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GENERALIZED QUASIVARIATIONAL INEQUALITIES ON FRÉCHET SPACES

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Abstract. In this paper generalized quasivariational inequalities on Fréchet spaces are deduced from new fixed point theory of Agarwal and O'Regan [1] and O'Regan [7].

1. Introduction

Quasivariational inequalities (or existence theorems for two variable functions) are discussed in this paper. In particular suppose $f : X\times Y \to R$ and $G : X \to 2^Y$ are upper semicontinuous maps (here 2^Y denote the family of nonempty subsets of Y) with X and Y closed, convex subsets of a Fréchet space E. Conditions are put on f, G, X and Y to guarantee that there exists $w_1 \in X$, $w_1 \in G(w_1)$ with $f(w_1, w_1) = \sup_{z\in G(w_1)} f(w_1, z)$ (or $f(w_1, w_1) = \inf_{z\in G(w_1)} f(w_1, z)$) or more generally to guarantee that there exists $w_1, w_2 \in X$, $w_1 \neq w_2$, $w_1 \in G(w_1)$, $w_2 \in G(w_2)$ with $f(w_1, w_1) = \sup_{z\in G(w_1)} f(w_1, z)$ and $f(w_2, w_2) = \sup_{z\in G(w_2)} f(w_2, z)$ (or $f(w_1, w_1) = \inf_{z\in G(w_1)} f(w_1, z)$ and $f(w_2, w_2) = \inf_{z\in G(w_2)} f(w_2, z)$). The results of this paper are new and they extend and complement many well known results in the literature [2, 3, 5, 8, 9, 11, 12]. Usually in the literature $Y \subseteq X$ or more generally ([8]) $G(\partial X) \subseteq X \cap Y$. In [5] we relaxed the condition $G(\partial X) \subseteq X \cap Y$ using a fixed point theorem of the author [5] of Furi–Pera type. Recently new fixed point results in Fréchet spaces have been established by Agarwal and O'Regan in [1] (single fixed point) and by O'Regan [7] (multiple fixed point). The fixed point theory established in [1, 7] is more general than the Furi–Pera type theory presented in [4, 5, 6]. Using the results in [1, 7] we are able to establish new quasivariational inequalities.

We now gather together some well known definitions. Let E_1 and E_2 be Fréchet spaces. A mapping $F : E_1 \to 2^{E_2}$ is upper semicontinuous (u.s.c.) if the set $F^{-1}(A) = \{x \in E_1 : F(x) \cap A \neq \emptyset\}$ is closed for any closed set A in E_2. Let (X, d) be a metric space and let Ω_X the bounded subsets of X. The

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Kuratowski measure of noncompactness is the map $\alpha : \Omega_X \to [0,\infty]$ defined by (here $B \in \Omega_X$),

$$\alpha(B) = \inf\{r > 0 : \ B \subseteq \cup_{i=1}^n B_i \ \text{ and } \ diam\,(B_i) \le r\}.$$

Let S be a nonempty subset of X and suppose $G : S \to 2^X$. Then (i). $G : S \to 2^X$ is k–set contractive (here $k \ge 0$) if $\alpha(G(A)) \le k\,\alpha(A)$ for all nonempty, bounded sets A of S, and (ii). $G : S \to 2^X$ is condensing if G is 1–set contractive and $\alpha(G(A)) < \alpha(A)$ for all bounded sets A of S with $\alpha(A) \ne 0$.

2. Quasivariational inequalities

Let $N_0 = \{1, 2, ...\}$. In this section we assume E is a Fréchet space endowed with a family of seminorms $\{|\,.\,|_n : \ n \in N_0\}$ with

$$|x|_1 \le |x|_2 \le \ldots\ldots\ldots. \quad \text{for all } \ x \in E.$$

Also for each $n \in N_0$ we assume that there are Banach spaces $(E_n, |\,.\,|_n)$ with

$$E_1 \supseteq E_2 \supseteq \ldots\ldots\ldots\ldots \ \text{ and } \ E = \cap_{n=1}^{\infty} E_n \ \text{ and } \ |x|_n \le |x|_{n+1} \ \text{ for all } \ x \in E_{n+1}.$$

