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ON THE STRUCTURE OF THE SOLUTION SET OF A FUNCTIONAL DIFFERENTIAL SYSTEM ON AN UNBOUNDED INTERVAL

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Abstract. It is proved that under some conditions the set of all solutions of an initial value problem for n-th order functional differential system on an unbounded interval is a compact R_δ.

The aim of this paper is to formulate a version of a theorem (see Proposition (1.1) below) proved originally by Vidossich in [8] (asserting that under certain conditions the set of all fixed points of an operator T defined on the Fréchet space X is a compact R_δ) for the case when T is considered only on a special subset of X (e.g., this is the case, when one can find an estimate $\|x\| \leq \gamma$ for fixed point x of T). This result (Theorem (1.2)) is then applied to an initial value problem for the n-th order functional differential system

$$x^{(n)}(t) = f(t, x_t, x'_t, \dots, x_t^{(n-1)}) , \quad t \in [b, \infty) .$$

For $n = 1$ this system was studied by Šeda and Kubáček in [7] and by Kubáček in [4], for $n = 2$ by Šeda and Belohorec in [5] and for n arbitrary by Šeda and Eliaš in [6] (with the growth condition

$$|f(t, x_t, \cdots, x_t^{(n-1)})| \leq \omega(t) h(\|x_t\|) \ ). \tag{1}$$

Our way of proof enables to avoid the construction of the sequence $\{T_n\}$ of approximating operators needed in [6] (as was shown by Vidossich, the existence of such a sequence can by guaranteed by properties (i), (ii) and (iii) of the operator T, see Proposition (1.1)) and, moreover, easy sufficient condition for the existence of an estimate for the norm of the solutions of this differential system is stated (see (21) for the growth condition of the form (1), and (17) for the growth condition of the form $|f(t, x_t, \cdots, x_t^{(n-1)})| \leq \omega(t) h(\|x_t^{(n-1)}\|)$).

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A more general approach to the study of the topological structure of solution sets of differential equations and inclusions on unbounded intervals and a detailed list of references can be found in [1], [2].

0. Definitions

(0.1) A non-empty subset F of a metric space X is said to be a *compact R_δ-set in the space* X if F is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.

(0.2) Recall that a function $g : [b,\infty) \times [0,\infty) \to [0,\infty)$ is said to *satisfy the Carathéodory conditions*, if $g(.,y)$ is measurable for each $y \in [0,\infty)$, $g(t,.)$ is continuous for almost all $t \in [b,\infty)$ and to every $(t_0,y_0) \in [b,\infty)\times[0,\infty)$ there exist numbers $\delta_1,\delta_2 > 0$ and an integrable function $\varrho : [t_0-\delta_1, t_0+\delta_1]\cap[0,\infty) \to \mathbf{R}$ such that $|g(t,y)| \le \varrho(t)$ for all $(t,y) \in \{(\tau,\upsilon) \in [b,\infty)\times[0,\infty); |\tau - t_0| \le \delta_1, |\upsilon - y_0| \le \delta_2\}$.

1. Theorems

Let K be a convex subset of a normed space $(Z,|.|)$; let $(Y,\|.\|)$ be a Banach space. Let X be the space of all continuous locally bounded maps $f : K \to Y$ (i.e. bounded on each bounded subset of K) equipped with the topology of locally uniform convergence. Let $t_0 \in K$, for $\varepsilon > 0$ denote by K_ε the set $\{t \in K; |t - t_0| \le \varepsilon\}$, for $x \in X$ denote by $x|K_\varepsilon$ the restriction of the map x to the set K_ε.

(1.1) Proposition (see [8] for a bounded K, [4] for an unbounded K). *Let $T : X \to X$ be a continuous map, $S := I - T$ (I denotes the identity on X). Suppose*

(i) $(\exists t_0 \in K)(\exists y_0 \in Y)(\forall x \in X)(Tx(t_0) = y_0)$;

(ii) *$T(X)$ is a set of locally equiuniformly continuous maps, i.e.*

$$(\forall \varepsilon > 0)(\forall \eta > 0)(\exists \delta > 0)(\forall x \in X)$$

$$(\forall t_1, t_2 \in K_\eta)(|t_1 - t_2| < \delta \Longrightarrow \|Tx(t_1) - Tx(t_2)\| < \varepsilon)\ ;$$

(iii) $(\forall \varepsilon > 0)(\forall x, y \in X)(x|K_\varepsilon = y|K_\varepsilon \Longrightarrow (Tx)|K_\varepsilon = (Ty)|K_\varepsilon)$.

Then it holds: if the map T satisfies the Palais-Smale condition (i.e., each sequence $\{x_n\} \subset X$ such that $\lim(x_n - Tx_n) = 0$, contains a convergent subsequence), then the set F of all its fixed points is a compact R_δ.

Remark. Recall that every compact map satisfies the Palais-Smale condition.

(1.2) Theorem. *Let $M := \{x \in X; \|x(t) - r(t)\| \le p(t), t \in K\}$, where $p : K \to \boldsymbol{R}$ is a non-negative locally bounded continuous function and $r \in X$. If a continuous map $T : M \to M$ satisfies*

(i') $(\exists t_0 \in K)(\exists y_0 \in Y, \|y_0 - r(t_0)\| \le p(t_0))(\forall x \in M)((Tx)(t_0) = y_0)$;

(ii') *$T(M)$ is a set of locally equiuniformly continuous maps;*

(iii') $(\forall \varepsilon > 0)(\forall x, y \in M)(x|K_\varepsilon = y|K_\varepsilon \Longrightarrow (Tx)|K_\varepsilon = (Ty)|K_\varepsilon)$,

and the Palais-Smale condition, then the set F of all its fixed points is a compact R_δ.

