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EXISTENCE OF SOLUTIONS FOR NONLINEAR PARABOLIC PROBLEMS

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Abstract. We consider nonlinear parabolic boundary value problems. First we assume that the right hand side term is discontinuous and nonmonotone and in order to have an existence theory we pass to a multivalued version by filling in the gaps at the discontinuity points. Assuming the existence of an upper solution ϕ and of a lower solution ψ such that $\psi \leq \phi$, and using the theory of nonlinear operators of monotone type, we show that there exists a solution $x \in [\psi, \phi]$ and that the set of all such solutions is compact in $W_{pq}(T)$. For the problem with a Caratheodory right hand side we show the existence of extremal solutions in $[\psi, \phi]$.

1. Introduction

Let $T = [0, b]$ and $Z \subseteq R^N$ a bounded open set with a C^1-boundary Γ. Let $D_k = \frac{\partial}{\partial z_k}, k \in 1, 2, ..., N$ and $D = grad$. In this paper we consider the following nonlinear parabolic boundary value problem:

$$(1)\quad \left\{ \begin{array}{l} \dfrac{\partial x}{\partial t} - \displaystyle\sum_{k=1}^{N} D_k \alpha_k(t, z, x, Dx) + \alpha_o(t, z, x, Dx) = f(t, z, x(t, z)) \text{ on } T \times Z\,. \\ x(0, z) = x_o(z) \text{ a.e. on } Z, x\mid_{T\times\Gamma} = 0\,. \end{array} \right\}$$

We do not assume that $f(t, z, \cdot)$ is continuous. Thus problem (1) need not have a (weak) solution. In order to have an existence theory, we pass to a multivalued version of (1) which roughly speaking is derived by filling in the gaps at the discontinuity points of the function $f(t, z, \cdot)$. For this purpose, we introduce the

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following two functions:

$$f_1(t,z,x) = \liminf_{x'\to x} f(t,z,x') = \liminf_{\varepsilon\downarrow 0\ \ |x'-x|<\varepsilon} f(t,z,x')$$
$$\text{and } f_2(t,z,x) = \limsup_{x'\to x} f(t,z,x) = \limsup_{\varepsilon\downarrow 0\ \ |x'-x|<\varepsilon} f(t,z,x')\,.$$

Then instead of (1) we deal with the following nonlinear parabolic partial differential inclusion:

$$(2)\qquad \left\{ \begin{array}{l} \dfrac{\partial x}{\partial t} - \displaystyle\sum_{k=1}^{N} D_k \alpha_k(t,z,x,Dx) + \alpha_o(t,z,x,Dx) \\ \in [f_1(t,z,x), f_2(t,z,x)] \text{ on } T\times Z\,. \\ x(0,z) = x_o(z) \text{ a.e. on } Z,\, x\,|_{T\times\Gamma} = 0\,. \end{array} \right\}$$

We solve (2) using the method of upper and lower solutions. More precisely assuming the existence of an upper solution ϕ and of a lower solution ψ, we show that problem (2) has a solution x in the order interval $[\psi,\phi]$ and the set of all such solutions is compact in $W_{pq}(T)$. Subsequently using stronger hypotheses on the functions $\alpha_k(t,z,x,y)$, we show that problem (1) with a Caratheodory f has extremal solutions in $K=[\psi,\phi]$.

Our work is related to those by Deuel-Hess [9], Boccardo-Murat-Puel [2] and Mokrane [19]. In these papers $\psi,\phi\in W^{1,\infty}(T,L^\infty(Z)), f=0$ (so there is no need to pass to a multivalued problem) and the authors do not address the question of extremal solutions in $[\psi,\phi]$. Problems with discontinuities have been studied primarly for stationary (elliptic) equations, using a variety of methods. These different approaches can be traced in the works of Chang [5], Costa-Goncalves [7] (where the nonsmooth critical point theory is used) and of Rauch [21] (which is based on Galerkin approximations). It should be pointed out that all these works treat semilinear elliptic equtions. The study of the corresponding dynamic problem is lagging behind. Only recently we had the works of Carl [4],Miettinen [18] and Kandilakis-Papageorgiou [14]. Carl considers a problem where $\alpha_o=0$ and the functions α_k are independent of x and monotone. Kandilakis-Papageorgiou have a more general differential operator (like the one considered here) but $\alpha_o=0$ and f is independent of $(t,z)\in T\times Z$. Both works use the method of upper and lower solutions but do not establish the existence of extremal solutions. Miettinen deals with semilinear problems and employs the Galerkin method. Finally we also mention the paper of Carl [3], where $f=0$ (no discontinuous term is present), the differential operator is independent of x and is strictly monotone. In that paper the author following some ideas of Dancer-Sweers proves the existence of extremal solutions. So it seems that no previous work considered the problem in the generality that we have in our formulation. We mention that problems with discontinuities, such as the one considered here, arise in solid mechanics in cases involving nonmonotone, possibly multivalued constitutive laws derived by nonconvex superpotentials. For such applications we refer to the thesis of Miettinen [17] and the book of Panagiotopoulos [20].

2. Preliminaries

In this section we fix our notation and the hypotheses on the data of problem (2) and we also recall some basic definitions and facts from the theory of operators of monotone type which we will need in the sequel.

Our hypotheses on the functions $\alpha_k(t,z,x,y), k \in \{1,2,...,N\}$, are the following:

$\mathbf{H}(\alpha)$: $\alpha_k : T \times Z \times R \times R^N \to R, k \in \{1,2,...,N\}$, are functions such that

(i) for every $(x,y) \in R \times R^N$, $(t,z) \to \alpha_k(t,z,x,y)$ is measurable;

(ii) for every $(t,z) \in T \times Z$, $(x,y) \to \alpha_k(t,z,x,y)$ is continuous;

(iii) for almost all $(t,z) \in T \times Z$, all $x \in R$ and all $y \in R^N$, we have

$$|\alpha_k(t,z,x,y)| \le \beta(t,z) + c(|x|^{p-1} + ||y||^{p-1})$$

with $\beta \in L^q(T \times Z), c > 0, 2 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$;

(iv) for almost all $(t,z) \in T \times Z$, all $x \in R$ and all $y, y' \in R^N, y \neq y'$ we have

$$\sum_{k=1}^{N} (\alpha_k(t,z,x,y) - \alpha_k(t,z,x,y'))(y_k - y'_k) > 0\,;$$

(v) for almost all $(t,z) \in T \times Z$, all $x \in R$ and all $y \in R^N$, we have

$$\sum_{k=1}^{N} \alpha_k(t,z,x,y) y_k \ge c_1 ||y||^p - \beta_1(t,z)$$

with $\beta_1 \in L^1(T \times Z), c_1 > 0$.

Recall that an "evolution triple" consists of a triple of spaces $X \subseteq H \subseteq X^*$ such that:

(a) X is a separable reflexive Banach space;

(b) H is a separable Hilbert space which is identified with its dual (pivot space);

(c) the embedding of X into H is continuous and dense (see Zeidler [22], p. 146).

Let $W^{1,p}(Z)$ be the usual Sobolev space and $W^{1,p}(Z)^*$ its dual. The spaces $W^{1,p}(Z) \subseteq L^2(Z) \subseteq W^{1,p}(Z)^*$ form an evolution triple and in addition all embeddings are compact. Also by $W_o^{1,p}(Z)$ we denote the subspace of $W^{1,p}(Z)$ whose elements have zero trace. As usual, by $W^{-1,q}(Z)$ we denote the dual of $W_o^{1,p}(Z)$. Then $W_o^{1,p}(Z) \subseteq L^2(Z) \subseteq W^{-1,q}(Z)$ is another evolution triple, with all embeddings again being compact.

We introduce two spaces which play a central role in this paper:

$$\widehat{W}_{pq}(T) = \{x \in L^p(T, W^{1,p}(Z)) : \frac{\partial x}{\partial t} \in L^q(T, W^{1,p}(Z)^*)\}$$

and

$$W_{pq}(T) = \{x \in L^p(T, W_o^{1,p}(Z)) : \frac{\partial x}{\partial t} \in L^q(T, W^{-1,q}(Z))\}\,.$$

In these definitions the time-derivative of x is understood in the sense of vector-valued distributions. Both spaces equipped with the obvious norm $||x||_{pq} = ||x||_p + ||\frac{\partial x}{\partial t}||_q$ become separable reflexive Banach spaces. Moreover, both spaces are embedded continuously in $C(T, L^2(Z))$ and compactly in $L^p(T \times Z)$ (see for example Lions [16], theorem 5.1, p. 58 or Zeidler [23], proposition 23.23, p. 422 and p. 450).