For each $n \in N_0$ let C_n be a cone in E_n and assume $|\,.\,|_n$ is increasing with respect to C_n. In addition assume

$$C_1 \supseteq C_2 \supseteq \ldots\ldots\ldots\ldots\ .$$

For $\rho > 0$ and $n \in N_0$ let

$$U_{n,\rho} = \{x \in E_n : \ |x|_n < \rho\} \quad \text{and} \quad \Omega_{n,\rho} = U_{n,\rho} \cap C_n.$$

Notice

$$\partial_{C_n} \Omega_{n,\rho} = \partial_{E_n} U_{n,\rho} \cap C_n \quad \text{and} \quad \overline{\Omega_{n,\rho}} = \overline{U_{n,\rho}} \cap C_n$$

(the first closure is with respect to C_n whereas the second is with respect to E_n). In addition notice since $|x|_n \le |x|_{n+1}$ for all $x \in E_{n+1}$ that

$$\Omega_{1,\rho} \supseteq \Omega_{2,\rho} \supseteq \ldots\ldots. \ \text{ and } \ \overline{\Omega_{1,\rho}} \supseteq \overline{\Omega_{2,\rho}} \supseteq \ldots\ldots.\ .$$

We first state a general result [7] that guarantees that the inclusion

$$y \in F\,y \tag{2.1}$$

has two solutions in E.

Definition 2.1. Fix $k \in N_0$. If $x, y \in E_k$ then we say $x = y$ in E_k if $|x-y|_k = 0$ (i.e. if $x - y = 0$; here 0 is the zero in E_k).

Definition 2.2. If $x, y \in E$ then we say $x = y$ in E if $x = y$ in E_k for each $k \in N_0$.

Definition 2.3. Fix $k \in N_0$. We say $x \in F\,y$ in E_k if there exists $w \in F\,y$ with $x = w$ in E_k.

Theorem 2.1. *Let L, γ, r, R be constants with $0 < L < \gamma < r < R$. Assume the following conditions are satisfied:*

$$(2.2)\qquad \begin{cases} \text{for each } n \in N_0,\ F_n : \overline{U_{n,R}} \cap C_n \to CK(C_n) \text{ is a u.s.c. map;} \\ \text{here } CK(C_n) \text{ denotes the family of nonempty, compact,} \\ \text{convex subsets of } C_n \end{cases}$$

$$(2.3)\quad \text{for each } n \in N_0,\ |y|_n \le |x|_n \text{ for all } y \in F_n(x) \text{ and } x \in \partial_{E_n} U_{n,L} \cap C_n$$

$$(2.4)\quad \text{for each } n \in N_0,\ |y|_n \le |x|_n \text{ for all } y \in F_n(x) \text{ and } x \in \partial_{E_n} U_{n,r} \cap C_n$$

$$(2.5)\quad \text{for each } n \in N_0,\ |y|_n \ge |x|_n \text{ for all } y \in F_n(x) \text{ and } x \in \partial_{E_n} U_{n,R} \cap C_n$$

$$(2.6)\qquad \begin{cases} \text{for each } n \in N_0, \text{ the map } \mathcal{K}_n : \overline{U_{n,R}} \cap C_n \to 2^{C_n} \text{ given by} \\ \mathcal{K}_n\, y = \cup_{m=n}^{\infty} F_m\, y \text{ is } k\text{-set contractive (here } 0 \le k < 1) \end{cases}$$

$$(2.7)\qquad \begin{cases} \text{for every } k \in N_0 \text{ and any subsequence } A \subseteq \{k, k+1, \ldots..\} \text{ if} \\ x \in C_n,\ n \in A, \text{ is such that } R \ge |x|_n \ge r \text{ then } |x|_k \ge \gamma \end{cases}$$