Proof. Let us assign to each $x \in X$ the map $Px \in M$ given by

$$Px(t) = P_{r(t),p(t)}(x(t)) \ ,$$

where $P_{r(t),p(t)}$ is the projection from Lemma (3.1). The map Px is continuous according to the inequality

$$\|Px(t) - Px(s)\| \le \|P_{r(t),p(t)}x(t) - P_{r(s),p(t)}x(t)\|$$

$$+\|P_{r(s),p(t)}x(t) - P_{r(s),p(s)}x(t)\| + \| - P_{r(s),p(s)}x(t) - P_{r(s),p(s)}x(s)\|$$

$$\le 3\|r(t) - r(s)\| + |p(t) - p(s)| + 2\|x(t) - x(s)\|$$

(see (23), (24), (22) from Lemma (3.1)); its local boundedness follows from the inequality

$$\|Px(t)\| \le \|r(t)\| + p(t)$$

and from the local boundedness of r and p.

The map $P : X \to M$ has the property

$$x|K_\varepsilon = y|K_\varepsilon \Longrightarrow (Px)|K_\varepsilon = (Py)|K_\varepsilon \tag{2}$$

and its continuity is a consequence of the inequality $\|Px(t) - Py(t)\| \le 2\|x(t) - y(t)\|$ (see (22) in Lemma (3.1)). For

$$\tilde{T} := T \circ P$$

we have $\tilde{T}(X) = T(M)$, thus, by (i') and (ii'), $\tilde{T}$ satisfies conditions (i) and (ii) from Theorem (1.1). By (2) and (iii'), for $\tilde{T}$ the condition (iii) is fulfilled, too.

It remains to verify that $\tilde{T}$ satisfies the Palais-Smale condition. So, let

$$x_n - \tilde{T}x_n \to 0 \ . \tag{3}$$

Then (as $\{\tilde{T}x_n\}_{n=1}^\infty \subset M$) it holds $\varrho(x_n, M) \to 0$, where ϱ is the standard metric on the Fréchet space X with the topology of locally uniform convergence. As $\varrho(x_n, M) = \varrho(x_n, Px_n)$ (see (25)), we have $\varrho(x_n, Px_n) \to 0$, i.e.,

$$x_n - Px_n \to 0 \ . \tag{4}$$

(3) and (4) imply $Px_n - \tilde{T}x_n = Px_n - T(Px_n) \to 0$. T satisfies the Palais-Smale condition, therefore $\{Px_n\}_{n=1}^\infty$ has a convergent subsequence $\{Px_{n(k)}\}_{k=1}^\infty$. By (4), the sequence $\{x_{n(k)}\}_{k=1}^\infty$ is convergent, too.

Thus, by Theorem (1.1) the set $\tilde{F}$ of all fixed points of $\tilde{T}$ is a compact R_δ. As $\tilde{F} \subset \tilde{T}(X) \subset T(M) \subset M$ and $Px = x$ for $x \in M$, we have $F = \tilde{F}$, which completes the proof.

Remark. Specially, for a closed convex set $K \subset \mathbf{R}^\nu$ ($\nu \in \mathbf{N}$) and a compact map $T : M \to M$ the condition (ii') can be omitted (in this case, the sets K_η from condition (ii) in Theorem (1.1) are compact, thus, due to the Arzelà-Ascoli theorem, the locally equiuniform continuity of $T(M)$ is a consequence of the relative compactness of the set $T(M)$) and the Palais-Smale condition is fulfilled (see Remark (1.1)).

(1.3) For the case when Y is a Hilbert space the following theorem is a generalization of Theorem (1.2).

Theorem. *Let Y be a real Hilbert space, $F : K \to cf\ Y$ be a continuous locally bounded map (where $cf\ Y$ is the space of all nonvoid convex closed subsets of Y equipped with the Hausdorff metric), let $M := \{x \in X\ ;\ x(t) \in F(t), t \in K\}$. If a continuous map $T : M \to M$ satisfies*

(i") $(\exists t_0 \in K)(\exists y_0 \in F(t_0))(\forall x \in X)(Tx(t_0) = y_0)$;
(ii') *$T(M)$ is a set of locally equiuniformly continuous maps;*
(iii") $(\forall \varepsilon > 0)(\forall x, y \in M)(x|K_\varepsilon = y|K_\varepsilon \Longrightarrow (Tx)|K_\varepsilon = (Ty)|K_\varepsilon)$

and the Palais-Smale condition, then the set of all its fixed points is a compact R_δ.

Proof. The idea of the proof is the same as in Theorem (1.2), now the set F of all fixed points of T is identical with the set $\tilde{F}$ of all fixed points of the map $\tilde{T}$, where $\tilde{T}(x) = T(P_x)$ and P_x is the map from Lemma (3.4). The proof of the fact that $\tilde{T}$ satisfies all assumptions of Theorem (1.1) is very similar to the proof of the preceding Theorem (1.2) (the continuity of $\tilde{T}$ is now a consequence of (27) in Proposition (3.2) and (26) instead of (25) is to be used to prove that $\tilde{T}$ satisfies the Palais-Smale condition).