Hypotheses $H(\alpha)$ allow us to define the semilinear form $\alpha : L^p(T, W^{1,p}(Z)) \times L^p(T, W^{1,p}(Z)) \to R$ by

$$\alpha(x, y) = \int_o^b \int_Z \sum_{k=1}^N \alpha_k(t, z, x, Dx) D_k y(t, z) dz dt\,.$$

In what follows by $((\cdot, \cdot))$ we denote the duality brackets for the pair

$$(L^p(T, W_o^{1,p}(Z)), L^p(T, W^{-1,q}(Z)))$$

and also of the pair

$$\left(L^p(T, W^{1,p}(Z)), L^q(T, W^{1,p}(Z)^*)\right)\,.$$

Recall that if X is a reflexive Banach space (or more generally if X^* has the Radon-Nikodym Property (RNP)) and $1 \leq p < \infty$, then $L^p(T, X)^* = L^q(T, X^*)$ (see Hu-Papageorgiou [12], theorem A.3.98, p. 918).

At this point we can introduce the notions of upper and lower solutions which will be basic in our subsequent considerations.

Definition. A function $\phi \in \widehat{W}_{pq}(T)$ is said to be an " upper solution" of problem (2) if

$$((\frac{\partial \phi}{\partial t}, u)) + \alpha(\phi, u) + \int_o^b \int_Z \alpha_o(t, z, \phi, D\phi) u dz dt \geq \int_o^b \int_Z f_2(t, z, \phi) u dz dt$$

for all $u \in L^p(T, W_o^{1,p}(Z)) \cap L^p(T \times Z)_+$ and $\phi(0, z) \geq x_o(z)$ a.e. on $Z, \phi\,|_{T\times\Gamma} \geq 0$.

Similarly $\psi \in \widehat{W}_{pq}(T)$ is said to be a "lower solution" of problem (2) if the inequalities are reversed and f_2 is replaced by f_1.

We assume that there exist upper and lower solutions.

$\mathbf{H}_o$: There exist an upper solution $\phi \in \widehat{W}_{pq}(T)$ and a lower solution $\psi \in \widehat{W}_{pq}(T)$ such that $\psi(t, z) \leq \phi(t, z)$ a.e. on $T \times Z$.

Finally our hypotheses on α_o and the discontinuity term f are the following:

$\mathbf{H}(\alpha_o)$: $\alpha_o : T \times Z \times R \times R^N \to R$ is a function such that

- **(i)** for every $(x, y) \in R \times R^N, (t, z) \to \alpha_o(t, z, x, y)$ is measurable;
- **(ii)** for all $(t, z) \in T \times Z, (x, y) \to \alpha_o(t, z, x, y)$ is continuous;
- **(iii)** for almost all $(t, z) \in T \times Z$, all $x \in [\psi(t, z), \phi(t, z)]$ and all $y \in R^N$, we have

$$|\alpha_o(t, z, x, y)| \leq \beta_2(t, z) + c_2||y||^{p-1}$$

with $\beta_2 \in L^q(T \times Z), c_2 > 0$.

$\mathbf{H(f)}$: $f : T \times Z \times R \to R$ is measurable function such that

(i) f_1, f_2 are superpositionally measurable (i.e. if $x : T \times Z \to R$ is measurable, then so are the functions $(t,z) \to f_k(t,z,x(t,z)), k = 1,2$;

(ii) for almost all $(t,z) \in T \times Z$ and all $x \in [\psi(t,z), \phi(t,z)]$, we have $|f(t,z,x)| \leq \beta_3(t,z)$ with $\beta_3 \in L^q(T \times Z)$.

Remark. If f is independent of $(t,z) \in T \times Z$, then hypotheses $H(f)(i)$ is automatically satisfied because f_1, f_2 are lower and upper semicontinuous functions of x respectively. Similarly if $f(t,z,\cdot)$ is monotone. Indeed if for example $f(t,z,\cdot)$ is nondecreasing, then $f_2(t,z,x) = f(t,z,x^+)$ and $f_1(t,z,x) = f(t,z,x^-)$. Then note that $f(t,z,x^+) = \lim_{n\to\infty} f(t,z,x+\frac{1}{n})$ and $f(t,z,x^-) = \lim_{n\to\infty} f(t,z,x-\frac{1}{n})$ and so f_1, f_2 are measurable functions, hence superpositionally measurable.

Let (X, H, X^*) be an evolution triple. By $|\cdot|$ (resp. $||\cdot||, ||\cdot||_*$) we denote the norm of H (resp. of X, X^*). Also by $(\cdot,\cdot)$ we denote the inner product of H and by $\langle\cdot,\cdot\rangle$ the duality brackets of (X, X^*). The two are compatible in the sense that $\langle\cdot,\cdot\rangle\,|_{H\times X} = (\cdot,\cdot)$.

Definition. An operator $A : X \to X^*$ is said to be of type $(S)_+$, if for every sequence $\{x_n\} \subseteq X$ such that $x_n \xrightarrow{w} x$ in X as $n \to \infty$ and $\limsup < A(x_n), x_n - x > \leq 0$, we have $x_n \to x$ in X as $n \to \infty$.

Remark. A demicontinuous operator of type $(S)_+$ is also generalized pseudomonotone (see Hu-Papageorgiou [13], proposition III.6.25, p. 371 or Zeidler [23], proposition 27.6, p. 586).

A related concept, useful in the context of dynamic (parabolic) problems is the following:

Definition. Let Y be a reflexive Banach space, $L : D \subseteq Y \to Y^*$ a linear, densely defined maximal monotone operator and $G : Y \to Y^*$ a nonlinear operator. We say that $G(\cdot)$ is of "type $L-(S)_+$", if for all sequences $\{y_n\}_{n\geq 1} \subseteq D$ such that $y_n \xrightarrow{w} y$ in Y, $L(y_n) \xrightarrow{w} L(y)$ in Y^* and $\limsup(G(y_n), y_n - y)_{Y^*,Y} \leq 0$, we have $y_n \to y$ in Y

This notion can be generalized as follows:

Definition. Let Y and L as in the previous definition. A set-valued map $F : Y \to 2^{Y^*} \setminus \{\emptyset\}$ with weakly compact and convex values, is said to be "L-pseudomonotone", if for all sequences $\{y_n\}_{n\geq 1} \subseteq D$ such that $y_n \xrightarrow{w} y$ in Y, $L(y_n) \xrightarrow{w} L(y)$ in Y^* and for $y_n^* \in F(y_n), n \geq 1$, satisfying $y_n^* \xrightarrow{w} y^*$ in Y^* and $\limsup(y_n^*, y_n - y) \leq 0$, we have $y^* \in F(y)$ and $(y_n^*, y_n) \to (y^*, y)$ as $n \to \infty$.

Our proof uses truncation and penalization techniques and so we introduce the truncation map $\tau : T \times Z \times R \to R$ and the penalty function $u : T \times Z \times R \to R$ defined by

$$\tau(t,z,x) = \begin{cases} \phi(t,z) & \text{if} \quad \phi(t,z) \leq x \\ x & \text{if} \quad \psi(t,z) \leq x \leq \phi(t,z) \\ \psi(t,z) & \text{if} \quad x \leq \psi(t,z) \end{cases}$$

and

$$u(t,z,x)=\begin{cases} (x-\phi(t,z))^{p-1} & \text{if } \phi(t,z)\le x\\ 0 & \text{if } \psi(t,z)\le x\le\phi(t,z)\\ -(\psi(t,z)-x)^{p-1} & \text{if } x\le\psi(t,z)\end{cases}$$

Let $\widehat{\tau}: L^p(T,W^{1,p}(Z))\to L^p(T,W^{1,p}(Z))$ be the Nemitsky operator corresponding to $\tau(t,z,x)$;

Proposition 1. $\widehat{\tau}: L^p(T,W^{1,p}(Z))\to L^p(T,W^{1,p}(Z))$ is continuous

Proposition 2. $u: T\times Z\times R\to R$ is a Caratheodory function (i.e. measurable in (t,z) and continuous in x),for almost all $(t,z)\in T\times Z$ and all $x\in R$, $|u(t,z,x)|\le \beta_4(t,z)+c_4|x|^{p-1}$ with $\beta_4\in L^q(T\times Z), c_4>0$ and $\int_o^b\int_Z u(t,z,x(t,z))dzdt\ge c_5||x||^p_{L^p(T\times Z)}-c_6$ with $c_5,c_6>0$.

3. Existence of solutions

First we define what we mean by a solution of problem (2):

Definition. A function $x\in W_{pq}(T)$ is said to be a "solution" (weak solution) of (2), if there exists $g\in L^q(T\times Z)$ such that $f_1(t,z,x(t,z))\le g(t,z)\le f_2(t,z,x(t,z))$ a.e. on $T\times Z$ and

$$((\frac{\partial x}{\partial t},u))+\alpha(x,u)+\int_o^b\int_Z \alpha_o(t,z,x,Dx)udzdt=\int_o^b\int_Z gudzdt\,.$$

for all $u\in L^p(T,W_o^{1,p}(Z))$ and $x(0,z)=x_o(z)$ a.e. on $Z, x\,|_{T\times\Gamma}=0$.