$$(2.8)\qquad \begin{cases} \text{if there exists a } v \in E, \text{ and for every } k \in N_0 \text{ there exists} \\ \text{a subsequence } S \subseteq \{k+1, k+2, \ldots..\} \text{ of } N_0 \text{ and a sequence} \\ \{u_n\}_{n \in S} \text{ with } u_n \in \overline{U_{n,L}} \cap C_n \text{ and } u_n \in F_n\, u_n \text{ in } E_n \\ \text{for } n \in S \text{ and with } u_n \to v \text{ in } E_k \text{ as } n \to \infty \\ \text{in } S, \text{ then } v \in F\, v \text{ in } E \end{cases}$$

and

$$(2.9)\qquad \begin{cases} \text{if there exists a } z \in E, \text{ and for every } k \in N_0 \text{ there exists} \\ \text{a subsequence } P \subseteq \{k+1, k+2, \ldots..\} \text{ of } N_0 \text{ and a sequence} \\ \{w_n\}_{n \in P} \text{ with } w_n \in \left(\overline{U_{n,R}} \setminus U_{n,r}\right) \cap C_n \text{ and } w_n \in F_n\, w_n \\ \text{in } E_n \text{ for } n \in P \text{ and with } w_n \to z \text{ in } E_k \text{ as } n \to \infty \\ \text{in } P, \text{ then } z \in F\, z \text{ in } E. \end{cases}$$

Then (2.1) *has at least two solutions x_0 and x_1 with*

$$x_0 \in \cap_{n=1}^{\infty} \left(\overline{U_{n,L}} \cap C_n\right) \quad \text{and} \quad x_1 \in \cap_{n=1}^{\infty} \left(\left(\overline{U_{n,R}} \setminus U_{n,\gamma}\right) \cap C_n\right).$$

Remark 2.1. The definition of $\mathcal{K}_n$ in (2.6) is as follows. If $y \in \overline{U_{n,R}} \cap C_n$ and $y \notin \overline{U_{n+1,R}} \cap C_{n+1}$ then $\mathcal{K}_n\, y = F_n\, y$, whereas if $y \in \overline{U_{n+1,R}} \cap C_{n+1}$ and $y \notin \overline{U_{n+2,R}} \cap C_{n+2}$ then $\mathcal{K}_n\, y = F_n\, y \cup F_{n+1}\, y$, and so on.

Remark 2.2. If F is defined on E_1 with $F_n = F|_{E_n}$ for each $n \in N_0$ then (2.8) and (2.9) are automatically satisfied.

Using Theorem 2.1 we are able to establish the following quasivariational inequality.

Theorem 2.2. *Let L, γ, r, R be constants with $0 < L < \gamma < r < R$. Assume the following conditions are satisfied:*

$$\text{for each } n \in N_0,\ f_n : \left(\overline{U_{n,R}} \cap C_n\right) \times C_n \to R \text{ is a u.s.c. function} \tag{2.10}$$

$$\begin{cases} \text{for each } n \in N_0,\ G_n : \overline{U_{n,R}} \cap C_n \to C(C_n) \text{ is a u.s.c. map; here} \\ C(C_n) \text{ denotes the family of nonempty, compact subsets of } C_n \end{cases} \tag{2.11}$$

and

$$\begin{cases} \text{for each } n \in N_0, \text{ the map } M_n \text{ (marginal function), defined by} \\ M_n(x) = \sup_{y \in G_n(x)} f_n(x,y) \text{ for } x \in \overline{U_{n,R}} \cap C_n \text{ is lower} \\ \text{semicontinuous (l.s.c.).} \end{cases} \tag{2.12}$$

For any $n \in N_0$, define the map Φ_n by

$$\Phi_n(x) = \{y \in G_n(x) :\ f_n(x,y) = M_n(x)\} \quad \text{for } x \in \overline{U_{n,R}} \cap C_n$$

and the map Φ by

$$\Phi(x) = \{y \in G(x) :\ f(x,y) = M(x)\} \quad \text{for } x \in \cap_{n=1}^{\infty} \left(\overline{U_{n,R}} \cap C_n\right)\ ;$$

here

$$f : \cap_{n=1}^{\infty} \left(\overline{U_{n,R}} \cap C_n\right) \times \cap_{n=1}^{\infty} C_n \to R \quad \text{and} \quad G : \cap_{n=1}^{\infty} \left(\overline{U_{n,R}} \cap C_n\right) \to 2^{\cap_{n=1}^{\infty} C_n}$$

together with

$$M(x) = \sup_{y \in G(x)} f(x,y) \quad \text{for } x \in \cap_{n=1}^{\infty} \left(\overline{U_{n,R}} \cap C_n\right).$$

