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SELF–DUALITY AND POINTWISE OSSERMAN MANIFOLDS

Dmitry Alekseevsky[1], Novica Blažić[2],
Neda Bokan[2], and Zoran Rakić[2]

Abstract. The main goal is to show that the pointwise Osserman four-dimensional pseudo-Riemannian manifolds (Lorentzian and manifolds of neutral signature $(--++)$) can be characterized as self dual (or anti-self dual) Einstein manifolds. Also, examples of pointwise Osserman manifolds which are not Osserman are discussed.

§0 Introduction and notational conventions

The Jacobi operator $\mathcal{K}_X : Y \mapsto R(Y,X)X$ is a very useful for understanding the relation between the curvature and the geometry of a pseudo-Riemannian manifold (M,g). Except the well known applications in the Riemannian geometry, Jacobi operator helps to describe dynamics of the pseudo-Riemannian manifold. For example, the family of free falling particles along a geodesic γ in a Lorentzian manifold is described by the normal variational Jacobi vector field V along γ. The Jacobi operator plays the role of the tidal force and V satisfies Newton's second law: $V'' - R_{V\gamma'}(\gamma') = 0$.

Assuming that $X \in T_pM$ is the unit vector, it is particularly important case when the eigenvalues of the Jacobi operator $\mathcal{K}_X$ are constant. Let M be a pseudo-Riemannian manifold of signature (p,q). Denote the metric tensor by $\langle \cdot, \cdot \rangle$. Let $S^\epsilon(p) := \{X \in T_pM \mid \langle X, X\rangle = \epsilon 1\}$ be the set of all unit spacelike $(\epsilon = +1)$ or timelike $(\epsilon = -1)$ tangent vectors at $p \in M$. Let $S^\epsilon(M) = \cup_p S^\epsilon(p)$ and $X \in S^\epsilon_p$. Since X is not a null vector, we have $\mathbb{R}X \oplus X^\perp = T_pM$. The Jacobi operator $\mathcal{K}_X$ induces a symmetric endomorphism of the vector space $T_X(S^\epsilon_p) = X^\perp = \{Y \in T_pM \mid \langle X, Y\rangle = 0\}$.

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We say that M is a *spacelike* (resp., *timelike*) *Osserman* at p if the eigenvalues of $\mathcal{K}_X$ are independent of $X \in S_p^+$ (resp., $X \in S_p^-$). M is *timelike* (resp., *spacelike*) *pointwise Osserman* if M is timelike (resp. spacelike) Osserman at each $p \in M$. We say that M is *spacelike* ($\epsilon = +1$) or *timelike* ($\epsilon = -1$) *Osserman* if the eigenvalues of $\mathcal{K}_X$ are constant on $S^\epsilon(M)$.

Let M be a Riemannian manifold. If M is locally a rank one symmetric space or locally flat, then M is locally a two-point homogeneous space. This means that local isometries of M act transitively on the unit sphere bundle $S^+(M)$. Conversely, any manifold which is locally a two-point homogeneous space is locally a rank one symmetric space or is flat. For these manifolds, the eigenvalues of the Jacobi operator $\mathcal{K}_X$ are constant on $S^+(M)$. Osserman [12] conjectured that the converse hold; we restate his conjecture as follows:

Conjecture. *If a Riemannian manifold M is Osserman, then M is locally a two-point homogeneous space.*

There are Riemannian four-dimensional manifolds which are pointwise Osserman but which are not Osserman manifolds. Construction of such an example, $K3$ surface, is based on the characterization of pointwise Osserman manifolds as self-dual (or anti-self-dual) Einstein manifolds obtained by Vanheceke and Sekigawa [15] (see also [10]).

Generally, pseudo-Riemannian non-flat Osserman manifolds are not necessarily locally rank-one symmetric space. There are examples which are not locally symmetric, even not locally homogeneous manifolds (see [2, 14, 8, 5, 3]). But, theorem of Sekigawa and Vanhecke can be generalized to the four-dimensional manifolds of arbitrary signature (Riemannian, Lorentzian and of neutral signature manifolds, Theorem 2). We have recently learned that the same result was obtained by Garcia-Rio independently.

