ARCHIVUM MATHEMATICUM (BRNO)
Tomus 35 (1999), 305 – 315

STOCHASTIC PARALLEL TRANSPORT AND CONNECTIONS OF H^2M

Pedro Catuogno

Abstract. In this paper we prove that there is a bijective correspondence between connections of H^2M, the principal bundle of the second order frames of M, and stochastic parallel transport in the tangent space of M. We construct in a direct geometric way a prolongation of connections without torsion of M to connections of H^2M. We interpret such prolongation in terms of stochastic calculus.

1. Introduction

The purpose of this paper is to study the intrinsic nature of the stochastic differential equation associated with the parallel transport in the tangent space. Firstly, we prove that the coefficients of the equation of parallel transport are the local components of a connection of the bundle of second order frames, with the opposite sign. This establishes a bijective correspondence between the set of connections of the bundle of second order frames of M, and stochastic parallel transport in the tangent space of M. In the section 2, we show an alternative construction of a canonical prolongation proposed by I. Kolář in [12], $\Gamma \to \Gamma^1$ of connections without torsion of M to connections of the bundle of second order frames. We give an interpretation of this prolongation in terms of stochastic parallel transport.

All manifolds are finite dimensional σ-compact and of class $\mathcal{C}^\infty$. As to manifolds, geometry and stochastic calculus, we shall use freely concepts and notations as Kobayashi-Nomizu [11], Ikeda-Watanabe [7] and Emery [4].

I thank my advisor, professor L. San Martin for his valuable suggestions and comments.

1991 *Mathematics Subject Classification*: 58G32, 53B15.
Key words and phrases: second order geometry, stochastic calculus, connections, parallel transport.
Received September 30, 1998.

2. Connections of H^2M and Stochastic Parallel Transport in TM.

In this section we study connections of H^2M. Firstly, we calculate the associated operator of covariant derivative in local coordinates, and the transformation coordinate formula for the local components of a connection in H^2M. After we calculate the transformation coordinate formula for the local components of a linear stochastic differential equation [14] and to show that this local components are the local components of a connection of H^2M with opposite sign. This establishes a bijective correspondence between connections of H^2M and stochastic parallel transport in TM. Finally, we prove that there is a bijective correspondence between connections of H^2M and 2-connections of H^1M [2].

We begin by recalling some fundamental facts about Schwartz geometry [13], [14] and [4].

If x is a point in a manifold M, the second order tangent space to M at x, denoted $\tau_x M$, is the vector space of all differential operators on M, at x, of order at most two, with no constant term. If $\dim M = n$, $\tau_x M$ has $n + \frac{1}{2}n(n+1)$ dimensions; using a local coordinate system (U, x^i) around x, every $L \in \tau_x M$ can be written in a unique way as

$$L = a_i^i \frac{\partial}{\partial x^i} + a^{ij} \frac{\partial}{\partial x^i \partial x^j} \qquad \text{with } a^{ij} = a^{ji}\,.$$

We use here and in other expressions in coordinates the convention of summing over the repeated indices. The elements of $\tau_x M$ are called second-order tangent vectors (or tangent vectors of order two) at x.

The disjoint union $\tau M = \bigcup_{x \in M} \tau_x M$ is canonically endowed with a vector bundle structure over M, called the second order tangent fiber bundle of M.

Let M be a manifold, and

$$H^2M = \left\{ j_0^2 s : s \text{ is a local difeomorphism of } 0 \in \mathbb{R}^n \text{in } M \right\}$$

the set of second order frames of M. This set is a principal fiber bundle with structural group

$$Gl^2 = \left\{ j_0^2 f : f \text{ is a local difeomorphism in } 0 \text{ and } f(0) = 0 \right\}$$

and projection $\pi^2 : H^2M \to M$ defined by $\pi^2(j_0^2 s) = s(0)$, where the right action of Gl^2 in H^2M is given by

$$j_0^2 s \cdot j_0^2 f = j_0^2 (s \circ f)\,.$$

We recall that Gl^2 is a semidirect product $Gl^2 = Gl(n) \propto S^2(n)$ where $S^2(n)$ is the real vector space of symmetric bilinear forms of $\mathbb{R}^n$. We identify the subgroup $Gl(n)$ of Gl^2 with $\{ j_0^2 f \in Gl^2 : D_{ij} f^r(0) = 0 \text{ for } r, i, j = 1, ..., n \}$, and $S^2(n)$ with $\left\{ j_0^2 f \in Gl^2 : D_i f^j(0) = \delta_i^j \text{ for } i, j = 1, ..., n \right\}$. In local coordinates if $\zeta = (\zeta_j^i, \zeta_{jk}^i)$, $\eta = (\eta_j^i, \eta_{jk}^i) \in Gl^2$ the product $\zeta\eta = (\lambda_j^i, \lambda_{jk}^i)$ is given by

$$\begin{aligned} \lambda_j^i &= \zeta_s^i \eta_j^s \\ \lambda_{jk}^i &= \zeta_{rs}^i \eta_j^r \eta_k^s + \zeta_s^i \eta_{jk}^s \end{aligned}$$

and

$$\zeta^{-1} = (\overline{\zeta}^i_j, -\overline{\zeta}^i_s \zeta^s_{pq} \overline{\zeta}^p_j \overline{\zeta}^q_k)$$

where $(\overline{\zeta}^i_j) = (\zeta^i_j)^{-1}$.