Also suppose the following conditions hold:

$$\text{for each } n \in N_0,\ \Phi_n(x) \text{ is convex for each } x \in \overline{U_{n,R}} \cap C_n \tag{2.13}$$

$$\text{for each } n \in N_0,\ |y|_n \le |x|_n \text{ for all } y \in \Phi_n(x) \text{ and } x \in \partial_{E_n} U_{n,L} \cap C_n \tag{2.14}$$

$$\text{for each } n \in N_0,\ |y|_n \le |x|_n \text{ for all } y \in \Phi_n(x) \text{ and } x \in \partial_{E_n} U_{n,r} \cap C_n \tag{2.15}$$

$$\text{for each } n \in N_0,\ |y|_n \ge |x|_n \text{ for all } y \in \Phi_n(x) \text{ and } x \in \partial_{E_n} U_{n,R} \cap C_n \tag{2.16}$$

$$\begin{cases} \text{for each } n \in N_0, \text{ the map } \mathcal{R}_n : \overline{U_{n,R}} \cap C_n \to 2^{C_n} \text{ given by} \\ \mathcal{R}_n\, y = \cup_{m=n}^{\infty} \Phi_m\, y \text{ is compact} \end{cases} \tag{2.17}$$

$$\begin{cases} \text{for every } k \in N_0 \text{ and any subsequence } A \subseteq \{k, k+1, ...\} \text{ if} \\ x \in C_n,\ n \in A, \text{ is such that } R \ge |x|_n \ge r \text{ then } |x|_k \ge \gamma \end{cases} \tag{2.18}$$

$$(2.19)\quad \begin{cases} \textit{if there exists a } \; v \in E, \;\; \textit{and for every } \; k \in N_0 \;\; \textit{there exists} \\ \textit{a subsequence } \; S \subseteq \{k+1, k+2, ...\} \;\; \textit{of } \; N_0 \;\; \textit{and a sequence} \\ \{u_n\}_{n \in S} \;\; \textit{with } \; u_n \in \overline{U_{n,L}} \cap C_n \;\;\; \textit{and } \; u_n \in \Phi_n\, u_n \;\; \textit{in } \; E_n \\ \textit{for } \; n \in S \;\; \textit{and with } \; u_n \to v \;\; \textit{in } \; E_k \;\; \textit{as } \; n \to \infty \\ \textit{in } \; S, \;\; \textit{then } \; v \in \Phi\, v \;\; \textit{in } \; E \end{cases}$$

and

$$(2.20)\quad \begin{cases} \textit{if there exists a } \; z \in E, \;\; \textit{and for every } \; k \in N_0 \;\; \textit{there exists} \\ \textit{a subsequence } \; P \subseteq \{k+1, k+2, ...\} \;\; \textit{of } \; N_0 \;\; \textit{and a sequence} \\ \{w_n\}_{n \in P} \;\; \textit{with } \; w_n \in \left(\overline{U_{n,R}} \setminus U_{n,r}\right) \cap C_n \;\;\; \textit{and } \; w_n \in \Phi_n\, w_n \\ \textit{in } \; E_n \;\; \textit{for } \; n \in P \;\; \textit{and with } \; w_n \to z \;\; \textit{in } \; E_k \;\; \textit{as } \; n \to \infty \\ \textit{in } \; P, \; \textit{then } \; z \in \Phi\, z \;\; \textit{in } \; E. \end{cases}$$

Then there exists $x_0 \in \cap_{n=1}^{\infty} \left(\overline{U_{n,L}} \cap C_n\right)$ *with* $x_0 \in G(x_0)$ *and* $f(x_0, x_0) = M(x_0)$ *(i.e. there exists* $x_0 \in \cap_{n=1}^{\infty} \left(\overline{U_{n,L}} \cap C_n\right)$ *with* $x_0 \in G(x_0)$ *and* $f(x_0, y) \le f(x_0, x_0)$ *for all* $y \in G(x_0)$*) and* $x_1 \in \cap_{n=1}^{\infty} \left(\left(\overline{U_{n,R}} \setminus U_{n,\gamma}\right) \cap C_n\right)$ *with* $x_1 \in G(x_1)$ *and* $f(x_1, x_1) = M(x_1)$*.*

Remark 2.3. Conditions (put on f_n and G_n) so that (2.13) holds may be found in [5] (and its references). The definition of $\mathcal{R}_n$ in (2.17) is as in Remark 2.1 with F_m replaced by Φ_m.