§1 Jacobi operator of manifolds of neutral signature

In the study of the Osserman type conditions for pseudo-Riemannian manifolds, the algebraic structure of Jacobi operator, specially its Jordan form, play important role. Let M be a timelike or spacelike Osserman manifold of signature $(--++)$. Then we naturally distinguish four different cases depending on the algebraic form of the endomorphism $\mathcal{K}_X$ of $\mathbb{R}^3$.

To describe symmetric operators A in pseudo-euclidean vector space V, with the metric $g = <\cdot,\cdot>$ of signature $(-++)$ first we introduce some basis and then prove the corresponding proposition.

We will denote by (t, x, y) an orthonormal basis of V, such that $t^2 = -1, x^2 = y^2 = 1$, and by (p, q, y) an isotropic basis, defined by the conditions

$$p^2 = q^2 = <p, y> = <q, y> = 0; \qquad <p, q> = y^2 = 1.$$

Proposition 1. *Any symmetric endomorphism A of V has one of the following three types and it is described below.*

(1) *Type I. If A has a timelike eigenvector t, then with respect to a suitable*

orthonormal basis (t, x, y) *it has the diagonal form*

$$A = \mathrm{diag}(\lambda, \mu, \nu).$$

(2) *Type II. If* A *has no timelike eigenvector, but it has a spacelike eigenvector* y, *then with respect to some isotropic basis* (p, q, y) *it has the matrix of the form*

$$\text{Type } II_a \quad \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & \mu \end{pmatrix}, \qquad \text{or Type } II_b \quad \begin{pmatrix} \mu & 1 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

(3) *Type III. If* A *has only isotropic eigenvector* p, *then with respect to some isotropic basis* (p, y, q) *the matrix of* A *has the form*

$$\begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}.$$

Proof. The proof follows from the following remarks:

(1) A has an eigenvector $v \in V$;
(2) The orthogonal complement $v^{\perp}$ is two-dimensional A-invariant subspace;
(3) An isotropic basis (p, q) of two-space of signature $(-+)$ is defined up to hyperbolic rotations $p \mapsto kp, \quad q \mapsto (1/k)q$.

§2 Self-dual Einstein and pointwise Osserman manifolds ot neutral type

Let M be a 4-dimensional pseudo-Riemannian manifold of the neutral signature $(--++)$ and E_0, E_1, E_2, E_3 a pseudoorthonormal basis of T_pM in which the first two vectors are timelike and the second two are spacelike. Let $\theta^0, \theta^1, \theta^2, \theta^3$, be the basis of T_p^*M dual to E_0, E_1, E_2, E_3 and $\omega = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$ the corresponding volume form. The metric on the space of two-forms is defined as

$$\langle \theta^i \wedge \theta^j, \theta^p \wedge \theta^q \rangle = \langle \theta^{ij}, \theta^{pq} \rangle = \det \begin{bmatrix} \langle \theta^i, \theta^p \rangle & \langle \theta^i, \theta^q \rangle \\ \langle \theta^j, \theta^p \rangle & \langle \theta^j, \theta^q \rangle \end{bmatrix},$$

where $\theta^{ij} := \theta^i \wedge \theta^j$.

The Hodge star operator $* : \bigwedge^2 T_pM \longrightarrow \bigwedge^2 T_pM$ is defined by

$$*(\xi \wedge \eta) = (\xi \wedge \eta) \lrcorner \omega$$

where $\lrcorner$ means the contraction. Then $*^2 = Id$ and $*(\theta^i \wedge \theta^j) = \epsilon_p \epsilon_q \epsilon_{ijpq} \theta^p \wedge \theta^q$, where $\epsilon_p = \langle E_p, E_p \rangle$ and ϵ_{ijpq} is the signature of the permutation (i, j, p, q).

Two forms

$$J_1 = \theta^{01} + \theta^{23}, \quad J_2 = \theta^{02} + \theta^{13}, \quad J_3 = \theta^{03} - \theta^{12} \tag{2.1}$$

defines a basis of the $+1$-eigenspace Λ_+ of $*$, called space of self-dual forms. Similarly, 2-forms

$$J_1' = \theta^{01} - \theta^{23}, \quad J_2' = \theta^{02} - \theta^{13}, \quad J_3' = \theta^{03} + \theta^{12} \tag{2.2}$$

defines a basis of the space Λ_- of anti-self-dual forms, that is the -1-eigenspace of $*$. These basis are called the standard basis associated to an orthonormal basis E_i of T_pM.

