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SOME GENERIC PROPERTIES OF NONLINEAR SECOND ORDER DIFFUSIONAL TYPE PROBLEM

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Abstract. We are interested of the Newton type mixed problem for the general second order semilinear evolution equation. Applying Nikolskij's decomposition theorem and general Fredholm operator theory results, the present paper yields sufficient conditions for generic properties, surjectivity and bifurcation sets of the given problem.

Introduction

Different problems describing dynamics of mechanical processes (bendding, vibration), physical-heating processes, reaction-diffusion processes in chemical and biological technologies or in the ecology are modelled by nonstationary parabolic or general evolution equations. The study of qualitative and quantitative properties of these models has a significant importance for the analysis of these processes.

In the present paper we deal with the set structure of classic solutions, bifurcation properties and surjectivity of operators associated to the given nonlinear evolution problems.

Recently such questions were studied by the authors L. Brüll and J. Mawhin in [1] and [5] and V. Šeda in [6] for ordinary differential equations and operator equations.

In the first part of this paper we formulate the Newton mixed problem for a semilinear evolution equation and there are presented general Fredholm operator results which will be applied in the other parts. The second part contains three fundamental lemmas proving that a linear operator is a Fredholm one, a Nemitskij operator is completely continuous and the Fredholm nonlinear operator is coercive. These lemmas are substantially employed for the investigation of the set structure and bifurcations of solutions of the given problem in the last section.

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1. The formulation of problem and basic results

Throughout this paper we assume that the set $\Omega \subset R^n$ for $n \in N$ is a bounded domain with the sufficiently smooth boundary $\partial\Omega =: S$. The real number T is positive and $Q := (0,T] \times \Omega, \Gamma := [0,T] \times S$.

We use the notation D_t for $\partial/\partial t$ and D_i for $\partial/\partial x_i$ and D_{ij} for $\partial^2/\partial x_i \partial x_j$ where $i,j = 1,\dots,n$ and $D_0 u$ for u. The symbol $\mathrm{cl}\, M$ means closure of the set M in R^n.

We consider the nonlinear differential equation (it does not need to be of a parabolic type)

$$D_t u - A(t,x,D_x)u + f(t,x,u,D_1 u,\dots,D_n u) = g(t,x) \tag{1.1}$$

for $(t,x) \in Q$, where the coefficients a_{ij}, a_i, a_0 for $i,j = 1,\dots,n$ of the second order linear operator

$$A(t,x,D_x)u = \sum_{i,j=1}^{n} a_{ij}(t,x) D_{ij} u + \sum_{i=1}^{n} a_i(t,x) D_i u + a_0(t,x)u$$

are continuous functions from the space $C(\mathrm{cl}\, Q, R)$. The function f is from the space $C(\mathrm{cl}\, Q \times R^{n+1}, R)$ and $g \in C(\mathrm{cl}\, Q, R)$.

Together with the equation (1.1) we consider the following homogeneous second boundary condition (Newton or for $b_0 = 0$ Neumann condition)

$$B_2(t,x,D_x)u|_\Gamma := \partial u/\partial\nu + b_0(t,x)u|_\Gamma = 0\,, \tag{1.2}$$

where $\nu := (0,\nu_1,\dots,\nu_n) : \mathrm{cl}\,\Gamma \to R^{n+1}$ is a vector function for which the value $\nu(t,x)$ means the inner normal vector to $\mathrm{cl}\,\Gamma$ at the point $(t,x) \in \mathrm{cl}\,\Gamma$ and $\partial/\partial\nu$ means derivative with respect to the normal ν. Here the coefficient b_0 is a function from $C(\mathrm{cl}\,\Gamma, R)$.

Together with the boundary condition we require for the solution of (1.1) to satisfy the homogeneous initial condition

$$u|_{t=0} = 0 \quad on \quad \mathrm{cl}\ \Omega\,. \tag{1.3}$$

We shall use the notation

$$< u >^{s}_{t,\mu,Q} := \sup_{\substack{(t,x)\in \mathrm{cl}\, Q \\ t\neq s}} \frac{|u(t,x) - u(s,x)|}{|t-s|^{\mu}}\,, \tag{1.4}$$

$$< u >^{y}_{x,\nu,Q} := \sup_{\substack{(t,x)\in \mathrm{cl}\, Q \\ t\neq s}} \frac{|u(t,x) - u(t,y)|}{|x-y|^{\nu}}\,. \tag{1.5}$$

where $x = (x_1,\dots,x_n), y = (y_1,\dots,y_n)$ are from R^n, $|x-y| = \left[\sum_{i=1}^{n}(x_i - y_i)^2\right]^{1/2}$ and $\mu,\nu \in R$.

In our considerations we need the uniform parabolicity of a operator of the type $D_t - A(t,x,D_x)$ (see S.D. Ivasisen [4], p. 12).

Definition 1.1. (The uniform parabolicity condition (P).) We say that the differential operator

$$D_t - A(t,x,D_x)$$

is *uniform parabolic on* $\mathrm{cl}\, Q$ in the sense of I.G.Petrovskij with the constant δ or shortly, the operator satisfies *the parabolicity condition (P)* if there is a constant $\delta > 0$ such that for all $(t,x) \in \mathrm{cl}\ Q$ and each $\sigma = (\sigma_1, \dots, \sigma_n) \in R^n$ the inequality

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\sigma_i\sigma_j \geq \delta \left[\sum_{i=1}^{n} \sigma_i^2\right]$$

holds.

The concept of a locally smooth boundary of domain is given in the following definition.

Definition 1.2. Let $r \in (1,\infty)$ and $\Omega \subset R^n$ be a bounded domain. We say that *the boundary* $\partial\Omega$ *belongs to the class* $C^r, r > 1$ if:

(i) There exists a tangential space to $\partial\Omega$ in any point from boundary $\partial\Omega$.