In this section we show that problem (2) has at least one solution in the order interval $K=[\psi,\phi]=\{x\in L^p(T\times Z):\psi(t,z)\le x(t,z)\le\phi(t,z)$ a.e. on $T\times Z\}$ and that the set of all such solutions, denoted henceforth by $S(x_o)\subseteq W_{pq}(T)$, is compact.

Let $\overline{f}(t,z,x)=f(t,z,\tau(t,z,x))$. For this function we define $\overline{f}_1(t,z,x)$ and $\overline{f}_2(t,z,x)$ as we did for f; i.e. $\overline{f}_1(t,z,x)=\liminf_{x'\to x}\overline{f}(t,z,x')$ and $\overline{f}_2(t,z,x)=\limsup_{x'\to x}\overline{f}(t,z,x')$.

Using $\overline{f}_1$ and $\overline{f}_2$, we introduce the following auxiliary problem:

$$(3)\quad \left\{\begin{array}{l} \frac{\partial x}{\partial t}-\sum_{k=1}^N D_k\alpha_k(t,z,\widehat{\tau}(x),Dx)+\alpha_o(t,z,\widehat{\tau}(x),D\widehat{\tau}(x))+\xi u(t,z,x)\\ \in[\overline{f}_1(t,z,x),\overline{f}_2(t,z,x)] \text{ on } T\times Z\,.\\ x(0,z)=x_o(z) \text{ a.e. on } Z, x\,|_{T\times\Gamma}=0,\xi>0\,.\end{array}\right\}$$

Solutions for (3) are defined as for problem (2). We denote the solution set of (3) by $S_\alpha(x_o)\subseteq W_{pq}(T)$. The next proposition establishes the nonemptiness of $S_\alpha(x_o)$.

Proposition 3. If hypotheses $H(\alpha),H(\alpha_o),H_o,H(f)$ hold, $x_o\in L^2(Z)$ and $\xi\ge\xi_o>0$, **then** $S_\alpha(x_o)$ is nonempty subset of $W_{pq}(T)$.

Proof. Let $A_1 : T \times W_o^{1,p}(Z) \to W^{-1,q}(Z)$ be defined by

$$\langle A_1(t,x), y\rangle = \int_Z \sum_{k=1}^N \alpha_k(t,z,\widehat{\tau}(x), Dx)D_k y dz \text{ for all } y \in W_o^{1,p}(Z)$$

Using hypotheses $H(\alpha)$, we can easily verify that for every $x \in W_o^{1,p}(Z), t \to A_1(t,x)$ is measurable, for every $t \in T$ $x \to A_1(t,x)$ is demicontinuous, $||A_1(t,x)||_* \leq \widehat{\beta}(t)+\widehat{c}||x||^{p-1}$ a.e. on T with $\widehat{\beta} \in L^q(T), \widehat{c} > 0$ and $\langle A_1(t,x), x\rangle \geq c_1||x||^p - \widehat{\beta}_1(t)$ a.e. on T with $\widehat{\beta}_1 \in L^1(T), c_1 > 0$ (here by $||\cdot||, ||\cdot||_*$ we denote the norms of $W_o^{1,p}(Z), W^{-1,q}(Z)$ respectively and by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair $(W_o^{1,p}(Z), W^{-1,q}(Z))$. Moreover, from proposition 3.3 of Kandilakis-Papageorgiou [14] we have that for all $t \in T, A_1(t,\cdot)$ is of type $(S)_+$.

Also let $h : T \times W_o^{1,p}(Z) \to L^q(T \times Z)$ be defined by

$$h(t,x)(z) = \alpha_o(t,z,\widehat{\tau}(x), D\widehat{\tau}(x))\,.$$

By virtue of hypotheses $H(\alpha_o)$ and proposition 1, for every $x \in W_o^{1,p}(Z)$, $t \to h(t,x)$ is measurable and for every $t \in T, x \to h(t,x)$ is continuous. Also we have:

$$
\begin{aligned}
(4) \qquad |\langle h(t,x), x\rangle| &= \Big|\int_Z \alpha_o(t,z,\widehat{\tau}(x), D\widehat{\tau}(x))x(z)dz \\
&\leq \Big|\int_{\{x<\psi\}\cup\{\phi<x\}} \alpha_o(t,z,\widehat{\tau}(x), D\widehat{\tau}(x))x(z)dz\Big| \\
&\quad + \Big|\int_{\{\psi\leq x\leq\phi\}} \alpha_o(t,z,\widehat{\tau}(x), D\widehat{\tau}(x))x(z)dz\Big|\,.
\end{aligned}
$$

Because of hypothesis $H(\alpha_o)(iii)$ we can find $\beta_5 \in L^q(T \times Z)$ such that

$$
\begin{aligned}
(5) \qquad \Big|\int_{\{\phi<x\}} \alpha_o(t,z,\widehat{\tau}(x), D\widehat{\tau}(x))x(z)dz\Big| &= \Big|\int_{\{\phi<x\}} \alpha_o(t,z,\phi,D\phi)x(z)dz\Big| \\
&\leq ||\beta_5(t,\cdot)||_q||x||_p
\end{aligned}
$$

and

$$
\begin{aligned}
(6) \qquad \Big|\int_{\{x<\psi\}} \alpha_o(t,z,\widehat{\tau}(x), D\widehat{\tau}(x))x(z)dz\Big| &= \Big|\int_{\{x<\psi\}} \alpha_o(t,z,\psi,D\psi)x(z)dz\Big| \\
&\leq ||\beta_5(t,\cdot)||_q||x||_p\,.
\end{aligned}
$$

Recall that $D\widehat{\tau}(x)(t,z) = \begin{cases} D\phi(t,z) & \text{if } \phi(t,z) \leq x(t,z) \\ Dx(t,z) & \text{if } \psi(t,z) \leq x(t,z) \leq \phi(t,z) \\ D\psi(t,z) & \text{if } x(t,z) \leq \psi(t,z) \end{cases}$;

(see for example Evans-Gariepy [10], theorem 4, pp. 129-130).

In addition if $\sigma(t) = max[||\psi(t,\cdot)||_p, ||\phi(t,\cdot)||_p] \in L^p(T)$, we have

$$(7)\quad \Big|\int_{\{\psi\leq x\leq\phi\}} \alpha_o(t,z,\widehat{\tau}(x), D\widehat{\tau}(x))x(z)dz\Big| \leq (||\beta_2(t,\cdot)||_q + c_2\sigma(t)^{p-1})\sigma(t) = \eta_1(t) \text{ a.e. on } T.$$

with $\eta_1(t) \in L^1(T)$.

Use Young's inequality on the right hand side of (5) and (6) and then together with (7) use them in (4), to obtain

$$|< h(t,x), x >| \leq \eta_{2\varepsilon}(t) + \frac{\varepsilon}{p}||x||^{p-1} \text{ a.e. on } T$$

with $\eta_{2\varepsilon} \in L^p(T), \alpha_2 \in L^2(T)$. In addition, using once more hypothesis $H(\alpha_o)(iii)$ we have

$$||h(t,x)||_q \leq ||\beta_2(t,\cdot)||_q + c_2||x||^{p-1} \text{ a.e. on } T.$$

Let $A : T \times W_o^{1,p}(Z) \to W^{-1,q}(Z)$ be defined by $A(t,x) = A_1(t,x) + h(t,x)$. Evidently, $t \to A(t,x)$ is measurable, $x \to A(t,x)$ is demicontinuous, $||A(t,x)||_* \leq \widehat{\beta}_2(t)+\widehat{c}_2||x||^{p-1}$ a.e. on T and choosing ε small enough $\langle A(t,x),x\rangle \geq \widehat{c}_o||x||^p - \widehat{\eta}(t)$ a.e. on T where $\widehat{\beta}_2 \in L^q(T), \widehat{c}_2, \widehat{c}_o > 0$ and $\widehat{\eta} \in L^1(T)_+$. In addition for all $t \in T$ $A(t,\cdot)$ is of type $(S)_+$. Indeed if $x_n \xrightarrow{w} x$ in $W_o^{1,p}(Z)$ and $\limsup\langle A(t,x_n), x_n - x\rangle \leq 0$, then $x_n \to x$ in $L^p(Z)$ and so $\langle h(t,x_n), x_n - x\rangle = \int_o^b \alpha_o(t,z,\widehat{\tau}(x_n), D\widehat{\tau}(x_n))$ $(x_n - x)(z)dz \to 0$ as $n \to \infty$. Hence $\limsup\langle A_1(t,x_n), x_n - x\rangle \leq 0$. But as we already mentioned $A_1(t,\cdot)$ is of type $(S)_+$. Hence by definition $x_n \to x$ in $W_o^{1,p}(Z)$ which proves that $A(t,\cdot)$ is of type $(S)_+$ as claimed.

