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ON ASYMPTOTIC DECAYING SOLUTIONS FOR A CLASS OF SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. The author considers the quasilinear differential equations

$$\left(r(t)\varphi(x')\right)' + q(t)f(x) = 0\,, \qquad t \geq a$$

and

$$\left(r(t)\varphi(x')\right)' + F(t,x) = \pm g(t)\,, \qquad t \geq a\,.$$

By means of topological tools there are established conditions ensuring the existence of nonnegative asymptotic decaying solutions of these equations.

1. Introduction

The purpose of the present paper is to study the existence of asymptotic decaying positive solutions of the quasilinear differential equation

$$\left(r(t)\varphi(x')\right)' + q(t)f(x) = 0, \qquad t \geq a \tag{1}$$

and of the more general ones

$$\left(r(t)\varphi(x')\right)' + F(t,x) = \pm g(t), \qquad t \geq a \tag{2$\pm$}$$

where a is a nonnegative constant, and

- a1) $r,\, q \in C\left([a,\infty),(0,\infty)\right)$;
- a2) $\varphi \in C\left(\mathbb{R},\mathbb{R}\right)$, $u\varphi(u) > 0$ for $u \neq 0$, φ increasing and $\varphi(\mathbb{R}) = \mathbb{R}$;
- a3) $f \in C\left(\mathbb{R},\mathbb{R}\right)$, $uf(u) > 0$ for $u \neq 0$, f nondecreasing;
- a3') $F \in C\left([a,\infty)\times\mathbb{R},\mathbb{R}\right)$, $uF(t,u) > 0$ for $u \neq 0$ and for each fixed $t \geq a$, F nondecreasing with respect to the second variable;
- a4) $g \in C\left([a,\infty),[0,\infty)\right)$.

As customary [10, p. 322] it will be assumed throughout this paper that a solution $x = x(t)$ of (1) (resp. of (2+) or (2−)) is a continuously differentiable function on $[T_x,\infty)$, $T_x \geq a$, such that $r(t)\varphi(x'(t))$ has a continuous derivative in $[T_x,\infty)$ satisfying (1) (resp. (2+) or (2−)) at all points $t \geq T_x$. When the function $r(t)$ is

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continuous but does not have a continuous derivative, Eqs. (1), (2+) and (2−) can be interpreted as a first order nonlinear differential system. For instance, Eq. (1) means the system

$$x'(t) = \varphi^{-1}\left(\frac{y(t)}{r(t)}\right)$$
$$y'(t) = -q(t)f(x(t))$$

for the vector $(x, y) = (x, r\varphi(x'))$.

By a proper solution of (1) (resp. of (2±)) we mean a solution of (1) (resp. of (2±)) such that $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq T_x$. A proper solution is called oscillatory if it has arbitrarily large zeros, and nonoscillatory otherwise.

A prototype of Eq. (1) satisfying assumptions a1), a2), a3) is

$$(r(t)(x')^{m_*})' + q(t)x^{n_*} = 0, \qquad t \geq a \tag{3}$$

where m and n are positive constants and use is made of the notation

$$u^{\alpha_*} = |u|^\alpha \operatorname{sgn} u, \quad \alpha > 0.$$

A great variety of behaviors is exhibited by solutions of equations of the type (3). Equation (3) with $m = 1$ is the generalized second order Emden-Fowler differential equation, which has been extensively investigated from various viewpoints, see for instance the excellent survey by J.S.W. Wong [27], and also the papers [5], [12]. Unlike the linear equation, obtained when $n = 1$ too, the generalized Emden-Fowler equation may at the same time possess oscillatory and nonoscillatory solutions. On the other hand, a striking similarity exists between equations of the form (3) with $n = m$, called half-linear equations (see [7]) and the linear equation obtained when $n = m = 1$. This similarity was observed by Mirzov [19, 20] and Elbert [7, 8], who showed in particular that Sturmian theory (e.g. separation and comparison theorems) for the linear equation can be extended in a natural way to the half-linear case. Thus it is shown that all solutions of the half-linear equation are either oscillatory or else nonoscillatory, so that the possibility of coexistence of oscillatory and nonoscillatory solutions is precluded. Criteria for oscillation and nonoscillation of the half-linear equation have been widely investigated ([6], [7], [8], [9], [13], [14], [15], [16], [19], [20]), characterizing also the phenomena of strong oscillation and nonoscillation of the differential equation