(ii) Assume $y \in \partial\Omega$ and let $(y, z_1, \dots, z_n)$ be a local orthonormal coordinate system with the center y and with the axis z_n oriented like the inner normal to $\partial\Omega$ at the point y. Then there exists a number $b > 0$ such that for every $y \in \partial\Omega$ there exists a neighbourhood $O(y) \subset R^n$ of the point y and a function $F \in C^r(\mathrm{cl}\, B, R)$ such that the part of boundary

$$\partial\Omega \cap O(y) = \{(z', F(z')) \in R^n, z' = (z_1, \dots, z_{n-1}) \in B\} =: S(y)\,,$$

where $B = \{z' \in R^{n-1}; |z'| < b\}$.

Here $C^r(\mathrm{cl}\, B, R)$ is a vector space of the functions $u \in C^l(\mathrm{cl}\, B, R)$ for $l = [r]$ with the finite norm

$$||u||_{l+\alpha} = \sum_{0 \leq k \leq l} \sup_{x \in clB} \left|D_x^k u(x)\right| + \sum_{k=l} \left\langle D_x^k u\right\rangle_{x,\alpha,B}^{y}\,,$$

whereby $\alpha = r - [r] \in [0,1)$ and $r = l + \alpha$.

Further, we shall need the following Hölder spaces (see [2], p. 147).

Definition 1.3. Let $\alpha \in (0,1)$

1. By the symbol $C_{t,x}^{(1+\alpha)/2,1+\alpha}(\mathrm{cl}\, Q, R)$ we denote the vector space of continuous functions $u : \mathrm{cl}\, Q \to R$ which have continuous derivatives $D_i u$ for $i = 1, \dots, n$ on $\mathrm{cl}\, Q$ and the norm

$$(1.6)\qquad \begin{aligned} ||u||_{(1+\alpha)/2,1+\alpha,Q} :=& \sum_{i=0}^{n} \sup_{\mathrm{cl}\, Q} |D_i u(t,x)| + < u >_{t,(1+\alpha)/2,Q}^{s} \\ &+ \sum_{i=1}^{n} \langle D_i u\rangle_{t,\alpha/2,Q}^{s} + \sum_{i=1}^{n} \langle D_i u\rangle_{x,\alpha/2,Q}^{y} \end{aligned}$$

is finite.

2. The symbol $C^{(2+\alpha)/2,2+\alpha}_{(t,x)}(\mathrm{cl}\ Q, R)$ means the space of continuous functions $u : \mathrm{cl}\ Q \to R$ for which there exist continuous derivatives $D_t u, D_i u, D_{ij} u$ on $\mathrm{cl}\, Q,\ i,j = 1,\dots n$ and the norm

$$\begin{aligned}||u||_{(2+\alpha)/2,2+\alpha,Q} &= \sum_{i=0}^{n} \sup_{\mathrm{cl}\ Q} |D_i u(t,x)| + \sup_{\mathrm{cl}\ Q} |D_t u(t,x)| \\ &+ \sum_{i,j=1}^{n} \sup_{\mathrm{cl}\ Q} |D_{ij} u(t,x)| + \sum_{i=1}^{n} < D_i u >^s_{t,(1+\alpha)/2,Q} + < D_t u >^s_{t,\alpha/2,Q} \\ &+ \sum_{i,j=1}^{n} < D_{ij} u >^s_{t,\alpha/2.Q} + < D_t u >^y_{x,\alpha,Q} + \sum_{i,j=1}^{n} < D_{ij} u >^y_{x,\alpha,Q}\end{aligned} \tag{1.7}$$

is finite.

3. The symbol $C^{(3+\alpha)/2,3+\alpha}_{t,x}(\mathrm{cl}\, Q, R)$ means the vector space of continuous functions $u : \mathrm{cl}\, Q \to R$ for which the derivatives $D_t, D_i u, D_t D_i u, D_{ij} u, D_{ijk} u,\ i,j,k = 1,\dots,n$ are continuous on $\mathrm{cl}\, Q$ and the norm

$$\begin{aligned}||u||_{(3+\alpha)/2,3+\alpha,Q} :=& \sum_{i=0}^{n} \sup_{\mathrm{cl}\ Q} |D_i u(t,x)| + \sum_{i,j=1}^{n} \sup_{\mathrm{cl}\ Q} |D_{ij} u(t,x)| \\ &+ \sum_{i=0}^{n} \sup_{\mathrm{cl}\ Q} |D_t D_i u(t,x)| + \sum_{i,j,k=1}^{n} \sup_{\mathrm{cl}\ Q} |D_{ijk} u(t,x)| \\ &+ \langle D_t u\rangle^s_{t,(1+\alpha)/2,Q} + \sum_{i,j=1}^{n} \langle D_{ij} u\rangle^s_{t,(1+\alpha)/2,Q} \\ &+ \sum_{i=1}^{n} \langle D_t D_i u\rangle^s_{t,\alpha/2,Q} + \sum_{i,j,k=1}^{n} \langle D_{ijk} u\rangle^s_{t,\alpha/2,Q} \\ &+ \sum_{i=1}^{n} \langle D_t D_i u\rangle^y_{x,\alpha,Q} + \sum_{i,j,k=1}^{n} \langle D_{ijk} u\rangle^y_{x,\alpha,Q}\end{aligned} \tag{1.8}$$

is finite.

The previous norm spaces are Banach ones.

Also we need the Hőlder space of functions defined on the manifold $\mathrm{cl}\,\Gamma$ (see [4], p. 10).

