ of (3.2).

**Example 3.** We consider the equation

$$y'(x) = a_1 y(x^{\gamma_1}) + a_2 y(x^{\gamma_2}) + b y(x), \qquad x \in [1, \infty), \tag{3.4}$$

where $a_1$, $a_2 \neq 0$, $0 < \gamma_1$, $\gamma_2 < 1$, $b < 0$. Functions $x^{\gamma_1}$, $x^{\gamma_2}$ can be embedded into a continuous iteration group $\{x^u, u \in \mathbb{R}\}$. Abel equation (1.5) then becomes

$$\psi(x^{\gamma_1}) = \psi(x) - 1, \qquad x \in [1, \infty)$$

and has the function $\psi^*(x) = \frac{\log \log x}{-\log \gamma_1}$ as the required solution. We note that delays considered in (3.4) intersect the identity function at the initial point $x_0 = 1$. Nevertheless, all assertions of this paper remain valid for equations with such delays as well.

Further, let $\alpha^*$ be a real root of equation (1.8) with $b < 0$, i.e., equation

$$|a_1| \exp(-\alpha) + |a_2| \exp\Big\{ -\frac{\log \gamma_2}{\log \gamma_1} \alpha \Big\} + b = 0$$

and put

$$\omega^*(x) = \exp(\alpha^* \psi^*(x)) .$$

Then, by Theorem 2,

$$y(x) = O(\omega^*(x)) \qquad \text{as } x \to \infty$$

for any solution $y(x)$ of (3.4).

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Department of Mathematics, Technical University of Brno
Technická 2, 616 69 Brno, CZECH REPUBLIC
*E-mail*: `cermakh@mat.fme.vutbr.cz`