$$(r(t)(x')^{m_*})' + \lambda q(t)x^{m_*} = 0$$

where $\lambda > 0$ is a parameter, and generalizing Nehari's oscillation theorem [22] stated for the linear case. The oscillatory behavior of Eq. (3) in the general case $n \neq m$ has been treated in [17, 18], [9], [21]. Finally in [15], [21] a re deduced nonoscillation theorems for Eq. (3) both in case $n = m$ and $n \neq m$, developing a nonoscillation theorem which is a natural generalization of nonoscillation criteria of Atkinson [1] and Heidel [11] for the Emden-Fowler equation.

Oscillatory and nonoscillatory behavior of solutions of the more general second order quasilinear differential equation

$$\left(|x'|^{\alpha-1}x'\right)' + f(t, x) = 0, \qquad t \geq 0 \tag{4}$$

is investigated in [25]; in this paper it is shown that all of Wong's results [26] can be generalized to (4). We refer to this work also for a good review of results on half-linear equations.

Oscillation and non-oscillation theorems for equations of type (2±) with $g \equiv 0$ are presented in the paper by Elbert and Kusano [9], under the assumption

$$\int_a^\infty \left|\varphi^{-1}\left(\frac{k}{r(t)}\right)\right| dt = \infty \quad \text{for every constant } k \neq 0.$$

Under this hypothesis any nonoscillatory solution x of (2±) with $g \equiv 0$ is one of the following three types:

i) $\lim_{t\to\infty} r(t)\varphi(x'(t)) = \text{ const } \neq 0$;
ii) $\lim_{t\to\infty} r(t)\varphi(x'(t)) = 0$ and $\lim_{t\to\infty} |x(t)| = \infty$;
iii) $\lim_{t\to\infty} r(t)\varphi(x'(t)) = 0$ and $\lim_{t\to\infty} x(t) = \text{ const } \neq 0$.

Thus in particular no nonoscillatory solution exists such that $\lim_{t\to\infty} x(t) = 0$ and eventually positive proper solutions are increasing, eventually negative proper solutions are decreasing. We give here a proof of this assertion slightly different from that in [9], fitting to Eq. (1) methods developped in [4].

Lemma 1.1. *Let $x = x(t)$ be a nonoscillatory proper solution of Eq. (1). Then x' is nonoscillatory.*

Proof. Let us consider the function

$$\Phi(t) := r(t)\varphi(x'(t))x(t)$$

defined in the existence interval I_x of the solution x. Owing to definition, Φ is a C^1-function in I_x, and

$$\begin{aligned}\Phi'(t) &= \left(r(t)\varphi(x'(t))\right)' x(t) + r(t)\varphi(x'(t))x'(t)\\ &= -q(t)f(x(t))x(t) + r(t)\varphi(x'(t))x'(t).\end{aligned}$$

As x is nonoscillatory, there exists a suitable time τ such that x is different from zero for $t > \tau$ and so $f(x)x$ is eventually positive by assumption a3). Let t_1 and t_2 be consecutive zeros of the function x', with t_1, $t_2 > \tau$. Since r is always different from zero (assumption a1)) and x is different from zero in (τ, ∞), then t_1, t_2 are consecutive zeros of the function Φ too. On the other hand it results

$$\Phi'(t_i) = -q(t_i)f(x(t_i))x(t_i) < 0, \quad i = 1, 2$$

and this is a contradiction, being F a continuous function. Thus the function x' is nonoscillatory. □

Theorem 1.1. *Let x be a nonoscillatory solution of Eq. (1) and let us assume that*

$$\int_0^\infty \left|\varphi^{-1}\left(\frac{k}{r(t)}\right)\right| dt = \infty \quad \textit{for every } k \in \mathbb{R},\ k \neq 0. \tag{5}$$