We will identify the curvature tensor R of M at a point p with a symmetric endomorphism of the space $\bigwedge^2 T_pM$ defined by

$$R(\theta^{pq}) = \tfrac{1}{2} R_{ijpq}\theta^{ij}\epsilon_i\epsilon_j. \tag{2.3}$$

Assume now that the manifold M is Einstein. Then its curvature tensor R can be written as

$$R = \frac{\tau}{12}\mathrm{Id} + W_+ + W_-,$$

where τ is the scalar curvature and W_+, W_- are self-dual and anti-self-dual parts of R characterized by conditions

$$W_\pm \bigwedge\nolimits_\mp = 0.$$

A manifold M is called self-dual (anti-self-dual) if $W_- = 0$ ($W_+ = 0$).

The problem of classification and characterization of four-dimensional Osserman type manifolds was studied in [2]. It is interesting to see some relations between the self-dual (anti-self-dual) manifolds and the pointwise Osserman conditions.

Theorem 2. *Let M be an oriented four dimensional Riemannian, Lorentzian or manifold of neutral signature. Then M is pointwise Osserman if and only if M is Einstein self-dual (or anti-self-dual).*

For the Riemannian manifolds it was proved by Sekigawa and Vanhecke [15] (see also [10]). In [4] it is proved that Lorentzian pointwise Osserman manifolds are of constant sectional curvature what is clarifying that case. The proof for manifold of neutral signature is given in the following two propositions.

Proposition 3. *Let M be an oriented four dimensional pointwise Osserman manifold of signature $(--++)$. Then, possibly after a change of orientation, M becomes a self-dual Einstein manifold with the curvature tensor*

$$R = \frac{\tau}{12}Id + W_+,$$

where the self-dual part $W_+ : \bigwedge_+ \to \bigwedge_+$ of R has one of the following forms, depending on the type of the Jacobi operator:

(1) *for Types I, II_a and II_b*

$$W_+ = \begin{bmatrix} -2a - \frac{\tau}{12} & 2\gamma & 0 \\ -2\gamma & -2b - \frac{\tau}{12} & 0 \\ 0 & 0 & -2c - \frac{\tau}{12} \end{bmatrix} \tag{2.4}$$

where $\tau = 4(a+b+c)$ is the scalar curvature, or

(2) *for Type III*

$$R = \begin{bmatrix} \frac{\tau}{12} & 0 & 2k \\ 0 & \frac{\tau}{12} & 2k \\ -2k & 2k & \frac{\tau}{12} \end{bmatrix},$$

where $\tau = 12\alpha$.

Proof. Let M be a pointwise Osserman manifold. First of all let us remark that it is Einstein ([2, Proposition 2.1]). The Jacobi operator of M at a point p has one of the types, described in Proposition 1.

(1) Types I, II_a, and II_b

These cases can be considered together, i.e., there exists an pseudoorthonormal basis E_0, E_1, E_2, E_3 such that the Jacobi operator $\mathcal{K}_{E_0}$ is of the form

$$\begin{bmatrix} -a & \gamma & 0 \\ -\gamma & -b & 0 \\ 0 & 0 & -c \end{bmatrix},$$

and the components of the curvature tensor are determined in ([2, §4.1 and §4.2]) as follows

$$R_{1221} = R_{4334} = a, \quad R_{1331} = R_{4224} = -b, \quad R_{1441} = R_{3223} = -c,$$
$$R_{2113} = R_{2443} = -\gamma, \quad R_{1224} = R_{1334} = \gamma,$$
$$R_{1234} = (-2a+b+c)/3, \quad R_{1423} = (a+b-2c)/3, \quad R_{1342} = (a-2b+c)/3.$$

Note that the scalar curvature is given by $\tau = -4(a+b+c)$.