Definition 1.4. Let the boundary $\partial\Omega =: S$ of a domain $\Omega \subset R^n$ belong to C^r for $r > 1$ (see Definition 1.2). We put $S_y := \partial\Omega \cap O(y)$ and $\Gamma_y = (0,T] \times S_y$ for $y \in \partial\Omega$, where $O(y)$ is a neighbourhood of the point y from Definition 1.2.

The symbol $C^{(2+\alpha)/2,2+\alpha}_{t,x}(\mathrm{cl}\,\Gamma, R)$ means the vector space of functions $u : \mathrm{cl}\,\Gamma \to R$ with the norm

$$||u||_{(2+\alpha)/2,2+\alpha,\Gamma} = \sup_{y\in S} ||u||_{(2+\alpha)/2,2+\alpha,\Gamma_y}\,,$$

where the norm on the right hand side of the last equality is defined by the formula (1.7) in which we write Γ_y instead of Q.

Definition 1.5. (The smoothness condition $(S_2^{1+\alpha})$.) Let $\alpha \in (0,1)$. We say that the differential operator $A(t,x,D_x)$ from (1.1) and $B_2(t,x,D_x)$ from (1.2), respectively satisfies *the smoothness condition* $(S_2^{1+\alpha})$ if

(i) the coefficients a_{ij}, a_i, a_0 from (3.1) for $i,j = 1,\dots,n$ belong to the space $C_{t,x}^{(1+\alpha)/2,1+\alpha}(\mathrm{cl}\, Q, R)$ and $\partial\Omega \in C^{3+\alpha}$ and

(ii) the coefficient b_0 from (1.2) belongs to the space $C_{t,x}^{(2+\alpha)/2,2+\alpha}(\mathrm{cl}\,\Gamma, R)$.

The classical and fundamental result for the solution of the second mixed problem (1.1), (1.2), (1.3) represents the following proposition (see [4], p. 21).

Proposition 1.1. *Let the assumptions (P), and $(S_2^{1+\alpha})$ be satisfied. The necessary and sufficient condition for the existence and uniqueness of the solution $u \in C_{t,x}^{(3+\alpha)/2,3+\alpha}(\mathrm{cl}\, Q, R)$ of the linear parabolic equation*

$$D_t u - A(t,x,D_x)u = f(t,x) \quad \text{on } \Omega\,,$$

where the operator A is from the equation (1.1) with the data (1.2), *(1.3) is $f \in C_{t,x}^{(1+\alpha)/2,1+\alpha}(\mathrm{cl}\, Q, R)$ and*

$$\frac{\partial f}{\partial \nu}(t,x) + b_0(t,x)f(t,x)|_{t=0,x\in S} = 0$$

Moreover, if this condition is satisfied then there exists a constant $K > 0$ independent of f such that

$$K^{-1}||f||_{(1+\alpha)/2,1+\alpha,Q} \le ||u||_{(3+\alpha)/2,3+\alpha,Q} \le K||f||_{(1+\alpha)/2,1+\alpha,Q}\,.$$

Proposition 1.2. *(S.M. Nikolskij [9], p. 233.) Let X and Y be Banach spaces either both real or complex. A linear bounded operator $A : X \to Y$ is Fredholm of the zero index if and only if $A = C + T$, where $C : X \to Y$ is a linear homeomorphism and $T : X \to Y$ is a linear completely continuous operator.*

Some frequent properties of the Fredholm operator are collected in the following propositions.

Proposition 1.3. *(V. Šeda [6], Theorem 3.1.) Let X and Y be Banach spaces over the same field of the real or complex numbers. If the assumptions*

(i) *$A : X \to Y$ is a linear bounded Fredholm operator of zero index and*

(ii) *$B : X \to Y$ is a completely continuous operator and*

(iii) *$F = A + B : X \to Y$ is a coercive operator*

hold, then:

(a) *For every $g \in Y$ the argument set $F^{-1}(\{g\})$ is compact (possibly empty);*

(b) *The range $R(F)$ of F is closed and connected in Y;*

(c) *The set* Σ *of all points* $u \in X$ *at which* F *is not locally invertible and the set of images* $F(\Sigma)$ *are closed subsets of* X *and* Y, *respectively and the set* $F(X - \Sigma)$ *is open in* Y;

(d) *The cardinal number* $card F^{-1}(\{g\})$ *is constant and finite (it may be zero) on each connected component of the open set* $Y - F(\Sigma)$;

(e) *If* $\Sigma = 0$, *then* F *is homeomorphism of* X *onto* Y.

Proposition 1.4. *(V. Šeda [6], Corollary 3.1, Corollary 3.3, Remark 3.1.) Let the assumptions (i), (ii), (iii) from Proposition 1.3. be fulfilled. Then each of the following conditions is sufficient for the surjectivity of* $F = A + B : X \to Y$:

(iv) $F(\Sigma) \subset F(X - \Sigma)$;

(v) $Y - F(\Sigma)$ *is a connected set and* $F(X - \Sigma) - F(\Sigma) \neq 0$;

(vi) *There exists a strict solvable field* $G = I - g : X \to X$ *and* $R > 0$ *such that each solution* $u \in X$ *of the equation*

$$F(u) = kC \circ G(x) \quad \textit{for all} \quad k < 0$$

satisfies the estimation $||u||_X < R$.

Here $A = C + T : X \to Y$, where C is a linear homeomorphism of X onto Y and $T : X \to Y$ is a linear completely continuous operator.