Then

$$x(t)x'(t) > 0.$$

Proof. Lemma 1.1 implies that for every proper solution x of Eq. (1) it results $x(t)x'(t) \neq 0$ definitively. Let x be a solution of Eq. (1) such that $x(t)x'(t) < 0$ for t large. Without loss of generality we may assume $x(t) > 0$, $x'(t) < 0$ for $t \geq \tau$. Since $(r(t)\varphi(x'(t)))' = -q(t)f(x(t)) < 0$ for $t \geq \tau$, the function G given by

$$G(t) = r(t)\varphi(x'(t))$$

is negative decreasing for $t \geq \tau$. Thus $G(t) < G(\tau)$ or

$$\varphi(x'(t)) < \frac{G(\tau)}{r(t)}, \qquad t \geq \tau$$

which implies

$$x'(t) < \varphi^{-1}\left(\frac{G(\tau)}{r(t)}\right), \qquad t \geq \tau.$$

Integrating the last inequality from τ to t, we obtain

$$x(t) < x(\tau) + \int_\tau^t \varphi^{-1}\left(\frac{G(\tau)}{r(s)}\right) ds.$$

Taking into account that $G(\tau) < 0$, as $t \to \infty$ x becomes negative, which is a contradiction. $\square$

Our main objective here is to investigate the existence of positive solutions of Eqs. (1), (2+) and (2−) asymptotically decreasing towards zero. Thus instead of assumption (5) we consider the following one

$$(6) \qquad -\int_0^\infty \varphi^{-1}\left(-\frac{k}{r(s)}\right) ds < \infty \text{ for some constant } k > 0$$

which is the assumption complementary to (5). We remark that the presence of the forcing term $g(t)$ in Eqs. (2±) implies that this kind of equations do not fall within the cases classically treated in literature. Indeed $\hat{f}(x,t) := F(t,x) \pm g(t)$ does not satisfy assumption a3), thus we can assert that at our knowledge the results here proved are the first ones about Eqs. (2±). Under additional hypotheses on q, f (or F), φ (and g) we show in the following two sections that condition (6) guarantees the existence of asymptotic decaying solutions of Eqs. (1), (2+) and (2−).

Finally we want to point out that an interest in the ordinary differential equations (1) and (2±) also arises in connection with the study of quasilinear elliptic partial differential equations of the type

$$(7) \qquad \operatorname{div}\ (\psi(|\nabla u|^2)\nabla u) + n(x,u) = 0$$

where $\nabla = (\partial/\partial x_1, \cdots, \partial/\partial x_N)$ with $N \geq 1$, the function $\psi : (0,\infty) \mapsto (0,\infty)$ is continuous and such that $\varphi(s) := \psi(s^2)s$ is an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and the function $n : \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{R}$ is continuous and radially symmetric with respect to the first variable, that is $n(x,\cdot) = \hat{n}(|x|,\cdot)$. A classical radial solution of (7) is a function $u \in C^1(\mathbb{R}^N)$ with $\psi(|\nabla u|^2)\nabla u \in C^1(\mathbb{R}^N)$, which is radially symmetric and satisfies the equation everywhere in $\mathbb{R}^N$ [24]. It follows from [23] that u is a classical radial solution of (7) if and only if it is a solution of the one dimensional problem

$$\left(t^{N-1}\varphi(x')\right)' + t^{N-1}\hat{n}(t,x) = 0$$

and so the results for (1) and (2±) can be a source of helpful informations about the qualitative behavior of (7).

2. Existence of asymptotic decaying nonnegative proper solutions of Eq. (1)

Denote by B_0 the subset of (bounded) proper solutions of (1) approaching zero as $t \to \infty$:

$$B_0 = \{x \text{ proper solution of } (1) : x(+\infty) = 0\}.$$

The existence of solutions of (1) in class B_0 will be given by using a topological tool. More precisely, we will use a fixed point theorem for operators defined by Schauder's linearization device. Such a theorem was proved in [2], and reduces the existence of solution of a boundary value problem for differential equations in noncompact intervals to the existence of suitable a priori bounds. We restate such theorem ([2, Th. 1.1]) in the form that will be used in the following:

Theorem 2.1 ([2]). *Consider the boundary value problem*

$$\begin{aligned} &(r(t)\varphi(x'(t)))' = H(t, x(t)), \qquad t \in [a, \infty) \\ &x \in S, \end{aligned} \tag{8}$$

where $H : [a,\infty) \times \mathbb{R} \mapsto \mathbb{R}$ is a continuous function, r and φ satisfy assumptions a1) and a2) respectively and S is a nonempty subset of the Fréchet space $C([a,\infty))$ of the continuous real functions defined in $[a,\infty)$. Assume that there exists a nonempty, closed, convex and bounded subset $\Omega \subset C([a,\infty))$, such that for every $u \in \Omega$ the boundary value problem

$$\begin{aligned} &(r(t)\varphi(x'(t)))' = H(t, u(t)), \quad t \geq a \\ &x \in S \end{aligned}$$

has a unique solution $x = T(u)$. If

(i) $T(\Omega) \subset \Omega$,
(ii) $\overline{T(\Omega)} \subset S$,

then the boundary value problem (8) has at least a solution.

We prove the existence of solutions of (1) in B_0 assuming that (6) is fulfilled.

Theorem 2.2. *Let assumptions a1)-a3) hold and there exists a constant $k > 0$ such that*

$$\begin{aligned} I_1(k) &= -\int_a^\infty \varphi^{-1}\left(-\frac{2k}{r(t)}\right) dt < \infty, \\ I_2(k) &= \int_a^\infty q(t) f\left(-\int_t^\infty \varphi^{-1}\left(-\frac{2k}{r(s)}\right) ds\right) dt < \infty. \end{aligned} \tag{9}$$

Then Eq. (1) has at least one nonnegative solution in class B_0.

Proof. In order to get the existence of a solution of (1) in B_0 choose a large t_0 such that

$$c(t_0) := \int_{t_0}^{\infty} q(s) f\left(-\int_s^{\infty} \varphi^{-1}\left(-\frac{2k}{r(\sigma)}\right) d\sigma\right) ds \le k. \tag{10}$$

Let $C[t_0,\infty)$ be the Fréchet space of real continuous functions defined in $[t_0,\infty)$, $t_0 \ge a$, and let Ω and S be such that:

$$\Omega = \left\{ u \in C[t_0,\infty) :\, 0 \le u(t) \le -\int_t^{\infty} \varphi^{-1}\left(-\frac{2k}{r(s)}\right) ds,\, t \in [t_0,\infty) \right\}$$
$$S = \left\{ y \in C^1[t_0,\infty) :\, y'(t_0) = \varphi^{-1}\left(-\frac{k}{r(t_0)}\right),\, y(\infty) = 0 \right\}.$$

For every $u \in \Omega$ consider the differential equation:

$$(r(t)\varphi(y'(t)))' = -q(t) f(u(t)). \tag{11}$$

As

$$\int_{t_0}^{\infty} q(s) f(u(s))\, ds < \infty,$$
$$-\int_{t_0}^{\infty} \varphi^{-1}\left(-\frac{k}{r(s)} - \frac{1}{r(s)} \int_{t_0}^{s} q(\sigma) f(u(\sigma))\, d\sigma\right) ds < \infty,$$

it is immediate to prove that for every $u \in \Omega$ there exists a unique solution y_u of (11) such that $y_u \in S$. Therefore we may define an operator T that associates to every $u \in \Omega$ the unique solution $y_u = T(u)$ of (11) in S:

$$T :\, \Omega \mapsto C[t_0,\infty),$$
$$(Tu)(t) = -\int_t^{\infty} \varphi^{-1}\left(-\frac{k}{r(s)} - \frac{1}{r(s)} \int_{t_0}^{s} q(\sigma) f(u(\sigma))\, d\sigma\right) ds.$$

Let us prove that T has at least a fixed point in Ω; a fixed point of the operator T will be a solution of (1) in S. From Theorem 2.1 it suffices to show that conditions (i) and (ii) are satisfied.