Proof. Fix $n \in N_0$. Now since f_n is u.s.c. and G_n is a u.s.c., compact valued map then [2 pp. 473] and (2.12) imply M_n is continuous. In addition [2 pp. 44] implies for each $x \in \overline{U_{n,R}} \cap C_n$ that $\Phi_n(x)$ is nonempty and compact. This together with (2.13) implies $\Phi_n : \overline{U_{n,R}} \cap C_n \to CK(C_n)$. Next we show the graph of Φ_n is closed. Let $\{(x_m, y_m)\}_{m=1}^{\infty}$ be a sequence in $graph\,(\Phi_n)$ with $(x_m, y_m) \to (x, y)$ in $\left(\overline{U_{n,R}} \cap C_n\right) \times C_n$. Then

$$f_n(x, y) \ge \limsup f_n(x_m, y_m) = \limsup M_n(x_m) = \liminf M_n(x_m) = M_n(x).$$

In addition $y_m \in G_n(x_m)$ together with $x_m \to x$, $y_m \to y$ and G_n u.s.c. implies [10] that $y \in G_n(x)$. Thus $y \in G_n(x)$ and $f_n(x, y) \ge M_n(x) = \sup_{z \in G_n(x)} f_n(x, z)$. Consequently $f_n(x, y) = M_n(x)$ so $(x, y) \in graph\,(\Phi_n)$. Hence $\Phi_n : \overline{U_{n,R}} \cap C_n \to CK(C_n)$ is a closed map. Now since Φ_n is a compact map (see (2.17)) we have, using a standard result [2 pp. 465], that $\Phi_n : \overline{U_{n,R}} \cap C_n \to CK(C_n)$ is u.s.c. Now we apply Theorem 2.1 with F_n repaced by Φ_n to deduce that there exists $x_0 \in \cap_{n=1}^{\infty} \left(\overline{U_{n,L}} \cap C_n\right)$ and $x_1 \in \cap_{n=1}^{\infty} \left(\left(\overline{U_{n,R}} \setminus U_{n,\gamma}\right) \cap C_n\right)$ with $x_0 \in \Phi(x_0)$ and $x_1 \in \Phi(x_1)$. The result is now immediate. □

Remark 2.4. If (2.10) and (2.17) are replaced by,

$$(2.21)\quad \text{for each } \; n \in N_0, \; f_n : \left(\overline{U_{n,R}} \cap C_n\right) \times C_n \to R \;\; \text{is a continuous function}$$

and

$$(2.22)\quad \begin{cases} \text{for each } \; n \in N_0, \;\; \text{the map } \; \mathcal{R}_n : \overline{U_{n,R}} \cap C_n \to 2^{C_n} \;\; \text{given by} \\ \mathcal{R}_n\, y = \cup_{m=n}^{\infty} \, \Phi_m\, y \;\; \text{is } \; k\text{-set contractive (here } \; 0 \le k < 1), \end{cases}$$

then the result of Theorem 2.2 is again true.