Also we say that $G = I - g : X \to X$ is a *strict solvable field* if it is a condensing field and there is a sequence $r_k \to \infty$ as $k \to \infty$ such that the degree of the mapping G $deg(G, U(0, r_k), 0) \neq 0$, where $U(0, r_k) \subset X$ is the sphere with the center 0 and the radius r_k for $k = 1, 2, \dots$

Proposition 1.5. *(V. Šeda [6], Corollary 3.3) Let* X *and* Y *be real Banach spaces and let* $F = A + B : X \to Y$. *Suppose that the hypotheses (i), (ii), (iii) from Proposition 1.3 and (vi) from Proposition 1.4 are satisfied. Here* $A = C + T : X \to Y$ *whereby* C *is a linear homeomorphism of* X *onto* Y *and* $T : X \to Y$ *is a linear completely continuous operator. Then*

(f) *The card* $F^{-1}(\{g\})$ *is constant, finite and different from zero on each component of the open set* $Y - F(\Sigma)$.

2. The fundamental lemmas

In the first lemma we establish sufficient conditions under which a linear differential operator will be the Fredholm type with the zero index.

Lemma 2.1. *Suppose*

(1.i) $\alpha \in (0, 1)$.

(1.ii) *Define the Hölder vector subspaces*

$$D(A_2) :=$$
$$\{u \in C^{(3+\alpha)/2, 3+\alpha}_{t,x}(\mathrm{cl}\, Q, R); \quad B_2(t, x, D_x)u|_\Gamma = 0, u|_{t=0}(x) = 0 \quad \textit{for } x \in \mathrm{cl}\, Q\}$$

and

$$H(A_2) := \{v \in C^{(1+\alpha)/2,1+\alpha}_{t,x}(\mathrm{cl}\, Q, R);\ B_2(t,x,D_x)v(t,x)|_{t=0,x\in S} = 0\}$$

and associated Banach spaces provided with the corresponding Hölder norms:

$$X_2 = \big(D(A_2), ||.||_{(3+\alpha)/2,3+\alpha,Q}\big)$$

and

$$Y_2 = \big(H(A_2), ||.||_{(1+\alpha)/2,1+\alpha,Q}\big)\ .$$

Assume that the operator $A_2 : X_2 \to Y_2$, *where*

$$A_2 u = D_t u - A(t,x,D_x),\ \ u \in X_2$$

and the operator $B_2(t,x,D_x)$ *satisfy the smoothness condition* $(S_2^{1+\alpha})$.

(1.iii) *There is a second order linear differential operator* $C_2 : X_2 \to Y_2$ *with*

$$C_2 u = D_t u - C(t,x,D_x)u,\ \ u \in X_2\ ,$$

where

$$C(t,x,D_x)u = \sum_{i,j=1}^{n} c_{ij}(t,x)D_{ij}u + \sum_{i=1}^{n} c_i(t,x)D_i u + c_0(t,x)u$$

satisfying the conditions of the parabolicity (P) and the smoothness $(S_2^{1+\alpha})$.

Then

(j) $\dim X_2 = +\infty$.

(jj) *The operator* $A_2 : X_2 \to Y_2$ *is a linear bounded Fredholm operator of the zero index.*

Proof. (j) To prove the first part of this lemma we use the decomposition theorem from [8], p. 139:

Let X be linear space and $x^* : X \to R$ be a linear functional on X such that $x^* \neq 0$. Further put $M = \{x \in X; x^*(x) = 0\}$ and $x_0 \in X - M$. Then every element $x \in X$ can be expressed by the formula

$$x = \left[\frac{x^*(x)}{x^*(x_0)}\right] x_0 + m\ , \quad m \in M\ ,$$

i.e. there is a one-dimensional subspace L_1 of X such that $X = L_1 \oplus M$.

If we put now

$$M_1 := \left\{u \in C^{(3+\alpha)/2,3+\alpha}_{t,x}(\mathrm{cl}\, Q, R) =: H^{3+\alpha}; B_2(t,x,D_x)|_\Gamma = 0\right\}\ ,$$

which is the linear subspace of $H^{3+\alpha}$, then there exists a linear subspace L_1 of $H^{3+\alpha}$ with $dim L_1 = 1$ such that $H^{3+\alpha} = L_1 \oplus M_1$. Similar, if we take $M_2 := \{u \in M_1; u|_{t=0} = 0$ *on* $\mathrm{cl}\, Q\}$, then there is a subspace L_2 of M_1 with $dim L_2 = 1$

such that $M_1 = L_2 \oplus M_2$. Hence, we have $H^{3+\alpha} = L_1 \oplus L_2 \oplus D(A_2)$. Since $dim H^{3+\alpha} = +\infty$ we get that $dim X_2 = +\infty$.

(jj) 1. In the first step we prove the boundedness of the linear operator A_2. For this aim we observe the norm $\|A_2 u\|_{(1+\alpha)/2,1+\alpha,Q}$ for $u \in D(A_2)$. From the assumption $(S_2^{1+\alpha})$ we get for $k = 0, 1, \ldots, n$

$$\sup_{\mathrm{cl}\, Q} |D_k A_2 u(t,x)| \le K_1 \|u\|_{(3+\alpha)/2,3+\alpha,Q}, \quad K_1 > 0\,. \tag{2.1}$$

Applying again the smoothness assumption $(S_2^{1+\alpha})$, the mean value theorem for the function u and $D_i u$ and the boundedness of Q we obtain for the second member of the above mentioned norm the following estimation:

$$\begin{aligned}\langle A_2 u\rangle^s_{t,(1+\alpha)/2,Q} &= \sup_{\mathrm{cl}\, Q, t\ne s} \frac{|A_2 u(t,x) - A_2 u(s,x)|}{|t-s|^{(1+\alpha)/2}} \\ &\le K_2 \|u\|_{(3+\alpha)/2,3+\alpha,Q}, \quad K_2 > 0\,.\end{aligned} \tag{2.2}$$