The group Gl^2 acts on the left on $\tau_0 \mathbb{R}^n$ as follows

$$\begin{aligned} \zeta \cdot a &= (\zeta^i_j, \zeta^i_{jk}) \cdot (a^i D_i + a^{ij} D_{ij}) \\ &= (\zeta^i_k a^k + \zeta^i_{jk} a^{jk}) D_i + (\zeta^i_r \zeta^j_s a^{rs}) D_{ij} \,. \end{aligned}$$

For this action τM is a vector bundle associated with H^2M.

The canonical epimorphism of principal fiber bundles $\pi^2_1 : H^2M \to H^1M$, is given by $\pi^2_1(j^2_0 s) = j^1_0 s$.

Let Γ be a connection in H^2M and ω its associated connection form. Then ω can be decomposed as

$$\omega = \omega_0 + \omega_1$$

where ω_0 is the $gl(n, \mathbb{R})$ component and ω_1 the $S^2(n)$ component of ω.

Let $\{E^i_j\}$, $\{E^{ik}_j = E^{ki}_j\}$ denote canonical bases of $gl(n, \mathbb{R})$ and $S^2(n)$ respectively, and σ the local section of H^2M given by $\sigma(x) = (x^i, Id, 0)$ over (U, x^i). We define the functions Γ^i_{jk}, Γ^i_{jlk} on U with $\Gamma^i_{jkl} = \Gamma^i_{jlk}$ by

$$\begin{aligned} \sigma^* \omega_0 &= (\Gamma^i_{jk} dx^j) E^k_i \,, \\ \sigma^* \omega_1 &= (\Gamma^i_{jkl} dx^j) E^{kl}_i \,. \end{aligned}$$

The functions Γ^i_{jk}, Γ^i_{jkl} are called the local components of Γ with respect (U, x^i).

Let $\widetilde{\Gamma}$ be a connection of H^2M. As τM is an associated vector bundle of H^2M, there is a covariant derivative operator $\widetilde{\nabla}$ in τM associated with $\widetilde{\Gamma}$ [6, proposition 1.10]. We are interested in compute in local coordinates the covariant derivative $\widetilde{\nabla}$.

Let (U, x^i) be a local chart in M, $(\pi^{-1}(U), (x^i, x^i_j))$ and $(\pi^{-1}(U), (x^i, x^i_j, x^i_{jk}))$ the induced local chart by (U, x^i) in H^1M and H^2M respectively. Let Γ be a connection in H^1M and $\widetilde{\Gamma}$ a connection in H^2M. It is known [3, pg. 203] that if $X = X^i \frac{\partial}{\partial x^i} \in T_x M$, then the horizontal lift of X have the following local expression in these coordinate systems

$$X^{H^1}(x^i, x^i_j) = X^i \left(\frac{\partial}{\partial x^i} - \Gamma^r_{il} x^l_j \frac{\partial}{\partial x^r_j} \right) ;$$

$$X^{H^2}(x^i, x^i_j, x^i_{jk}) = X^i \left(\frac{\partial}{\partial x^i} - \Gamma^r_{il} x^l_j \frac{\partial}{\partial x^r_j} - (\Gamma^r_{il} x^l_{jk} + \Gamma^r_{ilm} x^l_j x^m_k) \frac{\partial}{\partial x^r_{jk}} \right) .$$

where Γ^i_{jk} are the local components of Γ and Γ^i_{jk}, Γ^i_{jkl} are the local components of $\widetilde{\Gamma}$.

Proposition 2.1. *Let* (U, x^i) *be a local chart of* M *and* $A(x) = a^i(x)\frac{\partial}{\partial x^i} + a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} \in \tau_x M$. *Then*

$$\widetilde{\nabla}_{\frac{\partial}{\partial x^k}} A(x) = (\frac{\partial a^i}{\partial x^k} + \Gamma^i_{kj}a^j + \Gamma^i_{kjr}a^{jr})\frac{\partial}{\partial x^i} + (\frac{\partial a^{ij}}{\partial x^k} + \Gamma^j_{ks}a^{is} + \Gamma^i_{ks}a^{sj})\frac{\partial^2}{\partial x^i \partial x^j} .$$

Proof. Let $A \in \Gamma(\tau M)$ and $F_A : H^2 M \to \tau_0 I\!R^n$ given by $F_A(p) = p^{-1} \cdot A(\pi(p))$. If the local expression of A in U is $A(x) = a^i \frac{\partial}{\partial x^i} + a^{ij} \frac{\partial^2}{\partial x^i \partial x^j}$, then the local expression of F_A in $(\pi^{-1}(U), (x^i, x^i_j, x^i_{jk}))$ is

$$\begin{aligned}
F_A(x^i, x^i_j, x^i_{jk}) &= (x^i_j, x^i_{jk})^{-1} \cdot F_A(x^i, \delta^i_j, 0) \\
&= (x^i_j, x^i_{jk})^{-1} \cdot [a^i D_i + a^{ij} D_{ij}] \\
&= (\overline{x}^i_s a^s - \overline{x}^i_s x^s_{pq} \overline{x}^p_j \overline{x}^q_k a^{jk}) D_i \\
&\quad + (\overline{x}^i_r \overline{x}^j_s a^{rs}) D_{ij}
\end{aligned}$$

where $(\overline{x}^i_s) = (x^i_s)^{-1}$and $\{D_i, D_{ij}\}$ is the canonical base of $\tau_0 I\!R^n$. Then

$$\begin{aligned}
[\frac{\partial}{\partial x^k}]^{H^2}(x^i, \delta^i_j, 0) F_A &= [\frac{\partial}{\partial x^k} - \Gamma^\alpha_{k\beta}\frac{\partial}{\partial x^\alpha_\beta} - \Gamma^\alpha_{k\beta\gamma}\frac{\partial}{\partial x^\alpha_{\beta\gamma}}](x^i, \delta^i_j, 0) \cdot \\
&\quad \cdot ((\overline{x}^i_s a^s - \overline{x}^i_s x^s_{pq} \overline{x}^p_j \overline{x}^q_k a^{jk}) D_i + (\overline{x}^i_r \overline{x}^j_s a^{rs}) D_{ij}) \\
&= (\frac{\partial a^i}{\partial x^k} + \Gamma^i_{kj}a^j + \Gamma^i_{kjr}a^{jr}) D_i \\
&\quad + (\frac{\partial a^{ij}}{\partial x^k} + \Gamma^i_{kr}a^{rj} + \Gamma^i_{kr}a^{jr}) D_{ij} .
\end{aligned}$$