2. An application

Let $h > 0, b \in \mathbf{R}, n, \nu \in \mathbf{N}$ and let $|.|$ be a norm in $\mathbf{R}^\nu$. For $k \in \mathbf{N} \cup \{0\}$ denote by H_k the space $C^k([b-h, b], \mathbf{R}^\nu)$ with the supremum-norm $\|x\| + \|x'\| + \|x''\| + \cdots + \|x^{(k)}\|$, where $\|y\| := \max\{|y(s)|; s \in [b-h, b]\}$; let

$$X = C^{n-1}([b-h, \infty), \mathbf{R}^\nu)$$

be equipped with the topology of locally uniform convergence of the function and its derivatives up to the order $n-1$ on $[b-h, \infty)$. For $x \in X$ and $t \in [b, \infty)$ denote by x_t the function from H_{n-1} given by $x_t(s) = x(t+s-b), s \in [b-h, b]$.

Let $f : [b, \infty) \times \underbrace{H_0 \times H_0 \times \cdots \times H_0}_{n\text{-times}} \to \mathbf{R}^\nu$ satisfy the following conditions:

(iv) $f(\cdot, \chi_{n-1}, \chi_{n-2}, \ldots, \chi_0)$ is measurable on $[b, \infty)$ for each $\chi_0, \ldots, \chi_{n-2}, \chi_{n-1} \in H_0$;
(v) $f(t, \cdot, \ldots, \cdot)$ is continuous on $H_0 \times H_0 \times \cdots \times H_0$ for almost all $t \in [b, \infty)$.

Let $\psi \in H_{n-1}$, we shall consider the following initial value problem

$$\left\{ \begin{array}{rcl} x^{(n)}(t) & = & f\left(t, x_t, x'_t, x''_t, \ldots, x_t^{(n-1)}\right), \quad t \in [b, \infty), \\ x_b & = & \psi, x'_b = \psi', \ldots, x_b^{(n-1)} = \psi^{(n-1)}\ ; \end{array} \right. \tag{5}$$

under a solution of (5) we will understand any function $x \in X$ satisfying

$$x^{(n-1)}(t) = \psi^{(n-1)}(b) + \int_b^t f\left(s, x_s, \dots, x_s^{(n-1)}\right) ds\,, \quad t \in [b, \infty) \tag{6}$$

and

$$x_b^{(k)} = \psi^{(k)} \qquad k = 0, 1, \dots, n-1\,. \tag{7}$$

(In our considerations, the integrability of f will be guaranteed by the assumption (8) in the following theorem and by Lemma (3.6).)

(2.1) Theorem. *Let the function $g : [b, \infty) \times [0, \infty) \to [0, \infty)$ be non-decreasing in the second variable and satisfy the Carathéodory conditions, let*

$$|f(t, x_t, x_t', \dots, x_t^{(n-1)})| \le g(t, \|x_t^{(n-1)}\|), \quad t \in [b, \infty), x_t \in H_{n-1} \tag{8}$$

and suppose that for some $\varepsilon > 0$ there exists a solution $\gamma : [b, \infty) \to \boldsymbol{R}$ of the differential equation

$$\gamma'(t) = g(t, \gamma(t) + \|\psi^{(n-1)}\|), \quad \gamma(b) = \varepsilon\,. \tag{9}$$

Then the set of all solutions of the equation (5) is a compact R_δ in X.

Proof. *Step 1.* Let $x \in X$ be a solution of (5), set

$$\phi(t) := \max\{|x^{(n-1)}(s) - \psi^{(n-1)}(b)|, s \in [b, t]\}\,.$$

Then — using (6), (8) and the monotonicity and the non-negativity of the function $g(s, .)$ — we have for any $u \in [b, t]$

$$\begin{aligned} |x^{(n-1)}(u) - \psi^{(n-1)}(b)| &\le \int_b^u g(s, \|x_s^{(n-1)}\|) ds \\ &\le \int_b^u g(s, \phi(s) + \|\psi^{(n-1)}\|) ds \\ &\le \int_b^t g(s, \phi(s) + \|\psi^{(n-1)}\|) ds\,, \end{aligned}$$

therefore

$$\phi(t) \le \int_b^t g(s, \phi(s) + \|\psi^{(n-1)}\|) ds$$

and by Lemma (3.5) we have $\phi(t) \le \gamma(t)$, where γ is the solution of (9). Hence, for any solution x of (5) we have

$$|x^{(n-1)}(t) - \psi^{(n-1)}(b)| \le \gamma(t)\,, \quad t \ge b\,. \tag{10}$$

Step 2. Now, for $z \in C([b-h, \infty), \mathbf{R}^\nu)$ and $r \in \mathbf{R}^\nu$ denote

$$(P_r z)(t) := r + \int_b^t z(s) ds\,, \quad t \in [b-h, \infty)$$

(i.e., $P_r z$ is that function differentiable on $[b-h, \infty)$, for which it holds $(P_r z)' = z$ and $P_r z(b) = r$).