The third member of the norm (1.6) we estimate for $k = 1, \ldots, n$ as follows:

$$\begin{aligned}\langle D_k A_2 u\rangle^s_{t,\alpha/2,Q} &= \sup_{\mathrm{cl}\, Q, t\ne s} \frac{|D_k A_2 u(t,x) - D_k A_2 u(s,x)|}{|t-s|^{\alpha/2}} \\ &\le K_3 \|u\|_{(3+\alpha)/2,3+\alpha,Q}\,, \quad K_3 > 0\,.\end{aligned} \tag{2.3}$$

An estimation of the last member in (1.6) for $A_2 u$ is given by the following inequality for $k = 1, \ldots, n$

$$\begin{aligned}\langle D_k A_2 u\rangle^y_{x,\alpha/2,Q} &= \sup_{\mathrm{cl}\, Q, x\ne y} \frac{|D_k A_2 u(t,x) - D_k A_2 u(t,y)|}{|x-y|^{\alpha/2}} \\ &\le K_4 \|u\|_{(3+\alpha)/2,3+\alpha,Q}, \quad K_4 > 0\,.\end{aligned} \tag{2.4}$$

From the estimations (2.1), (2.2), (2.3) and (2.4) we can conclude that

$$\|A_2 u\|_{Y_2} = \|A_2 u\|_{(1+\alpha)/2,1+\alpha,Q} \le K(n, T, \alpha, \Omega, a_{ij}, a_i, a_0)\|u\|_{X_2}\,.$$

2. To prove that A_2 is a Fredholm operator with the zero index we express it in the form

$$A_2 u = C_2 u + [C(t,x,D_x) - A(t,x,D_x)]u =: C_2 u + Tu\,.$$

By the decomposition Nikolskij theorem from [9], p. 233, it is sufficient to show that $C_2 : X_2 \to Y_2$ is linear homeomorphism and $T : X_2 \to Y_2$ is the linear completely continuous operator.

The first requirement is a consequence of Proposition 1.1.

The complete continuity of T can be proved by the Ascoli-Arzela theorem (see [7], p. 141).

From $(S_2^{1+\alpha})$ the uniform boundedness of the operator

$$Tu = \sum_{i,j=1}^{n}[c_{ij}(t,x)-a_{ij}(t,x)]D_{ij}u+\sum_{i=1}^{n}[c_i(t,x)-a_i(t,x)]D_iu+[c_0(t,x)-a_0(t,x)]u$$

follows by the same way as the boundedness of operator A_2 in the part 1. Thus for all $u \in M \subset X_2$, where M is a set bounded by the constant $K_1 > 0$, we obtain the estimate

$$\|Tu\|_{Y_2} \le K(n, \alpha T, \Omega, a_{ij}, c_{ij}, a_i, c_i, a_0, c_0)\|u\|_{X_2} \le KK_1 \,.$$

Using the smoothness condition of the operators A and C we get inequality:

$$\begin{aligned}
|Tu(t,x) - Tu(s,y)| \le & \sum_{i,j=1}^{n} |[c_{ij} - a_{ij}](t,x) - [c_{ij} - a_{ij}](s,y)|\,|D_{ij}u(t,x)| \\
& + \sum_{i,j=1}^{n} |c_{ij}(s,y) - a_{ij}(s,y)|\,|D_{ij}u(t,x) - D_{ij}u(s,y)| \\
& + \sum_{i=1}^{n} |[c_i - a_i](t,x) - [c_i - a_i](s,y)|\,|D_iu(t,x)| \\
& + \sum_{i=1}^{n} |c_i(s,y) - a_i(s,y)|\,|D_iu(t,x) - D_iu(s,y)| \\
& + |[c_0 - a_0](t,x) - [c_0 - a_0](s,y)|\,|u(t,x)| \\
& + |c_0(s,y) - a_0(s,y)|\,|u(t,x) - u(s,y)| \\
\le & 4K_1Kn^2\left[|t-s|^{\alpha/2} + |x-y|^{\alpha}\right] \\
& + 2K_1Kn\left[\left(|t-s|^{\alpha/2} + |x-y|^{\alpha}\right) + \left(|t-s|^{(1+\alpha)/2} + |x-y|\right)\right] \\
& + 2K_1K\left[\left(|t-s|^{\alpha/2} + |x-y|^{\alpha}\right) + (|t-s| + |x-y|)\right] ,
\end{aligned}$$

where K_1, K are positive constants. Hence the equicontinuity of $TM \subset Y_2$ follows. This finishes the proof of Lemma 2.1.

The Lemma 2.1 implies the following alternative.

Corollary 2.1. *Let L be the set of all second order linear differential operators*

$$A_2 = D_t - A(t,x,D_x) : X_2 \to C_{t,x}^{(1+\alpha)/2,1+\alpha}(\mathrm{cl}\, Q, R)$$

satifying the condition $(S_2^{1+\alpha})$. Then for all $A_2 \in L$ the mixed homogeneous problem $A_2u = 0$ on Q, (1.2), (1.3) has a nontrivial solution or any $A_2 \in L$ is a linear bounded Fredholm operator of the zero index mapping X_2 onto Y_2.

Remark 2.1. The assumption (1.iii) of Lemma 2.1 is satisfied for the diffusion operator $C_2 : X_2 \to Y_2$, where

$$C_2 u = D_t u - \Delta u, \quad u \in X_2 .$$

Hence, we have very simple corollary of Lemma 2.1.

Corollary 2.2. *Every linear second order differential operator $A_2 : X_2 \to C_{t,x}^{(1+\alpha)/2, 1+\alpha}(\mathrm{cl}\, Q, R)$ satisfying the smoothness condition $(S_2^{1+\alpha})$ is a linear bounded Fredholm operator with the zero index.*

The following lemma establishes the complete continuity of the Nemitskij operator from the nonlinear part of the equation (1.1).