Thus

$$\begin{aligned}
\widetilde{\nabla}_{\frac{\partial}{\partial x^k}} A(x) &= (x^i, \delta^i_j, 0) \cdot [\frac{\partial}{\partial x^k}]^{H^2}(x^i, \delta^i_j, 0) F_A \\
&= (x^i, \delta^i_j, 0) \cdot [(\frac{\partial a^i}{\partial x^k} + \Gamma^i_{kj}a^j + \Gamma^i_{kjr}a^{jr}) D_i \\
&\quad + (\frac{\partial a^{ij}}{\partial x^k} + \Gamma^j_{ks}a^{is} + \Gamma^i_{ks}a^{sj}) D_{ij}] \\
&= (\frac{\partial a^i}{\partial x^k} + \Gamma^i_{kj}a^j + \Gamma^i_{kjr}a^{jr})\frac{\partial}{\partial x^i} \\
&\quad + (\frac{\partial a^{ij}}{\partial x^k} + \Gamma^j_{ks}a^{is} + \Gamma^i_{ks}a^{sj})\frac{\partial^2}{\partial x^i \partial x^j} .
\end{aligned}$$

□

Now, it is not difficult to compute the transformation coordinates formula for the local components of a connection in $H^2 M$.

Proposition 2.2. *Let* Γ *be a connection in* $H^2 M$. *If* $(\Gamma^i_{kj}, \Gamma^s_{kmn})$ *and* $(\overline{\Gamma}^i_{kj}, \overline{\Gamma}^s_{kmn})$ *are the local components of* Γ *in the local charts* (U, x^i) *and* $(U, \overline{x}^i)$ *of* M *respec-*

tively. Then

$$\begin{aligned}
\overline{\Gamma}^k_{rl} &= \Gamma^i_{jt}\frac{\partial\overline{x}^k}{\partial x^i}\frac{\partial x^t}{\partial\overline{x}^l}\frac{\partial x^j}{\partial\overline{x}^r}+\frac{\partial\overline{x}^k}{\partial x^i}\frac{\partial^2 x^i}{\partial\overline{x}^r\partial\overline{x}^l}\\
\overline{\Gamma}^k_{rab} &= \Gamma^i_{jmn}\frac{\partial\overline{x}^k}{\partial x^i}\frac{\partial x^j}{\partial\overline{x}^r}\frac{\partial x^m}{\partial\overline{x}^a}\frac{\partial x^n}{\partial\overline{x}^b}+\Gamma^i_{jt}\frac{\partial^2 x^k}{\partial\overline{x}^a\partial\overline{x}^b}\frac{\partial x^j}{\partial\overline{x}^r}\frac{\partial\overline{x}^k}{\partial x^i}\\
&+2\Gamma^i_{ml}\frac{\partial^2\overline{x}^k}{\partial x^n\partial x^i}\frac{\partial x^l}{\partial\overline{x}^k}\frac{\partial x^m}{\partial\overline{x}^a}\frac{\partial x^n}{\partial\overline{x}^b}+\frac{\partial^2\overline{x}^k}{\partial x^i\partial x^j}\Big(\frac{\partial^2 x^j}{\partial\overline{x}^a\partial\overline{x}^r}\frac{\partial x^i}{\partial\overline{x}^b}\\
&+\frac{\partial^2 x^i}{\partial\overline{x}^b\partial\overline{x}^r}\frac{\partial x^j}{\partial\overline{x}^a}\Big)+\frac{\partial\overline{x}^k}{\partial x^l}\frac{\partial^3 x^l}{\partial\overline{x}^a\partial\overline{x}^b\partial\overline{x}^r}
\end{aligned}$$

Let X_t be a semimartingale in M. We recall that Y_t, the stochastic parallel transport of $Y_0 \in T_{X_0}M$ along of X_t in the sense of P. Meyer (c.f. [14], page 181-182) is written as the solution of a linear stochastic differential equation. In a local chart (U, x^i) of M, we have that Y_t satisfies an equation of the following type

$$dY^i = a^i_{jl}Y^j\,dX^l + a^i_{jmn}Y^j\frac{1}{2}d[X^m, X^n]$$

where $(\pi^{-1}(U), x^i, y^j = dx^j)$ is the natural induced chart of TM. We call (a^i_{jl}, a^i_{jmn}) the local components of the linear stochastic differential equation. Now, we compute the transformation coordinate formula for these local components.