For $y \in C([b,\infty))$ define $\Psi_{n-1}y \in C([b-h,\infty))$ in the following way

$$\Psi_{n-1}y(t) = \begin{cases} \psi^{(n-1)}(t) - \psi^{(n-1)}(b) + y(b), & \text{if} \quad t \in [b-h,b] \\ y(t) & , \text{ if} \quad t \in [b,\infty) \end{cases}$$

and set

$$\begin{aligned} \Psi_{n-2}y &:= P_{\psi^{(n-2)}(b)}(\Psi_{n-1}y) \ , \\ \Psi_{n-3}y &:= P_{\psi^{(n-3)}(b)}(\Psi_{n-2}y) \ , \\ &\vdots \\ \Psi_0 y &:= P_{\psi(b)}(\Psi_1 y) \ , \end{aligned}$$

i.e. $\Psi_0 y = x$ means $x \in C^{n-1}([b-h,\infty),\mathbf{R}^\nu)$,

$$x^{(n-1)}(t) = \begin{cases} \psi^{(n-1)}(t) - \psi^{(n-1)}(b) + y(b), & \text{if} \quad t \in [b-h,b] \\ y(t) & , \text{ if} \quad t \in [b,\infty) \end{cases} \ ,$$

$$x^{(k)}(b) = \psi^{(k)}(b) \ , \quad k = 0,1,\cdots,n-2$$

and

$$x^{(k)} = \Psi_k y, \quad k = 1,\ldots,n-1 \ .$$

Specially, if $y \in C([b,\infty))$ and

$$y(b) = \psi^{(n-1)}(b) \ , \tag{11}$$

then $x := \Psi_0 y$ satisfies (7).

Further, $y \in C([b,\infty),\mathbf{R}^\nu)$ is a solution of

$$y(t) = \psi^{(n-1)}(b) + \int_b^t f(s,(\Psi_0 y)_s,(\Psi_1 y)_s,\ldots,(\Psi_{n-1}y)_s)ds \tag{12}$$

on $[b,\infty)$ if and only if $\Psi_0 y$ is a solution of (5) (i.e., if and only if $\Psi_0 y$ satisfies (6)); to prove this equivalence it suffices to realize that every solution of (12) fulfils (11).

Let

$$M := \{y \in C([b,\infty),\mathbf{R}^\nu); |y(t) - \psi^{(n-1)}(b)| \le \gamma(t), \quad t \ge b\} \ .$$

By (10) every solution of (5) belongs to $\Psi_0(M)$ and the map Ψ_0 is evidently a homeomorphismus of M onto $\Psi_0(M)$. So it suffices to show that the set F of all solutions $y \in M$ of (12) is a compact R_δ (i.e. F is homeomorphic to the intersection of a decreasing sequence $\{A_n\}$ of compact absolute retracts), as the set $\Psi_0(F)$ — which is the solution set of the equation (5) — is then also a homeomorphic image of the intersection of $\{A_n\}$ and therefore it is a compact R_δ, too.

Define the map $T : M \to C([b,\infty),\mathbf{R}^\nu)$ by

$$Ty(t) := \psi^{(n-1)}(b) + \int_b^t f(s,(\Psi_0 y)_s,\ldots,(\Psi_{n-1}y)_s)ds$$

(the continuity of Ty is a consequence of Lemma (3.6), the inequality (8) and the local integrability of $g(t,\|(\Psi_{n-1}y)_t\|)$).

We will show that T satisfies all assumptions of Theorem (1.2). Let us begin with the inclusion $T(M) \subset M$: for $y \in M$ we have $|y(t) - \psi^{(n-1)}(b)| \leq \gamma(t)$, therefore (according to (8))

$$\begin{aligned}
|Ty(t) - \psi^{(n-1)}(b)| &\leq \int_b^t g\big(s, \|(\Psi_{n-1}y)_s\|\big)\, ds \\
&\leq \int_b^t g(s, \gamma(s) + \|\psi^{(n-1)}\|) ds = \gamma(t) - \varepsilon < \gamma(t)\ .
\end{aligned}$$

The relative compactness of the set $T(M)$ (which is sufficient for T to satisfy the Palais-Smale condition) is equivalent to the uniform boundedness and the equiuniform continuity of the set

$$T(M)|[b, b+m] := \{(Ty)|[b, b+m]; y \in M\}$$

for each $m \in \mathbf{N}$ and this property is a consequence of the inequalities

$$\begin{aligned}
|Ty(t)| &\leq |\psi^{(n-1)}(b)| + \int_b^t g(s, \|(\Psi_{n-1}y)_s\|) \\
&\leq |\psi^{(n-1)}(b)| + \int_b^t g(s, \gamma(s) + \|\psi^{(n-1)}\|) ds
\end{aligned}$$

and

$$|Ty(t_1) - Ty(t_2)| \leq \left| \int_{t_1}^{t_2} g(s, \|(\Psi_{n-1}y)_s\|) ds \right| \leq \left| \int_{t_1}^{t_2} g(s, \gamma(s) + \|\psi^{(n-1)}\|) ds \right|\ .$$