Lemma 2.2. *Let $\alpha \in (0,1]$ and*

(2.i) the function

$$f := f(t, x, u_0, u_1, \dots, u_n) : (\mathrm{cl}\, Q) \times R^{n+1} \to R$$

is locally Hőlder continuous on $(\mathrm{cl}\, Q) \times R^{n+1}$ in the variables t, i.e. for any compact set $D \subset (\mathrm{cl}\, Q) \times R^{n+1}$ there exists nonnegative constant p such that

$$|f(t, x, u_0, u_1, \dots, u_n) - f(s, x, u_0, u_1, \dots, u_n)| \le p|t - s|^{(1+\alpha)/2} \tag{2.5}$$

for all $(t, x, u_0, u_1, \dots, u_n)$ and $(s, x, u_0, u_1, \dots, u_n)$ from D.

(2.ii) The derivatives $\partial f / \partial x_i : (\mathrm{cl}\, Q) \times R^{n+1} \to R$ for $i = 1, \dots, n$ and $\partial f / \partial u_j : (\mathrm{cl}\, Q) \times R^{n+1} \to R$ for $j = 0, 1, \dots, n$ are locally Hőlder continuous on $(\mathrm{cl}\, Q) \times R^{n+1}$ i.e. the inequalities

$$\begin{aligned} &|\partial f/\partial x_i (t, x, u_0, u_1, \dots, u_n) - \partial f/\partial x_i (s, y, v_0, v_1, \dots, v_n)| \\ &\qquad \le p|t-s|^{\alpha/2} + q|x-y|^\alpha + \sum_{j=0}^{n} p_j |u_j - v_j| \end{aligned} \tag{2.6}$$

for $i = 1, \dots, n$ and

$$\begin{aligned} &|\partial f/\partial u_j (t, x, u_0, u_1, \dots, u_n) - \partial f/\partial u_j (s, y, v_0, v_1, \dots, v_n)| \\ &\qquad \le p|t-s|^{\alpha/2} + q|x-y|^\alpha + \sum_{l=0}^{n} p_l |u_l - v_l| \end{aligned} \tag{2.7}$$

for $j = 1, \dots, n$ hold on any compact subset of $(\mathrm{cl}\, Q) \times R^{n+1}$ with the nonnegative constants $p, q, p_j, j = 0, \dots, n$.

(2.iii) The equality

$$\frac{\partial}{\partial \nu} f(t, x, 0, \dots, 0) + b_0(t, x) f(t, x, 0, \dots, 0)|_{t=0, x \in S} = 0 \tag{2.8}$$

holds.

Then the Nemitskij operator $N_2 : X_2 \to Y_2$ *defined by*

$$(N_2u)(t,x) = f\,[t, x, u(t,x), D_1u(t,x), \ldots, D_nu(t,x)]$$

for $u \in X_2$ *and* $(t,x) \in \mathrm{cl}\,Q$, *where* X_2 *and* Y_2 *are the Banach spaces from Lemma 2.1, is completely continuous.*

Proof. Let $M_2 \subset X_2$ be a bounded set. By the Ascoli-Arzela theorem it is sufficient to show that the set $N_2(M_2)$ is uniform bounded and equicontinuous.

Take $u \in M_2$. According to the assumption (2.5) and (2.6) we obtain the locally boundedness of the function f and its derivatives $\partial f/\partial x_i$ on $(\mathrm{cl}\,Q) \times R^{n+1}$ for $i = 1, \ldots, n$. Hence and from the equation

$$D_i(N_2u)(t,x) = \left\{ D_if[.] + \sum_{l=0}^{n} D_lf[.]D_iD_lu \right\} [.,.,u, D_1u, \ldots, D_nu]\,(t,x)$$

we have the estimation

$$\sup_{\mathrm{cl}\,Q} |D_i(N_2u)(t,x)| \le K_1$$

for $i = 0, 1, \ldots, n$ with a positive sufficiently large constant K_1 not depending of $u \in M_2$.

Using the inequality (2.5) and the mean value theorem in the variable t for the difference of the derivatives of u we can write with respect to (1.8)

$$\langle N_2u\rangle^s_{t,(1+\alpha)/2,Q} \le K_1 .$$

Similarly by (2.6) and (2.7) we have

$$\langle D_iN_2u\rangle^s_{t,\alpha/2,Q} \le K_1$$

and

$$\langle D_iN_2u\rangle^y_{x,\alpha,Q} \le K_1$$

for $i = 1, \ldots, n$ for $u \in M_2$. The previous estimations yield the inequality

$$||N_2u||_{Y_2} \le K_2, \quad K_2 > 0$$

for all $u \in M_2$.

With respect to (2.5) for any $u \in M_2$ and $(t,x), (s,y) \in \mathrm{cl}\,Q$ such that $|t-s|^2 + |x-y|^2 < \delta^2$ with a sufficiently small $\delta > 0$ we have

$$|N_2u(t,x) - N_2u(s,y)| < \epsilon\,, \quad \epsilon > 0\,,$$

which is the equicontinuity of $N_2(M_2)$. This finishes the proof of Lemma 2.2.

Remark 2.2. With respect to the local Hölder continuity the function f can have an arbitrarily strong growth, for example exponential one.

The following lemma deals with the coercivity of nonlinear problem (1.1), (1.2), (1.3).