Proposition 2.3. *Let M be a manifold, (a^i_{kj}, a^s_{kmn}) and $(\overline{a}^i_{kj}, \overline{a}^s_{kmn})$ the local components of a linear stochastic differential equation in the local charts (U, x^i) and $(U, \overline{x}^i)$ of M respectively. Then*

$$\overline{a}^k_{rl} = a^i_{jt}\frac{\partial\overline{x}^k}{\partial x^i}\frac{\partial x^t}{\partial\overline{x}^l}\frac{\partial x^j}{\partial\overline{x}^r}-\frac{\partial\overline{x}^k}{\partial x^i}\frac{\partial^2 x^i}{\partial\overline{x}^r\partial\overline{x}^l}$$

and

$$\begin{aligned}
\overline{a}^k_{rab} &= a^i_{jmn}\frac{\partial\overline{x}^k}{\partial x^i}\frac{\partial x^j}{\partial\overline{x}^r}\frac{\partial x^m}{\partial\overline{x}^a}\frac{\partial x^n}{\partial\overline{x}^b}+a^i_{jt}\frac{\partial^2 x^k}{\partial\overline{x}^a\partial\overline{x}^b}\frac{\partial x^j}{\partial\overline{x}^r}\frac{\partial\overline{x}^k}{\partial x^i}\\
&+2a^i_{ml}\frac{\partial^2\overline{x}^k}{\partial x^n\partial x^i}\frac{\partial x^l}{\partial\overline{x}^k}\frac{\partial x^m}{\partial\overline{x}^a}\frac{\partial x^n}{\partial\overline{x}^b}-\frac{\partial^2\overline{x}^k}{\partial x^i\partial x^j}\Big(\frac{\partial^2 x^j}{\partial\overline{x}^a\partial\overline{x}^r}\frac{\partial x^j}{\partial\overline{x}^b}\\
&+\frac{\partial^2 x^i}{\partial\overline{x}^b\partial\overline{x}^r}\frac{\partial x^j}{\partial\overline{x}^a}\Big)-\frac{\partial\overline{x}^k}{\partial x^l}\frac{\partial^3 x^l}{\partial\overline{x}^a\partial\overline{x}^b\partial\overline{x}^r}\,.
\end{aligned}$$

Proof. From the product rule we obtain that

$$d\overline{Y}^k = d\Big(\frac{\partial\overline{x}^k}{\partial x^i}Y^i\Big) = Y^i d\frac{\partial\overline{x}^k}{\partial x^i}+\frac{\partial\overline{x}^k}{\partial x^i}dY^i + d\Big[\frac{\partial\overline{x}^k}{\partial x^i}, Y^i\Big]\,.$$

Now, by an application of Itô formula

$$
\begin{aligned}
Y^i d\frac{\partial \overline{x}^k}{\partial x^i} &= Y^i\Big(\frac{\partial^2 \overline{x}^k}{\partial x^i \partial x^j} dX^j + \frac{\partial^3 \overline{x}^k}{\partial x^i \partial x^i \partial x^t}\frac{1}{2} d[X^j, X^t]\Big) \\
&= \frac{\partial^2 \overline{x}^k}{\partial x^i \partial x^j}\frac{\partial x^j}{\partial \overline{x}^l}\frac{\partial x^i}{\partial \overline{x}^r}\overline{Y}^r d\overline{X}^l + \frac{\partial^2 \overline{x}^k}{\partial x^i \partial x^j}\frac{\partial x^i}{\partial \overline{x}^r}\frac{\partial^2 x^j}{\partial \overline{x}^a \partial \overline{x}^b}\overline{Y}^r \frac{1}{2} d[\overline{X}^a, \overline{X}^b] \\
&\quad + \frac{\partial^3 \overline{x}^k}{\partial x^i \partial x^j \partial x^t}\frac{\partial x^j}{\partial \overline{x}^a}\frac{\partial x^t}{\partial \overline{x}^b}\frac{\partial x^i}{\partial \overline{x}^r}\overline{Y}^r \frac{1}{2} d[\overline{X}^a, \overline{X}^b],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \overline{x}^k}{\partial x^i} dY^i &= \frac{\partial \overline{x}^k}{\partial x^i}\Big(a^i_{jt} Y^j dX^t + a^i_{jmn}\frac{1}{2} d[X^m, X^n]\Big) \\
&= a^i_{jt}\frac{\partial x^t}{\partial \overline{x}^l}\frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^j}{\partial \overline{x}^r}\overline{Y}^r d\overline{X}^l + a^i_{jt}\frac{\partial^2 x^t}{\partial \overline{x}^a \partial \overline{x}^b}\frac{\partial x^j}{\partial \overline{x}^r}\frac{\partial \overline{x}^k}{\partial x^i}\overline{Y}^r \frac{1}{2} d[\overline{X}^a, \overline{X}^b] \\
&\quad + a^i_{jmn}\frac{\partial \overline{x}^k}{\partial x^i}\frac{\partial x^j}{\partial \overline{x}^a}\frac{\partial x^m}{\partial \overline{x}^b}\frac{\partial x^n}{\partial \overline{x}^r}\overline{Y}^r \frac{1}{2} d[\overline{X}^a, \overline{X}^b],
\end{aligned}
$$

$$
\begin{aligned}
d\Big[\frac{\partial \overline{x}^k}{\partial x^i}, Y^i\Big] &= d\Big[\frac{\partial^2 \overline{x}^k}{\partial x^i \partial x^j} dX^j + \frac{\partial^3 \overline{x}^k}{\partial x^i \partial x^j \partial x^t}\frac{1}{2} d[X^j, X^t], a^i_{jt} Y^j dX^t \\
&\quad + a^i_{jmn}\frac{1}{2} d[X^m, X^n]\Big] = 2a^i_{ml}\frac{\partial^2 \overline{x}^k}{\partial x^i \partial x^n}\frac{\partial x^n}{\partial \overline{x}^b}\frac{\partial x^m}{\partial \overline{x}^a}\frac{\partial x^l}{\partial \overline{x}^r}\overline{Y}^r \frac{1}{2} d[\overline{X}^a, \overline{X}^b].
\end{aligned}
$$