Let $\widehat{A} : L^p(T, W_o^{1,p}(Z)) \to L^q(T, W^{-1,q}(Z))$ be the Nemitsky operator corresponding to $A(t,x)$; i.e. $\widehat{A}(x)(\cdot) = A(\cdot, x(\cdot))$. Also let $\widehat{u} : T \times L^p(Z) \to L^q(Z)$ be the map defined by $\widehat{u}(t,x)(\cdot) = u(t,\cdot,x(\cdot))$. By virtue of proposition 2, $\widehat{u}(t,x)$ is a Caratheodory map (i.e. measurable in $t \in T$ and continuous in $x \in L^p(Z)$) and for almost all $t \in T$ and all $x \in L^p(Z)$, we have $||\widehat{u}(t,x)||_p \leq \widehat{\beta}_3(t) + \widehat{c}_3||x||^{p-1}$, where $\widehat{\beta}_3 \in L^q(T), \widehat{c}_3 > 0$. Finally let $F_1 : T \times L^p(Z) \to 2^{L^q(Z)} \setminus \{\emptyset\}$ be the multifunction with closed and convex values defined by

$$F_1(t,x) = \{g \in L^q(Z) : \overline{f}_1(t,z,x(z)) \leq g(z) \leq \overline{f}_2(t,z,x(z)) \text{ a.e. on } Z\}.$$

We rewrite problem (3) in the following equivalent evolution inclusion form:

$$(8)\qquad \left\{\begin{array}{l} \dot{x}(t) + A(t,x(t)) + \xi\widehat{u}(t,x(t)) \in F_1(t,x(t)) \text{ a.e. on } T \\ x(0) = x_o. \end{array}\right\}$$

We solve (8). To this end, first assume $x_o \in W_o^{1,p}(Z)$. Then we set

$$A_o(t,x) = A(t,x+x_o), u_o(t,x) = \widehat{u}(t,x+x_o) \text{ and } \overline{F}_1(t,x) = F_1(t,x+x_o)$$

It is easy to verify that all properties of $A(t,x), \widehat{u}(t,x)$ and $F_1(t,x)$ are passed to A_o, u_o and $\overline{F}_1$ respectively (with different constants). We consider the following Cauchy problem:

$$(9) \qquad \left\{ \begin{array}{l} \dot{y}(t) + A_o(t, y(t)) + \xi u_o(t, y(t)) \in \overline{F}_1(t, y(t)) \text{ a.e. on } T \\ y(0) = 0\,. \end{array} \right\}$$

Note that $y \in W_{pq}(T)$ solves (9) if and only if $x = y + x_o \in W_{pq}$ solves (8). So to obtain a solution for (8) it suffices to solve (9) which has zero initial condition. Let $L : D \subseteq L^p(T, W_o^{1,p}(Z)) \to L^q(T, W^{-1,q}(Z))$ be defined by $Ly = \dot{y}$ for all $y \in D = \{y \in L^p(T, W_o^{1,p}(Z)) : \dot{y} \in L^q(T, W^{-1,q}(Z)), y(0) = 0\}$. Here as before the time derivative of y is defined in the sense of vectorial distributions. It is well-known (see for example Hu-Papageorgiou [11], proposition 9.3, p. 419 or Zeidler [23], proposition 32.10, p. 855), that L is linear, densely defined and maximal monotone. Let $U_o : L^p(T \times Z) \to L^q(T \times Z)$ be defined by $U_o(x)(t, z) = u_o(t, z, x(t, z))$. Proposition 2 implies that U_o is continuous and bounded. Also let $G_o : L^p(T \times Z) \to 2^{L^q(T \times Z)} \setminus \{\emptyset\}$ be defined by $G_o(x) = \{g \in L^q(T \times Z) : \overline{f}_1(t, z, x(t, z)) \leq g(t, z) \leq \overline{f}_2(t, z, x(t, z)) \text{ a.e. on } T \times Z\}$. Then $G_o(\cdot)$ has closed and convex values. Problem (9) is equivalent to the following operator inclusion:

$$Ly + A_o(y) + \xi U_o(y) \in G_o(y)$$

Claim 1: The operator $\widehat{A}_o + \xi U_o - G_o$ is L-pseudomonone.

Let $x_n \xrightarrow{w} x$ in $W_{pq}(T)$ $v_n \in A_o(x_n) + \xi U_o(x_n) - g_n$, where $g_n \in G_o(x_n), n \geq 1$ and assume that $v_n \xrightarrow{w} v$ in $L^q(T, W^{-1,q}(Z)$. Because $W_{pq}(T)$ is embedded compactly in $L^p(T \times Z)$, we have $x_n \to x$ in $L^p(T \times Z)$. Since $U_o(\cdot)$ is continuous we have that $U_o(x_n) \to U_o(x)$ in $L^q(T \times Z)$. So $((U_o(x_n), x_n - x)) = (U_o(x_n), x_n - x)_{pq} \to 0$ as $n \to \infty$ (here by $(\cdot,\cdot)_{pq}$ we denote the duality brackets for the pair $(L^p(T \times Z), L^q(T \times Z)))$. Also since $g_n \in G_o(x_n)$ we have $|g_n(t, z)| \leq \beta_4(t, z)$ a.e. on $T \times Z$, $n \geq 1$ with $\beta_4 \in L^q(T \times Z)$ (see hypothesis $H(f)(iii)$). So we may assume that $g_n \xrightarrow{w} g$ in $L^q(T \times Z)$ as $n \to \infty$. Using proposition 3.9, p. 694 of Hu-Papageorgiou [11] and the fact that $\overline{f}_1(t, z, \cdot)$ is lower semicontinuous, while $\overline{f}_2(t, z, \cdot)$ is upper semicontinuous, we infer that $\overline{f}_1(t, z, x(t, z)) \leq g(t, z) \leq \overline{f}_2(t, z, x(t, z))$ a.e. on $T \times Z$; i.e. $g \in G_o(x)$. In addition $((g_n, x_n - x)) = (g_n, x_n - x)_{pq} \to 0$ as $n \to \infty$. Thus finally we can say that

$$\limsup((\widehat{A}_o(x_n), x_n - x)) \leq 0$$

From proposition 3.6 of Kandilakis-Papageorgiou [14] we know that $\widehat{A}_o$ is of type $L - (S)_+$. So we have $x_n \to x$ in $L^p(T, W_o^{1,p}(Z))$. Therefore it follows that $\widehat{A}_o(x_n) \xrightarrow{w} \widehat{A}_o(x)$ in $L^q(T, W^{-1,q}(Z))$ and so in the limit as $n \to \infty$, we have that $v = \widehat{A}_o(x) + \xi U_o(x) - g \in (\widehat{A}_o + \xi U_o - G_o)(x)$ and $((v_n, x_n)) \to ((v, x))$. This proves the claim.

Claim 2: For $\xi \geq \xi_o$, $(\widehat{A}_o + \xi U_o - G_o)(\cdot)$ is coercive.

We have

$$(10) \qquad ((\widehat{A}_1(x + x_o), x)) \geq \widehat{c}_4 ||x||_{L^p(T, W_o^{1,p}(Z))} - \widehat{c}_5$$

Also if $\widehat{h} : L^p(T, W_o^{1,p}(Z)) \to L^q(T \times Z)$ is the Nemitsky operator corresponding to $h(t,x)$ (i.e. $\widehat{h}(x)(t) = h(t,x(t))$) and we use Young's inequality, we have

$$\begin{aligned} ((\widehat{h}(x+x_o),x)) = (\widehat{h}(x+x_o,x)_{pq} &\leq ||\widehat{h}(x+x_o)||_{L^p(T\times Z)}||x||_{L^p(T\times Z)} \\ &\leq \frac{\varepsilon^q}{q}||\widehat{h}(x+x_o)||^q_{L^q(T\times Z)} + \frac{1}{\varepsilon^p p}||x||^p_{L^p(T\times Z)} \end{aligned} \tag{11}$$

From previous considerations we know that

$$||\widehat{h}(x+x_o)||^q_{L^q(T\times Z)} \leq \frac{\varepsilon^q}{q}(\widehat{c}_7 + \widehat{c}_8||x||^p_{L^p(T,W_o^{1,p}(Z))}), \quad \widehat{c}_7, \widehat{c}_8 > 0\,.$$