This implies that the anti-self-dual part W_- of the curvature operator $R = \frac{\tau}{12}\mathrm{id} + W_+$ vanishes and with respect to the standard basis J_1, J_2, J_3 of $\bigwedge_+$ the matrix of the operator W_+ has the form

$$W_+ = \begin{bmatrix} -2a - \frac{\tau}{12} & 2\gamma & 0 \\ -2\gamma & -2b - \frac{\tau}{12} & 0 \\ 0 & 0 & -2c - \frac{\tau}{12} \end{bmatrix}.$$

More precisely,

$$W_+ = \mathrm{diag}(-2a - \tfrac{\tau}{12}, -2b - \tfrac{\tau}{12}, -2c - \tfrac{\tau}{12})$$

for the type I,

$$W_+ = \begin{bmatrix} \frac{4\alpha-\gamma}{3} & 2\beta & 0 \\ -2\beta & \frac{4\alpha-\gamma}{3} & 0 \\ 0 & 0 & \frac{4\alpha-\gamma}{3} \end{bmatrix}$$

for the type II_a, and

$$W_+ = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \tau/4 \end{bmatrix}$$

for the type II_b.

(2) Type III

We will start with the pseudoorthonormal base E_0, E_1, E_2, E_3 where $\mathcal{K}_{E_0}$ has a form

$$\begin{bmatrix} -\alpha & 0 & k \\ 0 & -\alpha & k \\ -k & k & -\alpha \end{bmatrix}, \quad k = \tfrac{\sqrt{2}}{2}.$$

Then the components of the curvature tensor are

$$R_{1221} = R_{4334} = \alpha, \quad R_{1331} = R_{4224} = -\alpha, \quad R_{1441} = R_{3223} = -\alpha,$$
$$R_{2114} = R_{2334} = -k, \quad R_{3114} = -R_{3224} = k\ ,$$
$$R_{1223} = R_{1443} = R_{1332} = -R_{1442} = k,$$

([2, §4.3]). This implies

$$W_+ = \begin{bmatrix} -\alpha & 0 & 2k \\ 0 & -\alpha & 2k \\ -2k & 2k & -\alpha \end{bmatrix}.$$

□

The inverse statement is also true.

Proposition 4. *Any self-dual (or anti-self-dual) Einstein four-dimensional manifold M of signature $(--++)$ is pointwise timelike and spacelike Osserman manifold.*

Proof. The curvature tensor R of the manifold M has the decomposition

$$R = s\mathrm{Id} \oplus W_+,$$

where $\tau = 12s$ is the scalar curvature and W_+ is the self-dual part of R which acts trivially on $\bigwedge_-$ and, hence, may be identified with a symmetric operator of the pseudo-Euclidean space $\bigwedge_+$ of signature $(-,+,+)$.

According to the classification of symmetric operators in the Lorentzian signature $(-,+,+)$, see Proposition 1, W_+ has to have one of the following forms

Case 1

$$W_+ = \begin{bmatrix} -a & \gamma & 0 \\ -\gamma & -b & 0 \\ 0 & 0 & -c \end{bmatrix}$$

or

Case 2

$$W_+ = \begin{bmatrix} -\alpha & 0 & k \\ 0 & -\alpha & k \\ -k & k & -\alpha \end{bmatrix}, \quad k = \tfrac{\sqrt{2}}{2}.$$

One can see that Case 1 corresponds to the types I, II_a and II_b and Case 2 to the type III.

In Case 1 we have

$$RJ_\alpha = sJ_\alpha,$$

$$RJ'_1 = -aJ'_1 - \gamma J'_2, \quad RJ'_2 = \gamma J'_1 - bJ'_2, \quad RJ'_3 = -cJ'_3, \tag{2.5}$$

and in Case 2

$$RJ_\alpha = sJ_\alpha,$$

$$RJ'_1 = -\alpha J'_1 - kJ'_3, \quad RJ'_2 = -\alpha J'_1 + kJ'_3, \quad RJ'_3 = kJ'_1 + kJ'_2 - \alpha J'_3. \tag{2.6}$$

Now, we consider Case 1 in more details. We use (2.1)-(2.4) to see

$$R_{01ij}\theta^{ij}\epsilon_i\epsilon_j + R_{23pq}\theta^{pq}\epsilon_p\epsilon_q = s(\theta^{01} + \theta^{23}), \tag{2.7}$$

$$R_{02ij}\theta^{ij}\epsilon_i\epsilon_j + R_{13pq}\theta^{pq}\epsilon_p\epsilon_q = s(\theta^{02} + \theta^{13}), \tag{2.8}$$

$$R_{03ij}\theta^{ij}\epsilon_i\epsilon_j - R_{12pq}\theta^{pq}\epsilon_p\epsilon_q = s(\theta^{03} - \theta^{12}), \tag{2.9}$$