To prove claim (i) it suffices to show that $T(u) \in \Omega$ for every $u \in \Omega$.
As $k > 0$, property a2) of the function φ implies that $T(u(t)) > 0$, for every $t \ge t_0$. As φ is an increasing function (and so φ^{-1}), we get

$$(Tu)(t) \le -\int_t^{\infty} \varphi^{-1}\left(-\frac{k}{r(s)} - \frac{1}{r(s)} \int_{t_0}^{\infty} q(\sigma) f(u(\sigma))\, d\sigma\right) ds$$

and the monotonicity property a3) of the function f implies

$$(Tu)(t) \leq$$
$$\leq -\int_t^\infty \varphi^{-1}\left(-\frac{k}{r(s)} - \frac{1}{r(s)}\int_{t_0}^\infty q(\sigma) f\left(-\int_{t_0}^\infty \varphi^{-1}\left(-\frac{2k}{r(\tau)}\right) d\tau\right) d\sigma\right) ds$$

$$= -\int_t^\infty \varphi^{-1}\left(-\frac{k}{r(s)} - \frac{c(t_0)}{r(s)}\right) ds \leq -\int_t^\infty \varphi^{-1}\left(-\frac{2k}{r(s)}\right) ds$$

having made use of assumption (6) on t_0.

Claim (ii) follows immediately from the definition of the sets Ω and S and of the operator T. □

Remark 2.1. In Theorem 2.2 we have showed that, for any $k > 0$ satisfying (9), it is possible to determine a time t_0 such that problem (1) admits a nonnegative proper solution defined in $[t_0, \infty)$ which is decreasing to zero when $t \to \infty$. The constant k is strictly related to the value of the derivative of the solution at time t_0, as the definition of the set S shows; the relation between k and $x'(t_0)$ is 1-1. When such constant k and consequently t_0 are fixed, the initial value $x(t_0)$ of the solution is uniquely determined by the expression

$$x(t_0) = -\int_{t_0}^\infty \varphi^{-1}\left(-\frac{k}{r(s)} - \frac{1}{r(s)}\int_{t_0}^s q(\sigma) f(x(\sigma))\, d\sigma\right) ds.$$

This relation shows that the bigger is k the bigger is $x(t_0)$; this corresponds to the intuitive physical idea that when the initial velocity of an object is big, a lot of space is needed in order to let it decelerate and stop after a finite length (indeed, after an infinite interval of time).

Remark 2.2. Assumptions (9) in Theorem 2.2 are the natural extension of classical ones. Indeed it is known (see for instance [3]) that for the semi-linear equation

$$(r(t)x'(t))' + q(t)f(x(t)) = 0, \quad t \geq a$$

the assumptions

$$\int_a^\infty \frac{1}{r(t)}\, dt < \infty, \quad \int_a^{+\infty} q(s) f\left(\int_a^s \frac{1}{r(\sigma)}\, d\sigma\right) ds < \infty$$

assure the existence of proper solutions asymptotically decreasing towards zero.

3. Existence of asymptotic decaying nonnegative proper solutions of Eqs. (2±)

We are now able to prove an analogous of Th. 2.2 for nonlinear differential equations in general form (2±). Denote with $B_0^\pm$ the subset of (bounded) proper solution of Eq. (2±) approaching zero as $t \to \infty$, and

$$G(t, t_0) = \int_{t_0}^t g(s)\, ds$$

where $t_0 \geq a$ is a real constant.

Throughout the present section we make the following assumption without further mentioning

a5) $$\sup_{t\in[a,\infty)} G(t,a) = \int_a^\infty g(s)\,ds < \infty$$

Theorem 3.1. *Let assumptions a1), a2), a3'), a4) and a5) be fulfilled and there exists a constant $k>0$ such that*

$$(12)\qquad \begin{aligned} I_1(k) &= -\int_a^\infty \varphi^{-1}\left(-\frac{2k}{r(t)}\right)dt < \infty, \\ I_3(k) &= \int_a^\infty F\left(t, -\int_t^\infty \varphi^{-1}\left(-\frac{2k}{r(s)}\right)ds\right)dt < \infty \end{aligned}$$

Then Eqs. (2+) and (2−) have at least one nonnegative solution in class B_0^+ and B_0^- respectively.