Using this in (11), we have

$$((\widehat{h}(x+x_o),x)) \leq \frac{\varepsilon^q}{q}(\widehat{c}_7 + \widehat{c}_8||x||^p_{L^p(T,W_o^{1,p}(Z))}) + \frac{1}{\varepsilon^p p}||x||^p_{L^p(T\times Z)}\,. \tag{12}$$

Also from proposition 2 we know that

$$\begin{aligned} ((U_o(x+x_o),x)) &= (U_o(x+x_o),x)_{pq} \\ &\geq \widehat{c}_9||x||^p_{L^p(T\times Z)} - \widehat{c}_10, \qquad \widehat{c}_9, \widehat{c}_{10} > 0\,. \end{aligned} \tag{13}$$

Using (10), (11), (12) and (13), for any $g \in G_o(x)$ we have

$$\begin{aligned} ((\widehat{A}_o(x) + \xi U_o(x) - g, x)) \geq\ & (\widehat{c}_1 - \frac{\varepsilon^q}{q}\widehat{c}_8)||x||^p_{L^p(T,W_o^{1,p}(Z))} \\ &+ (\xi\widehat{c}_5 - \frac{1}{\varepsilon^p p})||x||^p_{L^p(T\times Z)} - \widehat{c}_{11}, \quad \widehat{c}_{11} > 0\,. \end{aligned} \tag{14}$$

Choose $\varepsilon > 0$ so that $\widehat{c}_1 > \frac{\varepsilon^q}{q}\widehat{c}_8$ and then based on this choise of $\varepsilon > 0$, we choose $\xi > 0$ large enough so that $\xi c_9 > \frac{1}{\varepsilon^p p}$. Then from (14) it follows that $(\widehat{A}_o + \xi U_o - G_o)(\cdot)$ is coercive as claimed.

Using claims 1 and 2, we can apply proposition 3.4 of Kandilakis - Papageorgiou [14] and obtain $y \in W_{pq}(T)$ a solution of $Ly + \widehat{A}_o(y) + \xi U_o(y) \in G_o(y)$. This solves (4), hence $y + x_o = x \in W_{pq}(T)$ solves problem (3). Thus we have established the nonemptiness of the solution set of (3) when $\xi \geq \xi_o > 0$ and with regular initial datum $x_o \in W_o^{1,p}(Z)$. Now we remove this last restriction and assume that $x_o \in L^2(Z)$. Let $x_{on} \in W_o^{1,p}(Z), n \geq 1$, be such that $x_{on} \to x_o$ in $L^2(Z)$ as $n \to \infty$. Let $x_n \in S_\alpha(x_{on}), n \geq 1$. By the same a priori estimation as in Aizicovici - Papageorgiou [1], we can show that $\{x_n\}_{n\geq 1} \subseteq W_{pq}(T)$ is bounded. So we may assume that $x_n \xrightarrow{w} x$ in $W_{pq}(T), x_n \to x$ in $L^p(T \times Z)$ and $x_n \xrightarrow{w} x$ in $C(T, L^2(Z))$ (recall that $W_{pq}(T)$ is embedded compactly in $L^p(T \times Z)$ and continuously in $C(T, L^2(Z))$. Therefore $Lx_n = \dot{x}_n \xrightarrow{w} \dot{x} = Lx$ in $L^q(T, W^{-1,q}(Z)), U(x_n) \to U(x)$ in $L^q(T \times Z)$ and if $g_n \in G(x_n), n \geq 1$, are such that $Lx_n + \widehat{A}(x_n) + \xi U(x_n) = g_n$, then we may assume that $g_n \xrightarrow{w} g$ in $L^q(T \times Z)$. We have

$$((\dot{x}_n, x_n - x)) + ((\widehat{A}(x_n), x_n - x)) + \xi((U(x_n), x_n - x)) = ((g_n, x_n - x))$$

Note that $((U(x_n), x_n - x)) = (U(x_n), x_n - x)_{pq} \to 0$ and $((g_n, x_n - x)) = (g_n, x_n - x)_{pq} \to 0$ as $n \to \infty$. Also from the integration by parts formula for functions in $W_{pq}(T)$ (see Zeidler [23], proposition 23.23, p. 422-423) we have

$$\begin{aligned}((\dot{x}_n, x_n - x)) &= ||x_n(b,\cdot) - x(b,\cdot)||_2^2 - ||x_{on} - x_o||_2^2 + ((\dot{x}, x_n - x)) \\ &\geq -||x_{on} - x_o||_2^2 + ((\dot{x}, x_n - x))\end{aligned}$$

We know that $||x_{on} - x_o||_2^2 \to 0$ as $n \to \infty$, while because $x_n \xrightarrow{w} x$ in $W_{pq}(T)$ we infer that $((\dot{x}, x_n - x)) \to 0$. Therefore $0 \leq \liminf((\dot{x}_n, x_n - x))$ and from this it follows that $\limsup((\widehat{A}(x_n), x_n - x)) \leq 0$. Since $\widehat{A}$ is of type $L - (S)_+$ we have that $x_n \to x$ in $L^p(T, W_o^{1,p}(Z))$ and so as before we can verify that $g \in G(x)$ and that $Lx + \widehat{A}(x) + \xi U(x) = g$, i.e. $x \in S_\alpha(x_o)$. □

Using proposition 3 we can show the nonemptiness of $S(x_o) \subseteq W_{pq}(T)$. We recall that by $S(x_o)$ we denote the set of solutions of (2) in the order interval $K = [\psi, \phi]$. Moreover, we show that $S(x_o)$ is compact in $W_{pq}(T)$. This appears to be the first such result in the literature.

Theorem 1. If hypotheses $H(\alpha), H(\alpha_o), H_o, H(f)$ hold, $x_o \in L^2(Z)$, **then** $S(x_o) \subseteq W_{pq}(T)$ is nonempty and compact.

Proof. Let $x \in S_\alpha(x_o)$ (see proposition 3). For all $w \in W_{pq}(T)$ we have

$$((\dot{x}, w)) + ((\widehat{A}_1(x), w)) + ((\widehat{h} + \xi U(x), w)) = ((g, w)) \quad \text{for some } g \in G(x) \tag{15}$$

Also since by hypothesis $\psi \in \widehat{W}_{pq}(T)$ is a lower solution, for all $w \in W_{pq}(T) \cap L^p(T \times Z)_+$, we have

$$((\dot{\psi}, w)) + ((\widehat{A}_1(\psi), w)) + ((\widehat{h}(\psi), w)) \leq ((\widehat{f}_1(\psi), w)) \tag{16}$$

where $\widehat{f}_1(\psi)(t,z) = f(t,z,\psi(t,z))$. Let $w = (\psi - x)^+ \in W_{pq}(T) \cap L^p(T \times Z)_+$. Subtracting (16) from (15) we obtain

$$\begin{aligned}((\dot{x} - \dot{\psi}, (\psi - x)^+)) \; &+ \; ((\widehat{A}_1(x) - \widehat{A}_1(\psi), (\psi - x)^+)) + ((\widehat{h}(x) - \widehat{h}(\psi), (\psi - x)^+)) \\ &+ \; \xi((U(x), (\psi - x)^+)) \geq ((g - \widehat{f}_1(\psi), (\psi - x)^+))\,.\end{aligned} \tag{17}$$

From the integration by parts formula for functions in $W_{pq}(T)$, we have

$$((\dot{x} - \dot{\psi}, (\psi - x)^+)) = -\frac{1}{2}||(\psi - x)^+(b,\cdot)||_2^2 + \frac{1}{2}||(\psi - x)^+(0,\cdot)||_2^2\,.$$

But from the definition of lower solution, we have $(\psi - x)^+(0,\cdot) = 0$. So

$$((\dot{x} - \dot{\psi}, (\psi - x)^+)) \leq 0\,. \tag{18}$$

Also we have

$$(19)\qquad ((\widehat{A}_1(x)-\widehat{A}_1(\psi),(\psi-x)^+))$$
$$=\int_o^b\int_Z\sum_{k=1}^N(\alpha_k(t,z,\widehat{\tau}(x),Dx)-\alpha_k(t,z,\psi,D\psi))D_k(\psi-x)^+dzdt$$
$$=\int\int_{\{\psi>x\}}\sum_{k=1}^N(\alpha_k(t,z,\psi,Dx)-\alpha_k(t,z,\psi,D\psi))D_k(\psi-x)dzdt\leq 0$$

(see hypothesis $H(\alpha)(iv)$).