The result is essentially the same as in Theorem 2.2. The only difference is to show $\Phi_n : \overline{U_{n,R}} \cap C_n \to CK(C_n)$ is u.s.c. for each $n \in N_0$. To see this fix $n \in N_0$ and notice

$$\Phi_n(x) = G_n(x) \cap \Lambda_n(x)$$

where

$$\Lambda_n(x) = \{y \in C_n : \ f_n(x,y) = M_n(x)\}.$$

We claim that the graph of Λ_n is closed. If the claim is true then G_n u.s.c. with compact values and [2 pp. 470] implies Φ_n is u.s.c. It remains to prove the claim. Let $\{(x_m, y_m)\}_{m=1}^{\infty}$ be a sequence in $graph\,(\Lambda_n)$ with $(x_m, y_m) \to (x,y)$ in $\left(\overline{U_{n,R}} \cap C_n\right) \times C_n$. Then since (2.21) holds,

$$f_n(x,y) = \limsup f_n(x_m, y_m) = \limsup M_n(x_m) = M_n(x).$$

Consequently $(x,y) \in graph\,(\Lambda)$.

Remark 2.5. Sometimes $f(x,y)$ is defined for all $(x,y) \in \left(\overline{U_{1,R}} \cap C_1\right) \times C_1$, $G(x)$ is defined for all $x \in \overline{U_{1,R}} \cap C_1$, and $\Phi_n = \Phi|_{\left(\overline{U_{n,R}} \cap C_n\right) \times C_n}$.

Our next result replaces sup in Theorem 2.2 with inf.

Theorem 2.3. *Let L, γ, r, R be constants with $0 < L < \gamma < r < R$. Assume the following conditions are satisfied:*

(2.23) *for each* $n \in N_0$, $f_n : \left(\overline{U_{n,R}} \cap C_n\right) \times C_n \to R$ *is a continuous function*

and

(2.24) *for each* $n \in N_0$, $G_n : \overline{U_{n,R}} \cap C_n \to C(C_n)$ *is a u.s.c. map.*

For any $n \in N_0$, *define the map* Ψ_n *by*

$$\Psi_n(x) = \left\{ y \in G_n(x) : \ f_n(x,y) = N_n(x) = \inf_{z \in G_n(x)} f_n(x,z) \right\} \quad \text{for } x \in \overline{U_{n,R}} \cap C_n$$

and the map Ψ *by*

$$\Psi(x) = \{y \in G(x) : \ f(x,y) = N(x)\} \quad \text{for } x \in \cap_{n=1}^{\infty} \left(\overline{U_{n,R}} \cap C_n\right);$$

here

$$f : \cap_{n=1}^{\infty} \left(\overline{U_{n,R}} \cap C_n\right) \times \cap_{n=1}^{\infty} C_n \to R \quad \text{and} \quad G : \cap_{n=1}^{\infty} \left(\overline{U_{n,R}} \cap C_n\right) \to 2^{\cap_{n=1}^{\infty} C_n}$$

together with

$$N(x) = \inf_{y \in G(x)} f(x,y) \quad \text{for } x \in \cap_{n=1}^{\infty} \left(\overline{U_{n,R}} \cap C_n\right).$$

Also suppose the following conditions hold:

(2.25) *for each* $n \in N_0$, $\Psi_n(x)$ *is convex for each* $x \in \overline{U_{n,R}} \cap C_n$

(2.26) *for each* $n \in N_0$, $|y|_n \le |x|_n$ *for all* $y \in \Psi_n(x)$ *and* $x \in \partial_{E_n} U_{n,L} \cap C_n$

(2.27) *for each* $n \in N_0$, $|y|_n \le |x|_n$ *for all* $y \in \Psi_n(x)$ *and* $x \in \partial_{E_n} U_{n,r} \cap C_n$

(2.28) *for each* $n \in N_0$, $|y|_n \geq |x|_n$ *for all* $y \in \Psi_n(x)$ *and* $x \in \partial_{E_n} U_{n,R} \cap C_n$

$$(2.29) \qquad \begin{cases} \textit{for each } n \in N_0, \textit{ the map } \mathcal{R}_n : \overline{U_{n,R}} \cap C_n \to 2^{C_n} \textit{ given by} \\ \mathcal{R}_n\, y = \cup_{m=n}^{\infty} \Psi_m\, y \textit{ is } k\textit{-set contractive (here } 0 \leq k < 1) \end{cases}$$

$$(2.30) \qquad \begin{cases} \textit{for every } k \in N_0 \textit{ and any subsequence } A \subseteq \{k, k+1, ...\} \textit{ if} \\ x \in C_n,\ n \in A, \textit{ is such that } R \geq |x|_n \geq r \textit{ then } |x|_k \geq \gamma \end{cases}$$