Proof. In order to get the existence of a solution of (2+) ((2−)) in B_0^+ (B_0^-) let us make the following positions

$$\begin{aligned} \overline{c}(\tau) &:= \int_\tau^\infty F\left(s, -\int_s^\infty \varphi^{-1}\left(-\frac{2k}{r(\sigma)}\right)d\sigma\right)ds \\ L(\tau) &:= \sup_{t\in[\tau,\infty)} G(t,\tau) = \int_\tau^\infty g(s)\,ds \end{aligned}$$

where $\tau \geq a$ is a real constant. They are well defined for every $\tau \geq a$ because of assumptions done and they are nonincreasing with respect to τ.

To prove the existence of at least one nonnegative solution of Eq. (2+) in B_0^+, choose a large t_0^+ such that

$$(13)\qquad \begin{aligned} L(t_0^+) &\leq k \\ \overline{c}(t_0^+) &\leq k \end{aligned}$$

and let Ω_+ and S_+ be the analogous of Ω and S defined in the proof of Th. 2.2, with t_0^+ instead of t_0. For every $u \in \Omega_+$ there exists a unique solution y_u^+ of the differential equation

$$(14)\qquad (r(t)\varphi(y'))' = -F(t,u(t)) + g(t)$$

such that $y_u^+ \in S_+$. Therefore we may define an operator T_+ which associates to every $u \in \Omega_+$ the unique solution $y_u^+ = T_+(u)$ of (14) in S_+:

$$\begin{aligned} &T_+ : \Omega_+ \mapsto C([t_0^+,\infty)) \\ &(T_+u)(t) = -\int_t^\infty \varphi^{-1}\left(-\frac{k - G(s,t_0^+)}{r(s)} - \frac{1}{r(s)}\int_{t_0^+}^s F(\sigma,u(\sigma))\,d\sigma\right)ds. \end{aligned}$$

Assumptions (13) assure that $T_+u \in \Omega_+$ for every $u \in \Omega_+$, and $\overline{T_+(\Omega_+)} \subset S_+$ follows immediately from the definition of the sets Ω_+ and S_+ and of the operator T_+.

In the same way, to prove the existence of at least one nonnegative solution of Eq. (2−) in B_0^-, choose a large t_0^- such that

$$L(t_0^-) + \overline{c}(t_0^-) \le k \tag{15}$$

and let Ω_- and S_- be the analogous of Ω and S defined in the proof of Th. 2.2, with t_0^- instead of t_0. For every $u \in \Omega_-$ there exists a unique solution y_u^- of the differential equation

$$(r(t)\varphi(y'))' = -F(t, u(t)) - g(t) \tag{16}$$

such that $y_u^- \in S_-$. Therefore we may define an operator T_- which associates to every $u \in \Omega_-$ the unique solution $y_u^- = T_-(u)$ of (16) in S_-:

$$T_- : \Omega_- \mapsto C([t_0^-, \infty))$$

$$(T_- u)(t) = -\int_t^\infty \varphi^{-1}\left(-\frac{k + G(s, t_0^-)}{r(s)} - \frac{1}{r(s)} \int_{t_0^-}^s F(\sigma, u(\sigma))\, d\sigma\right) ds.$$

Assumption (15) assures that $T_- u \in \Omega_-$ for every $u \in \Omega_-$, and $\overline{T_-(\Omega_-)} \subset S_-$ follows immediately from the definition of the sets Ω_- and S_- and of the operator T_-. □

Remark 3.1. If $g \equiv 0$, Th. 3.1 can be regarded as an extension of Th. 2.2 to the case in which t and x are not separable in the source term, see also [25], [9]. It is noteworthy to remark that conditions (13) and (15) both reduce to $\overline{c}(t_0) \le k$ in this case, which is exactly the analogous of condition (10) of Th. 2.2.

Remark 3.2. The forcing term g in (2+) and (2−) can be viewed as a given acceleration field, depending on the time, whose action is to make the speed increase in the first case, decrease in the second one. So it is natural from a physical point of view that in case of Eq. (2+) we need a condition assuring that solutions of (14) remains nonnegative (the first condition in (13)), in addition to condition assuring that solutions do not increase too much (the second condition in (13)), while in case of Eq. (2−) only condition (15) is needed in order to assure that solutions of (16) do not increase too much, exactly as in the case of absence of forcing term.

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