It remains to prove the continuity of T. Let $y_k \in M, y \in M$, $y_k \to y$ in $C([b,\infty), \mathbf{R}^\nu)$. Then, for each $s \in [b, \infty)$, $\{(\Psi_i y_k)_s\}_{k=1}^\infty$ converges to $(\Psi_i y)_s$ in H_0 $(i = 0, 1, \cdots, n-1)$, therefore (by assumption (v))

$$f(s, (\Psi_0 y_k)_s, \cdots, (\Psi_{n-1}y_k)_s) \to f(s, (\Psi_0 y)_s, \cdots, (\Psi_{n-1}y)_s) \qquad (\text{in } \mathbf{R}^\nu)$$

for each $s \in [b, \infty)$. As

$$\Big| f(s, (\Psi_0 x)_s, \cdots, (\Psi_{n-1}x)_s) \Big| \leq g(s, \gamma(s) + \|\psi^{(n-1)}\|)\ ,$$

for each $x \in M$, we have — due to the Lebesgue Dominated Convergence Theorem

$$(Ty_k)(t) \to (Ty)(t) \quad (\text{in} \quad \mathbf{R}^\nu) \quad \text{for each} \quad t \in [b, \infty)\ . \tag{13}$$

As the set $S := \{Ty_k\}_{k=1}^\infty$ is relatively compact (as a subset of $(T(M))$, we get that each sequence in S has a convergent subsequence, which has (according to (13)) converge to Ty. Thus, $\{Ty_k\}_{k=1}^\infty$ converges to Ty in $C([b,\infty), \mathbf{R}^\nu)$.

Due to Theorem (1.2) (with $r(t) :\equiv \psi^{(n-1)}(b), p(t) := \gamma(t), t_0 := b, y_0 := \psi^{(n-1)}(b)$) the set F of all fixed points of T is an R_δ, which completes the proof.

(2.2) Now some sufficient conditions for the solvability of the equation (9) will be stated.

Lemma (see [7]). *Let* $\omega : [b,\infty) \to [0,\infty)$ *be a locally integrable function and* $h : [0,\infty) \to (0,\infty)$ *be continuous and non-decreasing. Then a solution of the differential equation*

$$\alpha'(t) = \omega(t)h(\alpha(t)) \ , \quad \alpha(b) = \eta > 0 \tag{14}$$

exists on $[b,\infty)$ *iff*

$$\int_b^t \omega(s)ds < \int_\eta^\infty \frac{ds}{h(s)} \quad \text{for each} \quad t \in [b,\infty) \ . \tag{15}$$

Specially, a solution of

$$\gamma'(t) = \omega(t)h(\gamma(t) + \|\psi^{n-1}\|) \ , \qquad \gamma(b) = \varepsilon > 0 \tag{16}$$

(i.e., a solution of (9) for the case $g(t,y) = \omega(t)h(y)$*) exists on* $[b,\infty)$ *iff*

$$\int_b^t \omega(s)ds < \int_{\varepsilon + \|\psi^{(n-1)}\|}^\infty \frac{ds}{h(s)} \quad \text{for each} \quad t \in [b,\infty) \ . \tag{17}$$

Proof. (14) is an equation with separated variables, thus the solution can be found in the form

$$\alpha(t) = H^{-1}\left(H(\eta) + \int_b^t \omega(s)ds\right) \ ,$$

where $H(t) := \int_0^t \frac{ds}{h(s)}$; the aim of (15) is to ensure that the number $H(\eta) + \int_b^t \omega(s)ds$ belongs to the interval $[0, \int_0^\infty ds/h(s))$, where the function H^{-1} is defined, for each $t \in [b,\infty)$.

(16) is a special case of (14) with $\alpha(t) = \gamma(t) + \|\psi^{(n-1)}\|$ and $\eta = \varepsilon + \|\psi^{(n-1)}\|$, thus, (17) is a consequence of (15).

Remark. Evidently, for $\int_0^\infty \frac{ds}{h(s)} = \infty$ the condition (17) is satisfied.

Corollary. *Let* $\omega : [b,\infty) \to [0,\infty)$ *be locally integrable,* $h : [0,\infty) \to (0,\infty)$ *be continuous and non-decreasing, let* $A > 0, R \geq 0$ *be constants. If*

$$\int_0^\infty \frac{ds}{h(s)} = \infty \ ,$$

then the equation

$$\gamma'(t) = \omega(t)h(A\gamma(t) + R) \ , \quad \gamma(b) = \varepsilon > 0 \ ,$$

has a solution on $[b,\infty)$.

Proof. Setting $\alpha(t) = A\gamma(t) + R$, we get

$$\alpha'(t) = A\omega(t)h(\alpha(t)) \ , \quad \alpha(b) = A\varepsilon + R \ ,$$

due to the preceding Lemma this equation has a solution on $[b,\infty)$.

(2.3) The following generalization of Corollary (2.2) will be used in the proof of Theorem (2.4).

Lemma. *Let $\omega : [b,\infty) \to [0,\infty)$ be a locally integrable function, let $a : [b,\infty) \to [0,\infty)$ and $r : [b,\infty) \to [0,\infty)$ be continuous and $h : [0,\infty) \to (0,\infty)$ be a continuous and non-decreasing function. If*

$$\int_0^\infty \frac{ds}{h(s)} = \infty \,, \tag{18}$$

then the equation

$$\gamma'(t) = \omega(t)h(a(t)\gamma(t) + r(t)) \,, \quad \gamma(b) = \varepsilon > 0 \tag{19}$$

has a solution on $[b,\infty)$.