Then

$$
\begin{aligned}
d\overline{Y}^k &= \Big(a^i_{jt}\frac{\partial x^t}{\partial \overline{x}^l}\frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^j}{\partial \overline{x}^r} + \frac{\partial^2 \overline{x}^k}{\partial x^i \partial x^j}\frac{\partial x^j}{\partial \overline{x}^l}\frac{\partial x^i}{\partial \overline{x}^r}\Big)\overline{Y}^r d\overline{X}^l \\
&\quad + \Big(a^i_{jmn}\frac{\partial \overline{x}^k}{\partial x^i}\frac{\partial x^j}{\partial \overline{x}^a}\frac{\partial x^m}{\partial \overline{x}^b}\frac{\partial x^n}{\partial \overline{x}^r} + a^i_{jt}\frac{\partial^2 x^t}{\partial \overline{x}^a \partial \overline{x}^b}\frac{\partial x^j}{\partial \overline{x}^r}\frac{\partial \overline{x}^k}{\partial x^i} \\
&\quad + 2a^i_{ml}\frac{\partial^2 \overline{x}^k}{\partial x^i \partial x^n}\frac{\partial x^n}{\partial \overline{x}^b}\frac{\partial x^m}{\partial \overline{x}^a}\frac{\partial x^l}{\partial \overline{x}^r}\frac{\partial^2 \overline{x}^k}{\partial x^i \partial x^j}\frac{\partial x^i}{\partial \overline{x}^r}\frac{\partial^2 x^j}{\partial \overline{x}^a \partial \overline{x}^b} \\
&\quad + \frac{\partial^3 \overline{x}^k}{\partial x^i \partial x^j \partial x^t}\frac{\partial x^j}{\partial \overline{x}^a}\frac{\partial x^t}{\partial \overline{x}^b}\frac{\partial x^i}{\partial \overline{x}^r}\Big)\overline{Y}^r \frac{1}{2} d[\overline{X}^a, \overline{X}^b].
\end{aligned}
$$

However by partial derivations of $\frac{\partial x^r}{\partial x^s} = \delta^r_s$ and $\frac{\partial \overline{x}^a}{\partial \overline{x}^b} = \delta^a_b$, we have

$$
\begin{aligned}
\frac{\partial^2 \overline{x}^k}{\partial x^t \partial x^s}\frac{\partial x^s}{\partial \overline{x}^l}\frac{\partial x^t}{\partial \overline{x}^r} + \frac{\partial \overline{x}^k}{\partial x^i}\frac{\partial^2 x^i}{\partial \overline{x}^r \partial \overline{x}^l} &= 0 \\
\frac{\partial^3 \overline{x}^k}{\partial x^i \partial x^j \partial x^t}\frac{\partial x^j}{\partial \overline{x}^a}\frac{\partial x^t}{\partial \overline{x}^b}\frac{\partial x^i}{\partial \overline{x}^r} + \frac{\partial^2 \overline{x}^k}{\partial x^i \partial x^j}\frac{\partial x^i}{\partial \overline{x}^r}\frac{\partial^2 x^j}{\partial \overline{x}^a \partial \overline{x}^b} & \\
+ \frac{\partial^2 \overline{x}^k}{\partial x^i \partial x^j}\Big(\frac{\partial^2 x^j}{\partial \overline{x}^a \partial \overline{x}^r}\frac{\partial x^j}{\partial \overline{x}^b} + \frac{\partial^2 x^i}{\partial \overline{x}^b \partial \overline{x}^r}\frac{\partial x^j}{\partial \overline{x}^a}\Big) + \frac{\partial \overline{x}^k}{\partial x^l}\frac{\partial^3 x^l}{\partial \overline{x}^a \partial \overline{x}^b \partial \overline{x}^r} &= 0
\end{aligned}
$$

Therefore, we obtain that

$$\begin{aligned}
d\overline{Y}^k &= (a^i_{jt}\frac{\partial\overline{x}^k}{\partial x^i}\frac{\partial x^t}{\partial\overline{x}^l}\frac{\partial x^j}{\partial\overline{x}^r} - \frac{\partial\overline{x}^k}{\partial x^i}\frac{\partial^2 x^i}{\partial\overline{x}^r\partial\overline{x}^l})\overline{Y}^r d\overline{X}^l \\
&+(a^i_{jmn}\frac{\partial\overline{x}^k}{\partial x^i}\frac{\partial x^j}{\partial\overline{x}^r}\frac{\partial x^m}{\partial\overline{x}^a}\frac{\partial x^n}{\partial\overline{x}^b} + a^i_{jt}\frac{\partial^2 x^k}{\partial\overline{x}^a\partial\overline{x}^b}\frac{\partial x^j}{\partial\overline{x}^r}\frac{\partial\overline{x}^k}{\partial x^i} \\
&+2a^i_{ml}\frac{\partial^2\overline{x}^k}{\partial x^n\partial x^i}\frac{\partial x^l}{\partial\overline{x}^k}\frac{\partial x^m}{\partial\overline{x}^a}\frac{\partial x^n}{\partial\overline{x}^b} - \frac{\partial^2\overline{x}^k}{\partial x^i\partial x^j}(\frac{\partial^2 x^j}{\partial\overline{x}^a\partial\overline{x}^r}\frac{\partial x^j}{\partial\overline{x}^b} \\
&+\frac{\partial^2 x^i}{\partial\overline{x}^b\partial\overline{x}^r}\frac{\partial x^j}{\partial\overline{x}^a}) - \frac{\partial\overline{x}^k}{\partial x^l}\frac{\partial^3 x^l}{\partial\overline{x}^a\partial\overline{x}^b\partial\overline{x}^r})\overline{Y}^r\frac{1}{2}d[\overline{X}^a,\overline{X}^b]\,.
\end{aligned}$$