Lemma 2.3. *Let*

(3.i) *the operator* $A_2 : X_2 \to Y_2$ *satisfy the assumptions of Lemma 1.1 and let*

(3.ii) *the Nemitskij operator* $N_2 : X_2 \to Y_2$ *keeps the hypotheses of Lemma 2.2.*

(3.iii) *For any bounded set* $M_2 \subset Y_2$ *there exists a positive constant* K *such that for every solution* u *of the problem (1.1), (1.2), (1.3) with* $g \in M_2$, *one of the following conditions holds:*

(a)
$$||u||_{(1+\alpha)/2,1+\alpha,Q} \le K,$$
$$f := f(t,x,u_0) : (\mathrm{cl}\, Q) \times R \to R$$

and coefficients of the operators A_2 *and* C_2 *from Lemma 1.1 satisfy the relations*

$$a_{ij} = c_{ij}, a_i = c_i \quad \text{for } i,j = 1,\dots,n,\ a_0 \ne c_0 \text{ on } \mathrm{cl}\, Q$$

or

(b)
$$||u||_{(2+\alpha)/2,2+\alpha,Q} \le K, \text{ and}$$
$$f:\ f(t,x,u_0,u_1,\dots,u_n) : (\mathrm{cl}\, Q) \times R^{n+1} \to R$$

and the coefficients of operators A_2 *and* C_2 *satisfy the relations:* $a_{ij} = c_{ij}$ *for* $i,j = 1,\dots,n$ *and* $a_i \ne c_i$ *for at least one* $i = 1,\dots n$ *on* $\mathrm{cl}\, Q$.

Then the operator $F_2 := A_2 + N_2 : X_2 \to Y_2$ *is coercive.*

Proof. We need prove that if the set $M_2 \subset Y_2$ is bounded in Y_2, then the set of arguments $F^{-1}(M_2) \subset X_2$ is bounded in X_2.

In the both cases (a) and (b) we get for all $u \in F^{-1}(M_2)$

$$||N_2u||_{(1+\alpha)/2,1+\alpha,Q} \le K_1\,,$$

where $K_1 > 0$ is a sufficiently large constant. Hence and from inequality

$$||F_2u||_{Y_2} \ge ||A_2u||_{Y_2} - ||N_2u||_{Y_2}$$

we have

$$||A_2u||_{Y_2} \le K_2\,, \quad K_2 > 0$$

for $u \in F^{-1}(M_2)$.

The hypothesis (1.iii) of Lemma 1.1 ensures the existence and uniqueness of the solution $u \in X_2$ of the linear parabolic problem with the conditions (1.2), (1.3) for the equation

$$C_2u = y$$

and for any $y \in Y_2$ the estimation

(2.9)
$$||u||_{X_2} \le K_3||y||_{Y_2}, \quad K_3 > 0\,.$$

If we write

$$C_2u = A_2u + \sum_{i,j=1}^{n}[a_{ij}(t,x) - c_{ij}(t,x)]D_{ij}u$$

$$+\sum_{i=1}^{n}[a_i(t,x) - c_i(t,x)]D_iu + [a_0(t,x) - c_0(t,x)]u$$

then in the both cases and for each $u \in F^{-1}(M_2)$ we obtain

$$||y||_{Y_2} \leq ||C_2u||_{Y_2} \leq K_2 + K_4||u||_{X_2} \leq K_2 + K_4K, \quad K_4 > 0$$

whence by the inequality (2.9) we can conclude that the operator F_2 is coercive.

3. The generic properties

The assertions from the second section are three basic hypotheses required in our following considerations.

Let us start with the definition of bifurcation point.

Definition 3.1. 1. A couple $(u, g) \in X_2 \times Y_2$ will be called **the bifurcation point of the mixed problem** (1.1), (1.2), (1.3) if u is a solution of that mixed problem and there exists a sequence $\{g_k\} \subset Y_2$ such that $g_k \to g$ in Y_2 as $k \to \infty$ and the problem (1.1), (1.2), (1.3) for $g = g_k$ has at least two different solutions u_k, v_k for each $k \in N$ and $u_k \to u, v_k \to u$ in X_2 as $k \to \infty$.

2. The set of all solutions $u \in X_2$ of (1.1), (1.2), (1.3) (or the set of all functions $g \in Y_2$) such that (u, g) is a bifurcation point of the mixed problem (1,1), (1,2), (1,3) will be called **the domain of bifurcation (the bifurcation range)** of that mixed problem.

Using the notations of the previous part we immediately obtain the following equivalence lemma:

Lemma 3.1. *Let $A_2 : X_2 \to Y_2$ be the linear operator from Lemma 2.1 and let $N_2 : X_2 \to Y_2$ be the Nemitskij operator from Lemma 2.2 and $F_2 = A_2 + N_2 : X_2 \to Y_2$. Then*

- (j) *the function $u \in X_2$ is a solution of the mixed problem (1.1), (1.2), (1.3) for $g \in Y_2$ if and only if $F_2u = g$.*
- (jj) *The couple $(u, g) \in X_2 \times Y_2$ is the bifurcation point of the mixed problem (1.1), (1.2), (1.3) if and only if $F_2(u) = g$ and $u \in \sum$, where $\sum$ means the set of all points of X_2 at which F_2 is not locally invertible.*

Proof. (j) The first equivalence directly follows from the definition of operator F_2 and the mixed problem (1.1), (1.2), (1.3).

(jj) If (u, g) is a bifurcation point of the mixed problem (1.1), (1.2), (1.3) and u_k, v_k and g_k for $k = 1, 2, \dots$ have the same meaning as in Definition 3.1 then with respect to (j) we have $F(u) = g, F(u_k) = g_k = F(v_k)$. Thus F_2 is not locally injective at u. Hence, F_2 is not locally invertible at u, i.e. $u \in \sum$. Conversely, if F_2 is not locally invertible at u and $F_2(u) = g$, then F_2 is not locally injective at u. Indirectly, from Definition 3.1 we see that the couple (u, g) is a bifurcation point of (1.1), (1.2), (1.3).