Proof. First we will show that (19) has a solution on $[b,T]$ for any given $T > b$. Set $A := \max_{t\in[b,T]} a(t)$, $R := \max_{t\in[b,T]} r(t)$. According to Corollary (2.2), there exists a solution $\alpha : [b,T] \to \mathbf{R}$ of the differential equation

$$y' = \omega(t)h(Ay(t) + R) \,, \quad y(b) = \varepsilon \,.$$

Denote $M := \{x \in C([b,T],\mathbf{R}); 0 \le x(t) \le \alpha(t)\}$ and

$$Bx(t) := \varepsilon + \int_b^t \omega(s)h(a(s)x(s) + r(s))ds \,, \quad t \in [b,T], x \in M \,.$$

The inequalities $0 \le x(s) \le \alpha(s), a(s) \le A, r(s) \le R$ and the monotonicity of h imply

$$h(a(s)x(s) + r(s)) \le h(A\alpha(s) + R) \,,$$

therefore

$$\begin{aligned} Bx(t) &= \varepsilon + \int_b^t \omega(s)h(a(s)x(s) + r(s))ds \\ &\le \varepsilon + \int_b^t \omega(s)h(A\alpha(s) + R)ds = \alpha(t) \,, \end{aligned}$$

this together with the inequality $Bx(t) \ge 0$ (which is a consequence of the non-negativity of ω and h) implies that $B(M) \subset M$. As M is a closed and bounded set and B is a compact mapping (the proof of this fact is based on the same ideas as in the case of the operator T from Theorem (2.1)), there exists a fixed point of B, i.e. (19) has a solution on $[b,T]$.

On the set S of all solutions of (19) let us define the partial ordering

$$x \prec y \quad \Leftrightarrow \quad D(x) \subset D(y) \wedge y|D(x) = x \,.$$

Evidently, each linearly ordered set in S has an upper bound, therefore — according to the Kuratowski-Zorn lemma — there exists a maximal element ξ in S. We will show that $D(\xi) = [b,\infty)$. Let P be the right end-point of $D(\xi)$. Suppose $P \in \mathbf{R}$ and let $\beta : [b,P] \to \mathbf{R}$ be a solution of the equation

$$\beta'(t) = \omega(t)h(a(t)\beta(t) + r(t)) \,, \qquad \beta(b) = 2\varepsilon$$

(such a β exists due to the preceding part of this proof); then by Lemma (3.5) we have $\xi(t) \leq \beta(t)$ for $t \in [b, P)$ and due to this inequality

$$|\xi(t_1) - \xi(t_2)| \leq \left| \int_{t_1}^{t_2} \omega(s) h(a(t)\beta(t) + r(t)) ds \right| \,, \quad t_1, t_2 \in [b, P) \,.$$

Consequently, ξ satisfies the Bolzano-Cauchy condition in P, thus $\lim_{t \to P} \xi(t)$ is finite and $P \in D(\xi)$. By the Peano Existence Theorem, the equation

$$u'(t) = \omega(t) h(a(t)u(t) + r(t)) \,, \quad u(P) = \xi(P)$$

has a solution on $[P, P + \eta)$ for some $\eta > 0$, which implies the existence of a solution ξ_1 of (19) on $[b, P + \eta)$ such that $\xi_1 | [b, P] = \xi$, and this contradicts the maximality of ξ.

(2.4) Now we want to consider the case when the estimation (8) is replaced by

$$\left| f\left(t, x_t, x_t', \ldots, x_t^{(n-1)}\right) \right| \leq \omega(t) h(\|x_t\|) \,, \quad t \in [b, \infty), x_t \in H_{n-1} \tag{20}$$

(this condition was used by Šeda and Eliaš in [6]).

Theorem. *Let $\omega : [b, \infty) \to [0, \infty)$ be locally integrable, $h : [0, \infty) \to (0, \infty)$ be continuous and non-decreasing, let f satisfy (iv), (v) (from the beginning of paragraph 2)* and (20). If

$$\int_0^\infty \frac{ds}{h(s)} = \infty \,, \tag{21}$$

then the solution set of the equation (5) is a compact R_δ.

Proof. Let us consider the function

$$\phi(t) := \max\left\{ |x^{(n-1)}(s) - \psi^{(n-1)}(b)|; s \in [b, t] \right\}$$

from the proof of Theorem (2.1), where $x \in X$ is a solution of (5). Then, by (20),

$$|x^{(n-1)}(u) - \psi^{(n-1)}(b)| \leq \int_b^u \omega(s) h(\|x_s\|) ds \,.$$

As

$$\begin{aligned} x(s) &= \sum_{k=0}^{n-2} \frac{x^{(k)}(b)(s-b)^k}{k!} + \frac{x^{(n-1)}(b + \theta(s-b))(s-b)^{n-1}}{(n-1)!} \\ &= \sum_{k=0}^{n-2} \frac{\psi^{(k)}(b)(s-b)^k}{k!} + \frac{x^{(n-1)}(b + \theta(s-b))(s-b)^{n-1}}{(n-1)!} \,, \end{aligned}$$

we get

$$\begin{aligned} &\|x_s\| \leq \max\{|x(t)|; t \in [b, s]\} + \|\psi\| \\ &\leq \sum_{k=0}^{n-2} \frac{|\psi^{(k)}(b)|(s-b)^k}{k!} + \frac{(\phi(s) + |\psi^{(n-1)}(b)|)(s-b)^{n-1}}{(n-1)!} + \|\psi\| \,, \end{aligned}$$

and so, ϕ satisfies the inequality

$$\phi(t) \le \int_b^t \omega(s)h(a(s)\phi(s) + r(s))ds \ ,$$

where

$$a(s) = \frac{(s-b)^{n-1}}{(n-1)!} \ , \quad r(s) = ||\psi|| + \sum_{k=0}^{n-1} \frac{|\psi^{(k)}(b)|(s-b)^k}{k!} \ .$$