On the other hand $d\overline{Y}^k = \overline{a}^k_{rl}\overline{Y}^r d\overline{X}^l + \overline{a}^k_{rab}\overline{Y}^r \frac{1}{2}d[\overline{X}^a,\overline{X}^b]$. Comparing this equallity with 1 we get

$$\begin{aligned}
\overline{a}^k_{rl} &= a^i_{jt}\frac{\partial\overline{x}^k}{\partial x^i}\frac{\partial x^t}{\partial\overline{x}^l}\frac{\partial x^j}{\partial\overline{x}^r} - \frac{\partial\overline{x}^k}{\partial x^i}\frac{\partial^2 x^i}{\partial\overline{x}^r\partial\overline{x}^l} \\
\overline{a}^k_{rab} &= a^i_{jmn}\frac{\partial\overline{x}^k}{\partial x^i}\frac{\partial x^j}{\partial\overline{x}^r}\frac{\partial x^m}{\partial\overline{x}^a}\frac{\partial x^n}{\partial\overline{x}^b} + a^i_{jt}\frac{\partial^2 x^k}{\partial\overline{x}^a\partial\overline{x}^b}\frac{\partial x^j}{\partial\overline{x}^r}\frac{\partial\overline{x}^k}{\partial x^i} \\
&+2a^i_{ml}\frac{\partial^2\overline{x}^k}{\partial x^n\partial x^i}\frac{\partial x^l}{\partial\overline{x}^k}\frac{\partial x^m}{\partial\overline{x}^a}\frac{\partial x^n}{\partial\overline{x}^b} - \frac{\partial^2\overline{x}^k}{\partial x^i\partial x^j}(\frac{\partial^2 x^j}{\partial\overline{x}^a\partial\overline{x}^r}\frac{\partial x^j}{\partial\overline{x}^b} \\
&+\frac{\partial^2 x^i}{\partial\overline{x}^b\partial\overline{x}^r}\frac{\partial x^j}{\partial\overline{x}^a}) - \frac{\partial\overline{x}^k}{\partial x^l}\frac{\partial^3 x^l}{\partial\overline{x}^a\partial\overline{x}^b\partial\overline{x}^r}\,.
\end{aligned}$$

□

From the last proposition we get

Corollary 2.4. *There is a bijective correspondence between connections of H^2M and stochastic parallel transport of TM. Let Γ be a connection of H^2M and (U,x^i) a local chart of M. Then this correspondence is given by*

$$\Gamma = (\Gamma^i_{kj},\Gamma^s_{kmn}) \to dY^i = -\Gamma^i_{jl}Y^j dX^l - \Gamma^i_{jmn}Y^j\frac{1}{2}d[X^m,X^n]\,.$$

We recall [2] that a 2-surconnection of TM is a splitting $\lambda_2 : TM \to J_2TM$ of the following exact sequence of vector bundles:

$$0 \longrightarrow J_2^0TM \longrightarrow J_2TM \longrightarrow TM \longrightarrow 0\,.$$

Let (U,x^i) be a local chart of M, $(\pi^{-1}_{TM}U, x^i, y^\alpha = dx^\alpha)$ and $(\pi^{-1}_{J_2TM}U, x^i, y^\alpha, y^\alpha_i, y^\alpha_{ij})$ the induced chart for TM and J_2TM respectively. Then in this local coordinates

$$\lambda_2 : (x^i,y^j) \to (x^i, y^j, -a^j_{kr}(x^i)y^k, -a^j_{krs}(x^i)y^k)\,.$$

It is to easy to verify, that there is a bijective correspondence between connections of H^2M and 2-surconnections of TM. In local coordinates this correspondence is given by

$$\Gamma = (\Gamma^i_{kj},\Gamma^s_{kmn}) \to \lambda_\Gamma : (x^i,y^j) \to (x^i, y^j, -\Gamma^j_{kr}y^k, -\Gamma^j_{krs}y^k)$$

As a consequence of the above comments and [2, proposition 4.8], we have the following result wich is of the strictly geometric nature.

Proposition 2.5. *There is a bijective correspondence between connections of H^2M and 2-connections of H^1M.*

3. The Prolongation $\Gamma \to \Gamma^1$.

The basic purpose of the present section is to give a natural prolongation $\Gamma \to \Gamma^1$ of connections of H^1M to connections of H^2M. Such prolongation is built from the following result of S. Kobayashi [10]: There is a bijective correspondence between $Gl(n)$-reductions of H^2M (that is, a subbundle of H^2M with structural group $Gl(n)$) and connections without torsion of H^1M (affine connections without torsion of M). We give a new proof of this and build the prolongation $\Gamma \to \Gamma^1$, calculate the local components of Γ^1 and show that Γ^1 coincide with the prolongation $p(\Gamma)$ defined by I. Kolá ř [12]. We give an interpretation of this prolongation in terms of stochastic parallel transport. Finally, we remark that by the bijective correspondence of Proposition 2.4, Γ^1 is associated with the parallel transport of Dorhn-Guerra in TM.