Similarly we have

$$(20)\qquad ((\widehat{h}(x)-\widehat{h}(\psi),(\psi-x)^+))$$
$$=\int_o^b\int_Z(\alpha_o(t,z,\widehat{\tau}(x),D\widehat{\tau}(x))-\alpha_o(t,z,\psi,D\psi))(\psi-x)^+dzdt$$
$$=\int\int_{\{\psi>x\}}(\alpha_o(t,z,\psi,D\psi)-\alpha_o(t,z,\psi,D\psi))(\psi-x)dzdt=0\,.$$

Finally since $\overline{f}_1(t,z,x(t,z))=f(t,z,\psi(t,z))\geq f_1(t,z,\psi(t,z))$ a.e. on $T\times Z$. Hence

$$(21)\qquad ((g-\widehat{f}_1(\psi),(\psi-x)^+))\geq 0$$

Using (18) $\to$ (21) in (17), we obtain

$$\xi((U(x),(\psi-x)^+))\geq 0$$
$$\Rightarrow\int_o^b\int_Z u(t,z,x(t,z))(\psi-x)^+(t,z)dzdt\geq 0$$
$$\Rightarrow\int\int_{\{\psi>x\}}-(\psi-x)^{p-1}(\psi-x)dzdt\geq 0$$
$$\Rightarrow ||(\psi-x)^+||_p=0\text{ i.e. }\psi\leq x$$

Similarly we can show that $x\leq\phi$, i.e. $x\in K=[\psi,\phi]$. Hence $\widehat{\tau}(x)=x, U(x)=0$ and so $g(t,z)\in\widehat{f}(t,z,x(t,z))$ a.e. Therefore $x\in S(x_o)$.

Now we prove the compactness property of the solution set $S(x_o)$. For this purpose let $V=\{g\in L^q(T\times Z):|g(t,z)|\leq\beta_3(t,z)$ a.e. on $T\times Z\}$. This set furnished with the relative weak-$L^p(T\times Z)$ topology is compact and metrizable. Let $\Gamma:V\to 2^{C(T,L^2(Z))}$ be the map which to each $g\in V$ assigns the set of solutions of the following Cauchy problem:

$$(22)\qquad \left\{\begin{array}{l}\dot{x}(t)+A(t,x(t))=g(t)\text{ a.e. on }T\\ x(0)=x_o\,.\end{array}\right\}$$

We know that $\Gamma(g)\neq\emptyset$.

Claim 3: $\Gamma(V)$ is compact in $W_{pq}(T)$.

Let $\{x_n\}_{n\geq 1} \subseteq \Gamma(V)$. Then we can find $\{g_n\}_{n\geq 1} \subseteq V$ such that $x_n \in \Gamma(g_n)$, $n \geq 1$. From a priori estimation (see Aizicovici-Papageorgiou [1]) we know that $\{x_n\}_{n\geq 1} \subseteq W_{pq}(T)$ is bounded. So we may assume that $x_n \xrightarrow{w}$ in $W_{pq}(T), x_n \to x$ in $L^p(T\times Z), x_n(t,\cdot) \to x(t,\cdot)$ in $L^p(Z)$ for all $t \in T\setminus N, |N| = 0$ and $g_n \xrightarrow{w} g$ in $L^q(T\times Z)$. The sequence $t \to < \dot{x}_n(t,\cdot), x_n(t,\cdot) - x(t,\cdot) >, n \geq 1$, is uniformly integrable. So given $\varepsilon > 0$ we can find $t \in T\setminus N$ such that

$$\int_t^b |< \dot{x}_n(s,\cdot), x_n(s,\cdot) - x(s,\cdot) > |ds < \varepsilon \tag{23}$$

Denote by $((\cdot,\cdot))_t$ the duality brackets for the pair

$$(L^2([0,t], W_o^{1,p}(Z)), L^q([0,t], W^{-1,q}(Z)))\,.$$

Using the integration by parts formula for functions in $W_{pq}(T)$, we have

$$((\dot{x}_n, x_n - x))_t = \frac{1}{2}||x_n(t,\cdot) - x(t,\cdot)||_2^2 + ((\dot{x}, x_n - x))_t$$

Since $t \in T\setminus N$, we have $||x_n(t,\cdot) - x(t,\cdot)||_2^2 \to 0$. Also $((\dot{x}, x_n - x))_t \to 0$. Hence $((\dot{x}_n, x_n - x))_t \to 0$ as $n \to \infty$.

Note that

$$\begin{aligned}
&((\dot{x}_n, x_n - x)) = ((\dot{x}_n, x_n - x))_t + \int_t^b < \dot{x}_n(s,\cdot), x_n(s,\cdot) - x(s,\cdot) > ds\\
\Rightarrow\ &((\dot{x}_n, x_n - x)) \geq ((\dot{x}_n, x_n - x)) - \varepsilon \text{ (see (23))}\\
\Rightarrow\ &\liminf((\dot{x}_n, x_n - x)) \geq -\varepsilon
\end{aligned}$$

Let $\varepsilon \downarrow 0$, to obtain that

$$\liminf((\dot{x}_n, x_n - x)) \geq 0 \tag{24}$$

From (23) and (24) it follows that $((\dot{x}_n, x_n - x)) \to 0$ as $n \to \infty$. For every $n \geq 1$ we have

$$\begin{aligned}
&((\dot{x}, x_n - x)) + ((\widehat{A}(x_n), x_n - x)) = (g_n, x_n - x)_{pq}\\
\Rightarrow\ &\limsup((\widehat{A}(x_n), x_n - x)) \leq 0
\end{aligned}$$

Since $\widehat{A}$ is of type $L-(S)+$, we deduce that $x_n \to x$ in $L^p(T, W_o^{1,p}(Z))$. Note that $W_{pq}(T)$ is embedded densely in $L^p(T,X)$. So for every $n \geq 1$ we can find $u_n \in W_{pq}(T)$ with $||u_n||_p \leq 1$ such that

$$||\dot{x}_n - \dot{x}||_q - \frac{1}{n} \leq ((\dot{x}_n - \dot{x}, u_n)).$$

As before, since the sequence $t \to < \dot{x}_n(t,\cdot), u_n(t,\cdot) >, \quad n \geq 1$ is uniformly integrable, given $\varepsilon > 0$ we can find $t \in T\setminus N$, $|N| = 0$ such that

$$\int_t^b |< \dot{x}_n(t,\cdot) - \dot{x}(t,\cdot), u_n(t,\cdot) > |dt \leq \varepsilon.$$

So we have

$$\begin{aligned} &((\dot{x}_n - \dot{x}, u_n)) = ((\dot{x}_n - \dot{x}, u_n))_t + \int_t^b < \dot{x}_n(t,\cdot) - \dot{x}(t,\cdot), u_n(t,\cdot) > dt \\ \Rightarrow \quad &((\dot{x}_n - \dot{x}, u_n)) \geq ((\dot{x}_n - \dot{x}, u_n))_t - \varepsilon \\ \Rightarrow \quad &\liminf((\dot{x}_n - \dot{x}, u_n)) \geq 0 \text{ (since } \varepsilon > 0 \text{ is arbitrary).} \end{aligned}$$

Similarly we have

$$\limsup((\dot{x}_n - \dot{x}, u_n)) \leq 0\,.$$

Thus we deduce that $((\dot{x}_n - \dot{x}, u_n)) \to 0$ and so $\dot{x}_n \to \dot{x}$ in $L^q(T, W^{-1,q}(Z))$ and $\dot{x} + \widehat{A}(x) + \widehat{h}(x) = g, g \in V$. This proves that $x_n \to x$ in $W_{pq}(T)$ with $x \in R(V)$ and so $R(V)$ is compact in $W_{pq}(T)$. Since $S(x_o) \subseteq W_{pq}(T)$, we conclude that $S(x_o)$ is compact in $W_{pq}(T)$. □

4. Extremal solutions

In this section by strengthening our hypotheses on the function α_k, assuming that f is Caratheodory and using ideas from Chipot-Rodrigues [6], we show that the problem has extremal solutions in the order interval $K = [\psi, \phi]$. A similar result was proved by Carl [3], but in his problem α_k is independent of x and his approach is different. Since we no longer allow f to be discontinuous, we can absorb f in the α_o function and so our problem becomes:

$$\left\{ \begin{array}{l} \frac{\partial x}{\partial t} - \sum_{k=1}^{N} D_k \alpha_k(t, , z, x, Dx) + \alpha_o(t, z, x, Dx) = 0 \text{ on } T \times Z\,. \\ x(0, z) = x_o(z) \text{ a.e. on } Z, x\,|_{T\times\Gamma} = 0\,. \end{array} \right\} \tag{25}$$

Our hypotheses on the functions α_k are the following:

$\mathbf{H}(\alpha)_1$: $\alpha_k : T \times Z \times R \times R^N \to R, k \in \{1, 2, ..., N\}$, are functions which satisfy $H(\alpha)(i) \to (iii)$ and

(iv) for all $(t, z) \in T \times Z$ and all $x \in R$ and all $y, y' \in R^N$ we have

$$\sum_{k=1}^{N} (\alpha_k(t, z, x, y) - \alpha_k(t, z, x, y'))(y_k - y_k') \geq c_o||y - y'||^p$$

where $c_o > 0$;

(v) for almost all $(t, z) \in T \times Z$, all $x \in R$ and all $y \in R^N$ we have

$$\sum_{k=1}^{N} \alpha_k(t, z, x, y) y_k \geq c_1||y||^p - \beta_1(t, z)$$

where $\beta_1 \in L^1(T \times Z), c_1 > 0$;

(vi) for almost all $(t, z) \in T \times Z$, all $x, x' \in R$ and all $y \in R^N$, we have

$$|\alpha_k(t, z, x, y) - \alpha_k(t, z, x', y)| \leq [\eta(t, z) + |x'|^{p-1} + |x|^{p-1} + ||y||^{p-1}]\omega(|x - x'|)$$

where $\eta \in L^q(T \times Z)$ and $\omega : R_+ \to R_+$ is the modulus of continuity satisfying $\int_{0+}^{\infty} \frac{dr}{\omega^q(r)} = \infty$.