Then, by Lemma (3.5),

$$\phi(t) \le \gamma(t) \ , \quad t \in [b, \infty)$$

where $\gamma : [b, \infty) \to \mathbf{R}$ is a solution of

$$\gamma'(t) = \omega(t)h(a(t)\gamma(t) + r(t)) \ , \quad \gamma(b) = \varepsilon > 0 \ ,$$

and such a γ exists on $[b, \infty)$ according to Lemma (2.3). The rest of the proof is then identical with the Step 2 of the proof of Theorem (2.1).

3. Auxiliary Lemmas

(3.1) Lemma. *Let $(Y, ||.||)$ be a Banach space, $a \in Y$, $\rho \ge 0$. For $x \in Y$ denote*

$$P_{a,\rho}x := \begin{cases} x & \mathit{for} \quad ||x-a|| \le \rho \\ a + \frac{x-a}{||x-a||}\rho & \mathit{for} \quad ||x-a|| > \rho \end{cases} \ .$$

Then

$$||P_{a,\rho}x - P_{a,\rho}y|| \le 2||x-y|| \ , \tag{22}$$

$$||P_{a,\rho}x - P_{b,\rho}x|| \le 3||a-b|| \ , \tag{23}$$

$$||P_{a,\rho_1}x - P_{a,\rho_2}x\} \le |\rho_1 - \rho_2| \tag{24}$$

and

$$||x - P_{a,\rho}x|| = \min\{||x-y|| \ ; \ y \in Y \wedge ||y-a|| \le \rho\} \ . \tag{25}$$

Proof. Let us begin with (22). For $||x-a|| \le \rho, ||y-a|| \le \rho$ is the inequality obvious. For $||x-a|| \ge \rho, ||y-a|| \ge \rho$ we have

$$||P_{a,\rho}x - P_{a,\rho}y|| = \left\| a + \frac{x-a}{||x-a||}\rho - a - \frac{y-a}{||y-a||}\rho \right\|$$

$$\le \rho \left\| \frac{x-a}{||x-a||} - \frac{y-a}{||x-a||} \right\| + \rho \left\| \frac{y-a}{||x-a||} - \frac{y-a}{||y-a||} \right\|$$

$$= \frac{\rho}{||x-a||}||x-y|| + \frac{\rho}{||x-a||}\Big| ||y-a|| - ||x-a|| \Big| \le 2||x-y|| \ ,$$

the last inequality beeing a consequence of the inequalities $\frac{\rho}{||x-a||} < 1$ and $\big| ||y-a|| - ||x-a|| \big| \le ||x-y||$.

For $0 < ||x-a|| \leq \rho < ||y-a||$ we get (using the inequalities $\frac{||x-a||}{||y-a||} \leq \frac{\rho}{||y-a||} < 1$ and $||y-a|| - \rho \leq ||y-a|| - ||x-a|| \leq ||x-y||$)

$$||P_{a,\rho}x - P_{a,\rho}y|| = \left\| x - a - \frac{y-a}{||y-a||}\rho \right\|$$

$$\leq \left\| x - a - \frac{x-a}{||y-a||}\rho \right\| + \left\| \frac{x-a}{||y-a||}\rho - \frac{y-a}{||y-a||}\rho \right\|$$

$$= \frac{||x-a||}{||y-a||}(||y-a|| - \rho) + \frac{\rho}{||y-a||}||x-y|| \leq ||y-a|| - \rho + ||x-y|| \leq$$

$$\leq ||x-y|| + ||x-y|| = 2||x-y|| \ ,$$

and, finally, for $||x-a|| = 0 \leq \rho < ||y-a||$ it holds

$$||P_{a,\rho}x - P_{a,\rho}y|| = ||P_{a,\rho}y - a|| = \rho < ||y-a|| = ||y-x|| \ .$$

As to (23), we have to consider three cases: $||x-a|| \leq \rho \wedge ||x-b|| \leq \rho$; $||x-a|| \leq \rho < ||x-b||$ and $||x-a|| > \rho \wedge ||x-b|| > \rho$; the first one of them is trivial, in the second one we get

$$||P_{a,\rho}x - P_{b,\rho}x|| = ||x-b|| - \rho \leq ||x-b|| - ||x-a|| \leq ||a-b||$$

and in the third one it holds

$$||P_{a,\rho}x - P_{b,\rho}x|| \leq ||a-b|| + \rho \left\| \frac{x-a}{||x-a||} - \frac{x-b}{||x-a||} \right\| + \rho \left\| \frac{x-b}{||x-a||} - \frac{x-b}{||x-b||} \right\|$$

$$= ||a-b|| + \frac{\rho}{||x-a||}||a-b|| + \frac{\rho}{||x-a||}\Big| ||x-a|| - ||x-b|| \Big| \leq 3||a-b|| \ .$$

The proof of (24) (where the cases $||x-a|| \leq \rho_1 \leq \rho_2$, $\rho_1 < ||x-a|| \leq \rho_2$, $\rho_1 \leq \rho_2 \leq ||x-a||$ are to be considered) is analogous.