We remember that

Definition 3.1. [5] Let M and N be manifolds endowed with connections without torsion Γ_M and Γ_N respectively, $x \in M$ and $y \in N$. A linear mapping $F : \tau_x M \to \tau_y N$ is affine if

$$\Gamma_N \circ F = F \circ \Gamma_M .$$

Let Γ be an affine connection without torsion of M. Then we define

$$Q(\Gamma) = \left\{ j_0^2 s \in H^2M : s_* : \tau_0^n I\!R \to \tau_{s(0)} M \text{ is affine for } \Gamma \right\}$$

where $I\!R^n$ is taken with the usual flat connection.

Let (U, x^α) be a local coordinate system (U, x^α) for M then [4] $s_* : \tau_0 I\!R^n \to \tau_{s(0)} M$ is affine for Γ iff

$$D_{ij} s^\alpha = -D_i s^\beta D_j s^\gamma \Gamma_{\beta\gamma}^\alpha$$

where $\Gamma_{\beta\gamma}^\alpha$ are the components of Γ in (U, x^α).

Obviously, if $j_0^2 s \in Q(\Gamma)$ and $j_0^2 f \in Gl(n) \leq Gl^2$ then $j_0^k s \circ f \in Q(\Gamma)$. In fact, since

$$D_{ij}(s \circ f)^\alpha = (D_{rt} s^\alpha) \circ f . D_j f^r \cdot D_i f^t + (D_r s^\alpha) \circ f \cdot D_{ij} f^r$$

and $D_{ij} f^r = 0$, we have

$$\begin{aligned} D_{ij}(s \circ f)^\alpha &= (D_{rt} s^\alpha) \circ f . D_j f^r \cdot D_i f^t \\ &= -D_r s^\beta D_t s^\gamma \Gamma_{\beta\gamma}^\alpha \cdot D_j f^r \cdot D_i f^t \\ &= -D_i (s \circ f)^\beta D_j (s \circ f)^\gamma \Gamma_{\beta\gamma}^\alpha . \end{aligned}$$

And as, $j_0^2 s \in Q(\Gamma)$ and $j_0^2 s \cdot g \in Q(\Gamma)$ implies $g \in Gl(n)$. We conclude that $Q(\Gamma)$ is a $Gl(n)$-structure of H^2M.

Theorem 3.2. [S. Kobayashi] *The application*

$$Q : \{\textit{connections without torsion for } M\} \to \{Gl(n)\textit{-structures of } H^2M\}$$

is bijective.

Proof. Let Γ and Γ' be connections without torsion for M. It is not difficult to see that $Q(\Gamma) \neq Q(\Gamma')$ if $\Gamma \neq \Gamma'$. Now, let P be a $Gl(n)$-structure of H^2M and $j_0^2 s \in P$. Then by the inverse function theorem, there is a neighborhood U of $x = s(0) \in M$ such that $(U, (x^i) = s^{-1})$ is a local coordinate system. In this coordinate system $D_\alpha s^\theta = \delta_\alpha^\theta$, we define $\Gamma_{\beta\gamma}^\alpha(x) = -D_{\beta\gamma} s^\alpha$. These are the local components of Γ, an affine connection without torsion for M. Obviously, for this connection $P = \{j_0^2 s \in H^2M : s_* : \tau_0 I\!R^n \to \tau_{s(0)} M \text{ is affine for } \Gamma\}$. That is $P = Q(\Gamma)$. □

Let Γ be a connection without torsion of M. We can define a monomorphism of principal fiber bundles $i_\Gamma = i \circ [\pi_1^2 / Q(\Gamma)]^{-1} : H^1M \to H^2M$ where $i_\Gamma : Gl(n) \to Gl^2$ is the natural inclusion. Then, by [11, proposition 6.1] there is a unique connection Γ^1 in H^2M such that the horizontal subspaces of Γ are mapped into horizontal subspaces of Γ^1 by i_Γ. If ω_Γ is the connection form of Γ then $i_\Gamma^* \omega_{\Gamma^1} = (i_\Gamma)_* \omega_\Gamma$.

Let Γ be an affine connection without torsion. Our next aim is to seek the local components of Γ^1.

Let $p_1 = (x^i, \delta_j^i) \in H^1M$ and $p_2 = (x^i, \delta_j^i, 0) \in H^2M$. From $i_\Gamma(p_1) = (x^i, \delta_j^i, -\Gamma_{jk}^i)$ we have that

$$\begin{aligned} i_{\Gamma *}(p_1)\left(\frac{\partial}{\partial x^i}\right) &= \frac{\partial}{\partial x^i} - x_j^r x_k^t \frac{\partial \Gamma_{rt}^l}{\partial x^i} \frac{\partial}{\partial x_{jk}^l} \\ &= \frac{\partial}{\partial x^i} - \frac{\partial \Gamma_{jk}^l}{\partial x^i} \frac{\partial}{\partial x_{jk}^l} \end{aligned}$$

and

$$\begin{aligned} i_{\Gamma *}(p_1)\left(\frac{\partial}{\partial x_j^r}\right) &= \frac{\partial}{\partial x_j^r} - \frac{\partial (x_n^l x_m^k)}{\partial x_j^r} \Gamma_{lk}^i \frac{\partial}{\partial x_{nm}^i} \\ &= \frac{\partial}{\partial x_j^r} - x_m^k \Gamma_{rk}^i \frac{\partial}{\partial x_{jm}^i} - x_n^l \Gamma_{lr}^i \frac{\partial}{\partial x_{nj}^i} \\ &= \frac{\partial}{\partial x_j^r} - 2\Gamma_{rk}^i \frac{\partial}{\partial x_{jk}^i}. \end{aligned}$$