$$R_{01ij}\theta^{ij}\epsilon_i\epsilon_j - R_{23pq}\theta^{pq}\epsilon_p\epsilon_q = -a(\theta^{01} - \theta^{23}) - \gamma(\theta^{02} - \theta^{13}), \tag{2.10}$$

$$R_{02ij}\theta^{ij}\epsilon_i\epsilon_j - R_{13pq}\theta^{pq}\epsilon_p\epsilon_q = \gamma(\theta^{01} - \theta^{23}) - b(\theta^{02} - \theta^{13}), \tag{2.11}$$

$$R_{03ij}\theta^{ij}\epsilon_i\epsilon_j + R_{12pq}\theta^{pq}\epsilon_p\epsilon_q = -c(\theta^{03} + \theta^{12}). \tag{2.12}$$

One can combine (2.8) with (2.11), (2.9) with (2.12), and (2.7) with (2.10) to obtain respectively

$$2R_{01ij}\theta^{ij}\epsilon_i\epsilon_j = (s-a)\theta^{01} + (s+a)\theta^{23} - \gamma(\theta^{02} - \theta^{13}), \tag{2.13}$$

$$2R_{23ij}\theta^{ij}\epsilon_i\epsilon_j = (s+a)\theta^{01} + (s-a)\theta^{23} + \gamma(\theta^{02} - \theta^{13}), \tag{2.14}$$

$$2R_{02ij}\theta^{ij}\epsilon_i\epsilon_j = (s-b)\theta^{02} + (s+b)\theta^{13} + \gamma(\theta^{01} - \theta^{23}), \tag{2.15}$$

$$2R_{13ij}\theta^{ij}\epsilon_i\epsilon_j = (s+b)\theta^{02} + (s-b)\theta^{13} - \gamma(\theta^{01} - \theta^{23}), \tag{2.16}$$

$$2R_{03ij}\theta^{ij}\epsilon_i\epsilon_j = (s-c)\theta^{03} - (s+c)\theta^{12}, \tag{2.17}$$

$$2R_{12ij}\theta^{ij}\epsilon_i\epsilon_j = -(s+c)\theta^{03} + (s-c)\theta^{12}. \tag{2.18}$$

Hence, we can read an arbitrary component of the curvature tensor from (2.13)-(2.18). Since $\mathcal{K}_{E_0}E_i = R(E_i, E_0)E_0 = R_{i00j}\epsilon_j e_j$ it follows

$$\mathcal{K}_{E_0} = \begin{bmatrix} -R_{1001} & -R_{2001} & -R_{3001} \\ R_{1002} & R_{2002} & R_{3002} \\ R_{1003} & R_{2003} & R_{3003} \end{bmatrix} = \begin{bmatrix} -\frac{s-a}{2} & \frac{\gamma}{2} & 0 \\ -\frac{\gamma}{2} & -\frac{s-b}{2} & 0 \\ 0 & 0 & -\frac{s-c}{2} \end{bmatrix}. \tag{2.19}$$

It follows directly from (2.13)-(2.18) by long computations that the eigenvalues of $\mathcal{K}_X$, $|X|^2 = 1$ or $|X|^2 = -1$, do not depend on the choice of a direction X at a given point. Consequently, the manifold is pointwise Osserman.

Case 2 can be considered analogously. □

Let us state the following consequence of Theorem 2.

Corrolary 5. *A four-dimensional manifold of signature* $(- - ++)$ *is timelike Osserman if and only if it is spacelike Osserman.*

It is proved, in [9], that for pseudo-Riemannian manifold of signature (p,q), $p,q \geq 1$ timelike Osserman condition is equivalent with space-like Osserman condition.

§3 Examples

Since a Riemannian or of neutral signature pointwise Osserman manifold is the same as a self-dual Einstein manifold many examples of such manifolds can be obtained by using the twistor construction, see ([1]). In compact Riemannian case the following result is known.

Theorem. *(N. Hitchin, T. Friedrich, H. Kurke) Let M be a compact Riemannian self-dual Einstein 4-manifold. If it has positive scalar curvature τ then it is S^4 or $\mathbb{C}P^2$ with the standard metric.*

If $\tau = 0$, then the universal covering of M is a K3 surface with the Calabi-Yau metric.