$$(2.31) \qquad \begin{cases} \textit{if there exists a } v \in E, \textit{ and for every } k \in N_0 \textit{ there exists} \\ \textit{a subsequence } S \subseteq \{k+1, k+2, ...\} \textit{ of } N_0 \textit{ and a sequence} \\ \{u_n\}_{n \in S} \textit{ with } u_n \in \overline{U_{n,L}} \cap C_n \textit{ and } u_n \in \Psi_n\, u_n \textit{ in } E_n \\ \textit{for } n \in S \textit{ and with } u_n \to v \textit{ in } E_k \textit{ as } n \to \infty \\ \textit{in } S, \textit{ then } v \in \Psi\, v \textit{ in } E \end{cases}$$

and

$$(2.32) \qquad \begin{cases} \textit{if there exists a } z \in E, \textit{ and for every } k \in N_0 \textit{ there exists} \\ \textit{a subsequence } P \subseteq \{k+1, k+2, ...\} \textit{ of } N_0 \textit{ and a sequence} \\ \{w_n\}_{n \in P} \textit{ with } w_n \in \left(\overline{U_{n,R}} \setminus U_{n,r}\right) \cap C_n \textit{ and } w_n \in \Psi_n\, w_n \\ \textit{in } E_n \textit{ for } n \in P \textit{ and with } w_n \to z \textit{ in } E_k \textit{ as } n \to \infty \\ \textit{in } P, \textit{ then } z \in \Psi\, z \textit{ in } E. \end{cases}$$

Then there exists $x_0 \in \cap_{n=1}^{\infty} \left(\overline{U_{n,L}} \cap C_n\right)$ *with* $x_0 \in G(x_0)$ *and* $f(x_0, x_0) = N(x_0)$ *(i.e. there exists* $x_0 \in \cap_{n=1}^{\infty} \left(\overline{U_{n,L}} \cap C_n\right)$ *with* $x_0 \in G(x_0)$ *and* $f(x_0, y) \geq f(x_0, x_0)$ *for all* $y \in G(x_0)$*) and* $x_1 \in \cap_{n=1}^{\infty} \left(\left(\overline{U_{n,R}} \setminus U_{n,\gamma}\right) \cap C_n\right)$ *with* $x_1 \in G(x_1)$ *and* $f(x_1, x_1) = N(x_1)$.

Proof. Fix $n \in N_0$. Now [2 pp. 472, 473] implies N_n is continuous. As in Theorem 2.2 (with Remark 2.4) it is easy to check that $\Psi_n : \overline{U_{n,R}} \times C_n \to CK(C_n)$ is u.s.c. Apply Theorem 2.1 with F_n replaced by Ψ_n. □

Remark 2.6. As in [8, 9], Theorem 2.2 and Theorem 2.3 can be used to obtain variational–like inequalities (see also [5]).

To conclude this paper we indicate how one could obtain results for closed sets (which may have empty interior). In this case we establish the existence of a single solution to variational–like inequalities. Let E be a Fréchet space endowed with a family of seminorms $\{|\,.\,|_n : n \in N_0\}$ with

$$|x|_1 \leq |x|_2 \leq .... \quad \text{for all } x \in E.$$

Also for each $n \in N_0$ we assume that there are Banach spaces $(E_n, |\,.\,|_n)$ with

$$E_1 \supseteq E_2 \supseteq .... \quad \text{and} \quad E = \cap_{n=1}^{\infty} E_n \quad \text{and} \quad |x|_n \leq |x|_{n+1} \text{ for all } x \in E_{n+1}.$$

For each $n \in N_0$ let Q_n be a closed, bounded, convex subset of E_n with $0 \in Q_n$ and

$$Q_1 \supseteq Q_2 \supseteq ....$$

We now establish a result which guarantees that (2.1) has a solution in E.