Theorem 3.1. *Let the hypotheses of Lemmas 2.1, 2.2 and 2.3 be satisfied.*

Then for the mixed problem (1.1), (1.2), (1.3) the following statements hold:

(j) *For each $g \in Y_2$ the set S_{2g} of all solutions is compact (possibly empty).*

(jj) *The set $R(F_2) = \{g \in Y_2$; there exists at least one solution of the given problem $\}$ is closed and connected in Y_2.*

(jjj) *The domain of bifurcation D_{2b} is closed in X_2 and the bifurcation range R_{2b} is closed in Y_2.*

(jv) *If $Y_2 - R_{2b} \neq \emptyset$, then each component of $Y_2 - R_{2b}$ is a nonempty open set (i.e. a domain).*
The number n_{2g} of solutions is finite, constant (it may be zero) on each component of the set $Y_2 - R_{2b}$, i.e. for every g belonging to the same component of $Y_2 - R_{2b}$.

(v) *If $R_{2b} = 0$, then the given problem has a unique solution $u \in X_2$ for each $g \in Y_2$ and this solution continuously depends on g as a mapping from Y_2 onto X_2.*

Proof. Consider the operator $F_2 = A_2 + N_2 : X_2 \to Y_2$, where A_2 and N_2 are defined in the section 2. The mutual equivalence of the operator equation $F_2 u = g$ with the problem (1.1), (1.2), (1.3)(see Lemma 3.1) ensures that a property of F_2 imply the corresponding property of the mixed problem. The operator A_2 is a Fredholm operator with the zero index, N_2 is a completely continuous operator and F_2 is a coercive one.

Then the parts (a), (b) of Proposition 1.3 imply the assertions (j), (jj), respectively.

(jjj) Since D_{2b} is the set of all points $u \in X_2$ for which F_2 is not locally invertible and $R_{2b} = F_2(D_{2b})$, the part (c) of Proposition 1.3 implies the statement (jjj).

(jv) The set $Y_2 - R_{2b} \neq \emptyset$ is an open subset of the Banach space Y_2. Then each its component is nonempty and open. The second part of (jv) follows by the assertion (d) of Proposition 1.3.
The assertion (v) is a corollary of (e) from Proposition 1.3.

Theorem 3.2. *Let the operators $A_2 : X_2 \to Y_2, N_2 : X_2 \to Y_2$ and $F_2 : A_2 + N_2 : X_2 \to Y_2$ satisfy hypotheses of Lemmas 2.1, 2.2, 2.3, respectively.*

Then each of the following conditions is sufficient for the surjectivity of the operator F_2 (we use the notation of Theorem 3.1):

(vi) *For each $g \in R_{2b}$ there is a solution u of (1.1), (1.2), (1.3) such that $u \in X_2 - D_{2b}$.*

(vii) *The set $Y_2 - R_{2b}$ is connected and there is a $g \in R(F_2) - R_{2b}$.*

(viii) *There exists a constat $K > 0$ such that all solutions u of the mixed problem for the equation*

$$C_2u + \mu[A_2u - C_2u + N_2u] = 0\,, \quad \mu \in (0,1) \tag{3.1}$$

with data (1.2), (1.3) fulfil one of the following conditions:

(a_1) *The condition (a) of Lemma 2.3.*

(b_1) *The condition (b) of Lemma 2.3.*

Proof. In the case (vi) the assertion follows from (iv) and in the case (vii) from (v) of Proposition 1.4.

(viii) Here we show that the condition (viii) implies (vi) of Proposition 1.4 for $G(u) = u, \quad u \in X_2$.

It is sufficient to prove that each solution u of the operator equation $F_2u = kC_2u$ for all $k < 0$ satisfies the estimation

$$||u||_{X_2} < K_1, \quad K_1 > 0. \tag{3.2}$$

The equation $F_2u = kC_2u, \quad k < 0$ can be written in the form

$$C_2u + (1-k)^{-1}[A_2u - C_2u + N_2u] = 0\,,$$

where $(1-k)^{-1} \in (0,1)$ and thus each solution of

$$F_2u = kC_2u\,, \quad k < 0$$

satisfies the equation (3.1) with the conditions (1.2), (1.3).

In the case (a_1) there is a constant $K > 0$ such that

$$||u||_{(1+\alpha)/2,1+\alpha,Q} \leq K.$$

In the case (b_1)

$$||u||_{(2+\alpha)/2,2+\alpha,Q} \leq K\,.$$

Further, by the same method as in Lemma 2.3 we get the estimation (3.2) and hence the surjectivity of the operator F_2. This completes the proof of the given theorem.

The following theorem says on the nonzero number of solutions of (1.1), (1.2), (1.3). It follows directly from Proposition 1.5.

Theorem 3.3. *Assume that operators $A_2 : X_2 \to Y_2$, $N_2 : X_2 \to Y_2$ and $F_2 = A_2 + N_2 : X_2 \to Y_2$ satisfy the hypothesis of Lemma 2.1, Lemma 2.2 and Lemma 2.3, respectively and the condition (viii) of Theorem 3.2. Then*

(jv') the number n_{2g} of solutions of (1.1), (1.2), (1.3) is finite, constant and different from zero on each component of the open set $Y_2 - R_{2b}$ (for all g belonging to the same component of $Y_2 - R_{2b}$).

Remark 3.1. The results of this section can be interpreted and applied to the different diffusion and heat problems for the nonlinear parabolic equations of the type (1.1). The reaction-diffusion problems modelled by the equation

$$u_t = D\Delta u + f(u), \quad t > 0, \; x \in \Omega \subset R^n$$

where D is a real (positive or negative) constant and f is a sufficient smooth function (it may be with a strong growth), also, shock waves and entropy models as

$$u_t + [f(u)]_x = 0\,, \quad t > 0\,,\ x \in (a, b)$$

can be studied together with other physical and technical models by this method.

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