The problem of description of compact self-dual Einstein manifolds of neutral signature $(- - ++)$ is still open.

First non-trivial examples of compact Ricci flat self dual metrics of neutral signature on the torus was constructed by Petean [13].

Kamada and Machida [11] constructed some examples of non-compact Ricci flat self-dual manifolds of neutral signature. In [11] using the decomposition of the curvature tensor, notion of Bianchi type is used to study self-dual manifolds. Particular attention is devoted to the Kähler self-dual manifolds.

The interesting problem is to constract non-Ricci flat Osserman (i.e. self-dual Einstein) manifolds of neutral signature. Remarks that rank one symmetric 4-manifolds of neutral signature $(- - ++)$ are exhaused by non-compact manifolds $SO_{2,3}/SO_{2,2}$ and $SU_{1,2}/SU_{1,1}$.

Problem. *Is there a compact non-Ricci flat Osserman manifold of neutral signature* $(--++)$*?*

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References

[1] Besse, A., *Einstein manifolds*, Ergebnisse der Math., vol. 3, Springer-Verlag, 1987.

[2] Blažić, N., Bokan, N. and Rakić, Z., *Characterization of 4-dimensional Osserman pseudo-Riemannian manifolds*, preprints 1995, 1997.

[3] ———, *A note on Osserman conjecture and isotropic covariant derivative of curvature*, Proc. Amer. Math. Soc., (to appear).

[4] Blažić, N., Bokan, N. and Gilkey, P., *A Note on Osserman Lorentzian manifolds*, Bull. London Math. Soc. **29** (1997), 227–230.

[5] Bonome, A., Castro, R., Garcia-Rio, E., Hervella, L. and Vázquez-Lorenzo, R., *On Osserman semi-Riemannian manifolds*, Proc. Amer. Math. Soc., (to appear).

[6] Galicki, K. and Lawson, B., *Quaternionic reduction and quaternionic orbifolds*, Math. Ann. **282** (1988), 1–21.

[7] Galicki, K. and Nitta, T., *Non-zero scalar curvature generalization of the ALE hyperkähler metrics*, J. Math. Phys. **33** (1992), 1765–1771.

[8] Garcia-Rio, E., Vázquez-Abal, M. E. and Vázquez-Lorenzo, R., *Nonsymmetric Osserman pseudo-Riemannian manifolds*, Proc. Amer. Math. Soc. **126** (1998), 2763–2769.

[9] Gilkey, P., Stanilov, G. and Videv, V., *Pseudo Riemannian manifolds whose generalized Jacobi operator has constant characteristic polynomial*, Journal of Geometry **62** (1998), 144–153.

[10] Gilkey, P., Swann, A. and Vanhecke, L., *Isoparametric geodesic spheres and a conjecture of Osserman concerning the Jacobi operator*, Quart. J. Math. Oxford. **46** (1995), 299–320.

[11] Kamada, H. and Machida, Y., *Self-duality of type (2,2) on four-dimensional manifolds*, Tohôku Math. J. **49** (1997), 259–275.

[12] Osserman, R., *Curvature in the eighties*, Amer. Math. Monthly **97** (1990), 731–756.

[13] Petean, J., *Indefinite Kähler-Einstein metrics on compact complex surfaces*, Comm. Math. Phys. **189** (1997), 227–235.

[14] Rakić, Z., *Rank 2 symmetric Osserman spaces*, Bull. Austral. Math. Soc. **56** (1988), 517–521.

[15] Sekigawa, K. and Vanhecke, L., *Volume preserving geodesic symmetries on four dimensional Kähler manifolds*, Differential geometry Penścola, 1985, Proceedings (A. M. Naveira, A. Fernandez and F. Mascaro,eds.), Lecture Notes in Math. **1209**, **Springer**, 275–290.

D. Alekseevsky
Center "Sophius Lie", gen. Antonova 2-99
117279 Moscow, RUSSIA
E-mail: daleksee@esi.ac.at; fort@rsuh.ru

N. Blažić, N. Bokan and Z. Rakić
University of Belgrade, Faculty of Mathematics
Studentski trg 16, p.p. 550
11000 Belgrade, YUGOSLAVIA
E-mail: blazicn@matf.bg.ac.yu
neda@matf.bg.ac.yu
zrakic@matf.bg.ac.yu