Remark. Hypothesis $H(vi)$ was introduced by Chipot-Rodrigues [6]. In that paper the authors use it to prove new comparison and uniqueness results for boundary value problems with and without obstacles. Note that hypothesis $H(\alpha)_1(vi)$ is satisfied for example when $\omega(r) = \widehat{c} r^{\frac{1}{q}}$ with $\widehat{c} > 0$ (i.e. $\alpha_k(t, z, \cdot, y)$ is Holder continuous).

The next proposition is crucial in establishing the existence of extremal solution in the order interval $K = [\psi, \psi]$. Recall that a subset C of a vector lattice (Riesz space) is "directed upward" (resp. "directed downward "), if for each pair $a, b \in C$ there exists some $c \in A$ satisfying $max\{a, b\} \leq c$ (resp. $min\{a, b\} \geq c$).

Proposition 4. If hypotheses $H(\alpha)_1, H(\alpha_o), H_o$ hold and $x_o \in L^2(Z)$, **then** $S(x_o)$ directed upward and downward.

Proof. Let $x_1, x_2 \in S(x_o)$ and set $y = x_1 \vee x_2 = max\{x_1, x_1\} \in L^p(T, W_o^{1,p}(Z))$. We show that y is a lower solution.

By virtue of hypothesis $H(\alpha)_1(vi)$, given $\varepsilon > 0$ we can find $\delta(\varepsilon) \in (0, \varepsilon)$ such that $\int_{\delta(\varepsilon)}^{\varepsilon} \frac{dr}{\omega^q(r)} = 1$. Then we introduce the function

$$\xi_\varepsilon(r) = \begin{cases} 0 & \text{if} \quad r < \delta(\varepsilon) \\ \int_{\delta(\varepsilon)}^{r} \frac{ds}{\omega^q(s)} & \text{if} \quad \delta(\varepsilon) \leq r \leq \varepsilon \\ 1 & \text{if} \quad \varepsilon < r \end{cases}$$

(see Chipot-Rodrigues [6]). Evidently the function $\xi_\varepsilon(\cdot)$ is Lipschitz continuous, nondecreasing and $\xi_\varepsilon \to \chi_{\{r>0\}}$ as $\varepsilon \downarrow 0$. Moreover, we have

$$\xi_\varepsilon^{'}(r) = \begin{cases} \frac{1}{\omega^q(r)} & \text{for} \quad \delta(\varepsilon) < r < \varepsilon \\ 0 & \quad \text{otherwise} \end{cases}$$

Let $\theta \in C_o^\infty(T \times Z), \theta \geq 0$ and set

$$\theta_1^\varepsilon = (1 - \xi_\varepsilon(x_2 - x_1))\theta \text{ and } \theta_2^\varepsilon = \xi_\varepsilon(x_2 - x_1)\theta, \theta_1^\varepsilon, \theta_2^\varepsilon \geq 0\,.$$

Using the chain rule for Sobolev functions (see for example Kesavan [15], Appendix 4) we obtain

$$D_k\theta_1^\varepsilon = D_k\theta - \xi_\varepsilon^{'}(x_2 - x_1)D_k(x_2 - x_1)\theta - \xi_\varepsilon(x_2 - x_1)D_k\theta$$
$$\text{and} \quad D_k\theta_2^\varepsilon = \xi_\varepsilon^{'}(x_2 - x_1)D_k(x_2 - x_1)\theta + \xi_\varepsilon(x_2 - x_1)D_k\theta, \quad k \in \{1, 2, ..., N\}$$

Since by hypothesis $x_1, x_2 \in S(x_o)$, we have

$$((\dot{x}_1, \theta_1^\varepsilon)) + ((\widehat{A}_1(x_1), \theta_1^\varepsilon)) + (\widehat{h}(x_1), \theta_1^\varepsilon)_{pq} = 0 \tag{26}$$

$$\text{and } ((\dot{x}_2, \theta_2^\varepsilon)) + ((\widehat{A}_1(x_2), \theta_2^\varepsilon)) + (\widehat{h}(x_2), \theta_2^\varepsilon)_{pq} = 0 \tag{27}$$

We have

$$((\dot{x}_1, \theta_1^\varepsilon)) \to ((\dot{x}_1, \chi_{\{x_1 \geq x_2\}}\theta)) \text{ and } ((\dot{x}_2, \theta_2^\varepsilon)) \to ((\dot{x}_2, \chi_{\{x_2 > x_1\}}\theta)) \text{ as } \varepsilon \downarrow 0\,.$$

Note that $((\dot{x}_1, \theta_1^\varepsilon)) + ((\dot{x}_2, \theta_2^\varepsilon)) \to ((\dot{y}, \theta))$ as $\varepsilon \downarrow 0$.
Also we have

$$
\begin{aligned}
(28) \qquad & ((\widehat{A}_1(x_1), \theta_1^\varepsilon)) + ((\widehat{A}_1(x_2), \theta_2^\varepsilon)) \\
= \ & \int_o^b \int_Z \sum_{k=1}^N \alpha_k(t, z, x_1, Dx_1) \\
& (D_k\theta - \xi_\varepsilon^{'}(x_2 - x_1)D_k(x_2 - x_1)\theta - \xi_\varepsilon(x_2 - x_1)D_k\theta)dzdt \\
& + \int_o^b \int_Z \sum_{k=1}^N \alpha_k(t, z, x_2, Dx_2) \\
& (\xi_\varepsilon^{'}(x_2 - x_1)D_k(x_2 - x_1)\theta + \xi_\varepsilon(x_2 - x_1)D_k\theta))dzdt
\end{aligned}
$$

We examine the terms containing the expression $\xi_\varepsilon^{'}(x_2 - x_1)D_k(x_2 - x_1)\theta$. We have

$$
\begin{aligned}
(29) \qquad & \int_o^b \int_Z \sum_{k=1}^N (\alpha_k(t, z, x_2, Dx_2) - \alpha_k(t, z, x_1, Dx_1)) \\
& \xi_\varepsilon^{'}(x_2 - x_1)D_k(x_2 - x_1)\theta dzdt \\
= \ & \int_o^b \int_Z \sum_{k=1}^N (\alpha_k(t, z, x_2, Dx_2) - \alpha_k(t, z, x_2, Dx_1)) \\
& \xi_\varepsilon^{'}(x_2 - x_1)D_k(x_2 - x_1)\theta dzdt \\
& + \int_o^b \int_Z \sum_{k=1}^N (\alpha_k(t, z, x_2, Dx_1) - \alpha_k(t, z, x_1, Dx_1)) \\
& \xi_\varepsilon^{'}(x_2 - x_1)D_k(x_2 - x_1)\theta dzdt\,.
\end{aligned}
$$

Because of hypothesis $H(\alpha)_1(iv)$ and since $0 \leq \xi_\varepsilon^{'} \leq 1,\ \theta \geq 0$, we have

$$
\begin{aligned}
(30) \qquad & \int_o^b \int_Z \sum_{k=1}^N (\alpha_k(t, z, x_2, Dx_2) - \alpha_k(t, z, x_2, Dx_1)) \\
& \xi_\varepsilon^{'}(x_2 - x_1)D_k(x_2 - x_1)\theta dzdt \\
\geq \ & c_o \int_o^b \int_Z ||D(x_2 - x_1)||^p \xi_\varepsilon^{'}(x_2 - x_1)\theta dzdt\,.
\end{aligned}
$$