Then

$$\begin{aligned} i_{\Gamma *}(p_1)(X^{H^1}) &= X^i\left(i_{\Gamma *}(p_1)(\frac{\partial}{\partial x^i}) - \Gamma^r_{ij} i_{\Gamma *}(p_1)(\frac{\partial}{\partial x^r_j})\right) \\ &= X^i\left(\frac{\partial}{\partial x^i} - \frac{\partial \Gamma^r_{jk}}{\partial x^i}\frac{\partial}{\partial x^r_{jk}} - \Gamma^r_{ij}(\frac{\partial}{\partial x^r_j} - 2\Gamma^s_{rk}\frac{\partial}{\partial x^s_{jk}})\right) \\ &= X^i\left(\frac{\partial}{\partial x^i} - \Gamma^r_{ij}\frac{\partial}{\partial x^r_j} + (2\Gamma^r_{ij}\Gamma^l_{rk} - \frac{\partial \Gamma^l_{jk}}{\partial x^i})\frac{\partial}{\partial x^l_{jk}}\right) . \end{aligned}$$

Let $g = (\delta^i_j, \Gamma^s_{mn}) \in Gl^2$. Since $i_\Gamma(p_1) \cdot g = (x^i, \delta^i_j, -\Gamma^i_{jk}) \cdot (\delta^i_j, \Gamma^i_{jk}) = (x^i, \delta^i_j, 0)$, we have that

$$\begin{aligned} X^{H^2}(p_2) &= R_{g*} \circ i_{\Gamma *}(p_1)(X^{H^1}) \\ &= X^i\left(\frac{\partial}{\partial x^i} - \Gamma^r_{ij}\frac{\partial}{\partial x^r_j} + (2\Gamma^r_{sn}\Gamma^s_{im} - \frac{\partial \Gamma^r_{mn}}{\partial x^i} - \Gamma^r_{ij}\Gamma^j_{mn})\frac{\partial}{\partial x^r_{mn}}\right) \end{aligned}$$

Let $\overline{\Gamma}^i_{jk}$, $\overline{\Gamma}^i_{jkl} = \overline{\Gamma}^i_{jlk}$ be the local components of Γ^1. Then

$$\begin{aligned} \overline{\Gamma}^i_{jk} &= \Gamma^i_{jk} \\ \overline{\Gamma}^r_{imn} &= (\frac{\partial \Gamma^r_{mn}}{\partial x^i} + \Gamma^r_{ij}\Gamma^j_{mn} - (\Gamma^r_{sn}\Gamma^s_{im} + \Gamma^r_{sm}\Gamma^s_{in})) \end{aligned}$$

This last formula implies

Proposition 3.3. *Let (U, x^i) be a local chart of M and Γ a connection without torsion for M with local components Γ^i_{jk}. Then the local components of Γ^1, $\overline{\Gamma}^i_{jk}$ and $\overline{\Gamma}^r_{imn}$ are given by*

$$\begin{aligned} \overline{\Gamma}^i_{jk} &= \Gamma^i_{jk} \\ \overline{\Gamma}^r_{imn}\ X &= (\frac{\partial \Gamma^r_{mn}}{\partial x^i} + \Gamma^r_{ij}\Gamma^j_{mn} - (\Gamma^r_{sn}\Gamma^s_{im} + \Gamma^r_{sm}\Gamma^s_{in})) \end{aligned}$$

As this are the local components of $p(\Gamma)$ [9], where p is the natural operator defined by I. Kolář in [12]. We have proved the following

Corollary 3.4. *Let Γ be a connection without torsion for M. Keeping the above notations, we have that*

$$p(\Gamma) = \Gamma^1$$

Now, we give a description of Γ^1 in terms of stochastic parallel transport. Let Γ be a connection without torsion of M and X_t a semimartingale. It is known that Γ determinates a stochastic parallel transport in TM along X_t, $\mathrm{P}_{X_s X_t} : T_{X_s}M \to T_{X_t}M$. By [5, lemma 11] we have that there is a unique affine extension $\mathrm{P}^\Gamma_{X_s X_t} : \tau_{X_s}M \to \tau_{X_t}M$. It is not difficult to prove, that $\mathrm{P}^\Gamma_{X_s X_t}$ determinates the stochastic horizontal lift of X_t in H^2M associated with $(\Gamma^1)^{\mathcal{S}}$ (this is the Stratonovich prolongation of Γ^1, [2]).

We remember that the stochastic differential equations associated with the parallel transport of Dorhn-Guerra in TM [14] have the following expression in local coordinates $(\pi^{-1}(U), x^i, y^j = dx^j)$:

$$dY_t^r = -\Gamma_{mn}^r Y_t^m dX_t^n + \left(-\frac{\partial \Gamma_{mn}^r}{\partial x^i} - \Gamma_{ij}^r \Gamma_{mn}^j + (\Gamma_{sn}^r \Gamma_{im}^s + \Gamma_{sm}^r \Gamma_{in}^s)\right) Y_t^i \frac{1}{2} d[X^m, X^n]_t \,.$$

Hence by the above proposition

$$dY_t^r = -\overline{\Gamma}_{im}^r Y_t^i dX_t^m - \overline{\Gamma}_{imn}^r Y_t^i \frac{1}{2} d[X^m, X^n]_t \,.$$

Corollary 3.5. *Let Γ be a connection without torsion of M. Then by the correspondence of* 2.4 Γ^1*is associated with the parallel transport of Dorhn-Guerra.*

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Departamento de Matemática, FCEYN. UNMDP
Funes 3350, (7600) Mar del Plata
ARGENTINA
E-mail: pedrojc@mdp.edu.ar