(25) is obvious for $||x-a|| \leq \rho$; for $||x-a|| > \rho$ it is a consequence of the inequality

$$||x - P_{a,\rho}x|| = ||x-a|| - \rho \leq ||x-a|| - ||y-a|| \leq ||x-y|| \ .$$

(3.2) Proposition (see [3, p. 23, Corollary 1]). *Let C be a nonvoid closed convex subset of a real Hilbert space $(Y, (.,.))$. Then for each $x \in Y$ there exists a unique $\pi_C(x) \in C$ such that*

$$||\pi_C(x) - x|| = \min_{c \in C} ||x - c|| \ . \tag{26}$$

Moreover, it holds

$$||\pi_C(x) - \pi_C(y)|| \leq ||x-y|| \quad \textit{for all} \quad x, y \in Y \ . \tag{27}$$

(3.3) Proposition (see [3, p.70, Theorem 1]). *Let (X, d) be a metric and $(Y, (.))$ a real Hilbert space. Let $F : X \to cf\ Y$ (where $cf\ Y$ denotes the space of all nonvoid closed convex subsets of Y equipped with the Hausdorff metric) be a continuous map. Then the map $P : X \to Y$ defined by $P(x) = \pi_{F(x)}(0)$ is continuous.*

(3.4) Lemma. *Let (X,d) be a metric and $(Y,(.))$ a real Hilbert space. Let $F : X \to cf\ Y$ and $f : X \to Y$ be continuous. Then the map $P_f : X \to Y$ defined by $P_f(x) := \pi_{F(x)}(f(x))$ is continuous.*

Proof. The map $G : X \to cf\ Y$ defined by $G(x) := F(x) - f(x)$ is continuous, by Proposition (3.3) this implies the continuity of the map $x \mapsto \pi_{G(x)}(0)$, which is identical with our map $P_f(x)$.

(3.5) Lemma. *Let $g : [b,\infty) \times [0,\infty) \to [0,\infty)$ be nondecreasing in the second variable and satisfy the Carathéodory conditions. Let $y \in C([b,B),\boldsymbol{R})$ (where $B \in (b,\infty) \cup \{\infty\}$) satisfy the inequality*

$$y(t) \le \eta + \int_b^t g(s,y(s))ds\ , \qquad t \in [b,B)$$

let $x \in C([b,B),\boldsymbol{R})$ be the solution of the equation

$$x(t) = \varepsilon + \int_b^t g(s,x(s))ds\ , \qquad t \in [b,B)$$

for $\varepsilon > \eta$. Then $y(t) \le x(t)$ on $[b,B)$.

Proof. Evidently, $y(b) \le \eta < \varepsilon = x(b)$. Suppose $x(t_0) < y(t_0)$ for some $t_0 \in (b,B)$ and denote $t_1 := \inf\{t \in (b,B); x(t) < y(t)\}$. Then we have $t_1 > b$, $x(t) > y(t)$ for $t \in [b,t_1)$ and

$$x(t_1) = y(t_1)\ . \tag{28}$$

But,

$$\begin{aligned} y(t_1) &\le \eta + \int_b^{t_1} g(s,y(s))ds \le \eta + \int_b^{t_1} g(s,x(s))ds \\ &< \varepsilon + \int_b^{t_1} g(s,x(s))ds = x(t_1)\ , \end{aligned}$$

which contradicts (28).

(3.6) Lemma. *Let $f : [b,\infty) \times H_0 \times H_0 \times \cdots \times H_0 \to \boldsymbol{R}^\nu$ satisfy the conditions (iv) and (v) from the beginning of paragraph 2, let $x \in C^{n-1}([b-h,T])$. Then the function $g(t) = f(t, x_t, x'_t, \ldots, x_t^{(n-1)})$ is measurable on $[b,T]$.*

Proof. As the functions $x^{(i)}$, $i = 0,1,\ldots,n-1$, are uniformly continuous on $[b-h,T]$, we have

$$t \to s \quad (\text{in } \mathbf{R}) \quad \Longrightarrow \quad x_t^{(i)} \to x_s^{(i)} \quad (\text{in } H_0) \tag{29}$$

for $t,s \in [b,T]$ and $i = 0,1,\ldots,n-1$.

Let us define the function $g_n : [b,T] \to \mathbf{R}$ by

$$g_n(t) = f(t, x_{\xi_i}, x'_{\xi_i}, \ldots, x_{\xi_i}^{(n-1)}) \quad \text{for} \quad t \in [\xi_i, \xi_{i+1})\ , \quad i = 0,1,\ldots,n-1\ ,$$

where $\xi_i := b + i\frac{T-b}{n}$, $i = 0, 1, \dots, n$. Then (according to the assumption (iv)), g_n is measurable on $[\xi_i, \xi_{i+1}]$ for $i = 0, 1, \dots, n-1$, thus, g_n is measurable. Due to (v) and (29), we have

$$\lim_{n\to\infty} g_n(t) = g(t) \quad \text{for almost all} \quad t \in [b, T]\,,$$

so, g is measurable as a limit of a sequence of measurable functions.

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