Also using hypothesis $H(\alpha)_1(vi)$ and Young's inequality we obtain

$$
\begin{aligned}
(31)\qquad & \Big|\int_o^b \int_Z \sum_{k=1}^N (\alpha_k(t,z,x_2,Dx_1) - \alpha_k(t,z,x_1,Dx_1)) \\
& \xi_\varepsilon^{'}(x_2-x_1) D_k(x_2-x_1)\theta dz dt\Big| \\
\leq\ & \int_o^b \int_Z \sum_{k=1}^N (\eta(t,z) + |x_1|^{p-1} + |x_2|^{p-1} + ||Dx_1||^{p-1}) \\
& \omega(|x_2-x_1|)|D_k(x_2-x_1)|\xi_\varepsilon^{'}(x_2-x_1)\theta dz dt \\
\leq\ & \frac{c_o}{2}\int_o^b \int_Z ||D(x_2-x_1)||^p \xi_\varepsilon^{'}(x_2-x_1)\theta dz dt \\
& +k\int \int_{\{\delta(\varepsilon)<x_2-x_1<\varepsilon\}} (\eta(t,z) + |x_1|^{p-1} + |x_2|^{p-1} + ||Dx_1||^{p-1})^q \theta dz dt\,,
\end{aligned}
$$

(here $k > 0$ and we have used Young's inequality $ab \leq \frac{\varepsilon^p}{p}|a|^p + \frac{1}{\varepsilon^q q}|b|^q \quad a,b \in R$ with appropriately chosen $\varepsilon > 0$ so that we produce the coefficient $\frac{c_o}{2}$ in the first summand (actually any coefficient strictly less than c_o will do the job); also we have used the fact that $\xi_\varepsilon^{'}(r) = \frac{1}{\omega^q(r)}$ for $\delta(\varepsilon) < r < \varepsilon$).

We use (30) and (31) in (29). So we have

$$
\begin{aligned}
(32)\qquad & \int_o^b \int_Z \sum_{k=1}^N (\alpha_k(t,z,x_2,Dx_2) - \alpha_k(t,z,x_1,Dx_1)) D_k(x_2-x_1) \\
& \xi_\varepsilon^{'}(x_2-x_1)\theta dz dt \\
\geq\ & \frac{c_o}{2}\int_o^b \int_Z ||D(x_2-x_1)||^p \xi_\varepsilon^{'}(x_2-x_1)\theta dz dt \\
& -k\int \int_{\{\delta(\varepsilon)<x_2-x_1<\varepsilon\}} (\eta(t,z) + |x_1|^{p-1} + |x_2|^{p-1} + ||Dx_1||^{p-1})^q \theta dz dt \\
\geq\ & -k\int \int_{\{\delta(\varepsilon)<x_2-x_1<\varepsilon\}} (\eta(t,z) + |x_1|^{p-1} + |x_2|^{p-1} + ||Dx_1||^{p-1})^q \theta dz dt
\end{aligned}
$$

Going back to (28) and using (32) we obtain

$$
\begin{aligned}
& ((\widehat{A}_1(x_1), \theta_1^\varepsilon)) + ((\widehat{A}_2(x_2), \theta_2^\varepsilon)) \\
\geq\ & \int_o^b \int_Z \sum_{k=1}^N \alpha_k(t,z,x_1,Dx_1) D_k \theta dz dt
\end{aligned}
$$

$$+\int_o^b\int_Z\sum_{k=1}^N(\alpha_k(t,z,x_2,Dx_2)-\alpha_k(t,z,x_1,Dx_1))\xi_\varepsilon(x_2-x_1)D_k\theta dzdt$$

$$-k\int\int_{\{\delta(\varepsilon)<x_2-x_1<\varepsilon\}}(\eta(t,z)+|x_1|^{p-1}+|x_2|^{p-1}+||Dx_1||^{p-1})\theta dzdt$$

Passing to the limit as $\varepsilon\downarrow 0$ on the right hand side of this inequality and since $|\{\delta(\varepsilon)<x_2-x_1<\varepsilon\}|\to 0$ as $\varepsilon\downarrow 0$, we have

$$\begin{aligned}
&\int_o^b\int_Z\sum_{k=1}^N\alpha_k(t,z,x_1,Dx_1)D_k\theta dzdt\\
&+\int_o^b\int_Z\sum_{k=1}^N(\alpha_k(t,z,x_2,Dx_2)-\alpha_k(t,z,x_1,Dx_1)\chi_{\{x_2>x_1\}}D_k\theta dzdt\\
=&\int_o^b\int_Z\sum_{k=1}^N\alpha_k(t,z,x_1,Dx_1)\chi_{\{x_1\geq x_2\}}D_k\theta dzdt\\
&+\int_o^b\int_Z\sum_{k=1}^N\alpha_k(t,z,x_2,Dx_2)\chi_{\{x_2>x_1\}}D_k\theta dzdt\\
=&\int_o^b\int_Z\sum_{k=1}^N\alpha_k(t,z,y,Dy)D_k\theta dzdt=((\widehat{A}_1(y),\theta))
\end{aligned}$$

Moreover, we have

$$(\widehat{h}(x_1),\theta_1^\varepsilon)_{pq}+(\widehat{h}(x_2),\theta_2^\varepsilon)pq\to(\widehat{h}(y),\theta)_{pq}\text{ as }\varepsilon\downarrow 0\,.$$

Hence from (26) and (27) and the above limits, we infer that

$$\begin{aligned}
&((\dot{y},\theta))+((\widehat{A}_1(y),\theta))+(\widehat{h}(y),\theta)_{pq}\leq 0\\
\Rightarrow\quad&((\frac{\partial y}{\partial t},\theta))+\alpha(y,\theta)+\int_o^b\int_Z\alpha_o(t,z,y,Dy)\theta dzdt\leq 0
\end{aligned}$$

Since $\theta\in C_o^\infty(T\times Z)_+$ was arbitrary and $C_o^\infty(T\times Z)_+$ is dense in $L^p(T,W_o^{1,p}(Z))\cap L^p(T\times Z)_+$, we deduce that $y\in W_{pq}(T)$ is a lower solution.

Then by considering truncation and penalty functions for the pair $\{y,\phi\}$, as in theorem 4 (via an auxiliary problem like (3)), we obtain a solution $x\in[y,\phi]$. So $S(x_o)$ is directed upward. Similarly by taking $y=x_1\wedge x_2=min\{x_1,x_2\}$ and by showing that in this case that y is an upper solution, we have that $S(x_o)$ is directed downward. □

Using this proposition, we can now prove the existence of extremal solutions in $K=[\psi,\phi]$, for problem (25).

Theorem 2. If hypotheses $H(\alpha)_1,H(\alpha_o),H_o$ hold and $x_o\in L^2(Z)$, **then** problem (25) has extremal solutions in $K=[\psi,\phi]$.

Proof. Let C be a chain of $S(x_o)$ and let $x = supC$. By virtue of corollary 7,p. 336 of Dunford-Schwartz [9] we can find $\{x_n\}_{n\geq 1} \subseteq C$ such that $x_n \to x$ in $L^p(T \times Z)$ as $n \to \infty$. Also from theorem 4 we know that $x_n \to x$ in $W_{pq}(T)$ as $n \to \infty$. Since $\dot{x}_n + \widehat{A}_1(x_n) + \widehat{h}(x_n) = 0, n \geq 1$, in the limit as $n \to \infty$ we have $\dot{x} + \widehat{A}_1(x) + \widehat{h}(x) = 0, x(0,z) = x_o(z)$ a.e. on Z. Thus $x \in S(x_o)$. By Zorn's lemma $S(x_o)$ has a maximal element $x^* \in S(x_o)$. Proposition 5 implies that $x \leq x^*$ for all $x \in S(x_o)$. Similarly exploiting the fact that $S(x_o)$ is directed downward we can show that there exists $x_* \in S(x_o)$ such that $x_* \leq x$ for all $x \in S(x_o)$. Evidently $x_*, x^* \in S(x_o)$ are the desired extremal solutions in K. $\square$

Remark. To extend theorem 2 to problems with discontinuities, i.e. to problem (2), we need to weaken the notions of upper and lower solutions for such problems. Indeed we need to replace in the definition of upper (resp. lower) solution $f_2(t,z,\phi)$ (resp. $f_1(t,z,\psi)$)) by v_2 (resp. v_1) in $L^q(T \times Z)$ such that

$$f_1(t,z,\phi(t,z)) \leq v_2(t,z) \leq f_2(t,z,\phi(t,z)) \text{ a.e. on } T \times Z$$

$$(\text{resp } f_1(t,z,\psi(t,z)) \leq v_1(t,z) \leq f_2(t,z,\psi(t,z)) \text{ a.e. on } T \times Z)\,.$$

These new definitions lead to a new auxiliary problem with different multivalued right hand side. Some ideas in this direction can be found in the work of Halidias-Papageorgiou [13] on multivalued ordinary differential equations of second order.

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