Theorem 2.4. *Assume the following conditions are satisfied:*

$$\left\{\begin{array}{l}\text{for each } n \in N_0,\ F_n : Q_n \to CD(E_n) \text{ is a closed map;}\\ \text{here } CD(E_n) \text{denotes the family of nonempty,}\\ \text{compact, acyclic subsets of } E_n\end{array}\right. \tag{2.33}$$

$$\left\{\begin{array}{l}\text{for each } n \in N_0,\ \text{ if } \{(x_j, \lambda_j)\}_{j=1}^{\infty} \text{ is a sequence}\\ \text{in } \partial Q_n \times [0,1] \text{ converging to } (x, \lambda) \text{ with } x \in \lambda\, F_n(x)\\ \text{and } 0 \le \lambda < 1 \text{ then there exists } j_0 \in \{1, 2, ...\}\\ \text{with } \{\lambda_j\, F_n(x_j)\} \subseteq Q_n \text{ for each } j \ge j_0\end{array}\right. \tag{2.34}$$

$$\left\{\begin{array}{l}\text{for each } n \in N_0, \text{ the map } \mathcal{K}_n : Q_n \to 2^{E_n} \text{ given by}\\ \mathcal{K}_n\, y = \cup_{m=n}^{\infty} F_m\, y \text{ (see Remark 2.1) is compact}\end{array}\right. \tag{2.35}$$

and

$$\left\{\begin{array}{l}\text{if there exists a } v \in E, \text{ and for every } k \in N_0 \text{ there exists}\\ \text{a subsequence } S \subseteq \{k+1, k+2, ...\} \text{ of } N_0 \text{ and a sequence}\\ \{u_n\}_{n \in S} \text{ with } u_n \in Q_n \text{ and } u_n \in F_n\, u_n \text{ in } E_n\\ \text{for } n \in S \text{ and with } u_n \to v \text{ in } E_k \text{ as } n \to \infty\\ \text{in } S, \text{ then } v \in F\, v \text{ in } E.\end{array}\right. \tag{2.36}$$

Then (2.1) *has at least one solution in* E *(in fact in* $\cap_{n=1}^{\infty} Q_n$*).*

Proof. Fix $n \in N_0$. Now [6] guarantees that $y \in F_n\, y$ has a solution $y_n \in Q_n$. Essentially the same reasoning as in [1] establishes the result. □

Remark 2.7. For each $n \in N_0$ if we can take sets Q_n so that the nearest point projection $r_n : E_n \to Q_n$ is 1–set contractive then we can replace (2.35) with: for each $n \in N_0$, the map $\mathcal{K}_n : Q_n \to 2^{E_n}$ given by $\mathcal{K}_n\, y = \cup_{m=n}^{\infty} F_m\, y$ is condensing.

We now establish the analogue of Theorem 2.2 for the situation described above.

Theorem 2.5. *Assume the following conditions are satisfied:*

$$\text{for each } n \in N_0,\ f_n : Q_n \times E_n \to R \text{ is a u.s.c. function} \tag{2.37}$$

$$\text{for each } n \in N_0,\ G_n : Q_n \to C(E_n) \text{ is a u.s.c. map} \tag{2.38}$$

and

$$\left\{\begin{array}{l}\text{for each } n \in N_0, \text{ the map } M_n \text{ (marginal function), defined by}\\ M_n(x) = \sup_{y \in G_n(x)} f_n(x, y) \text{ for } x \in Q_n \text{ is l.s.c.}\end{array}\right. \tag{2.39}$$

For any $n \in N_0$*, define the map* Φ_n *by*

$$\Phi_n(x) = \{y \in G_n(x) :\ f_n(x, y) = M_n(x)\} \quad \text{for } x \in Q_n$$

and the map Φ *by*

$$\Phi(x) = \{y \in G(x) :\ f(x, y) = M(x)\} \quad \text{for } x \in \cap_{n=1}^{\infty} Q_n\,;$$

here

$$f : \cap_{n=1}^{\infty} Q_n \times E \to R \quad and \quad G : \cap_{n=1}^{\infty} Q_n \to 2^E$$

together with

$$M(x) = \sup_{y \in G(x)} f(x,y) \quad for \quad x \in \cap_{n=1}^{\infty} Q_n.$$

Also suppose the following conditions hold:

$$\text{(2.40)} \qquad for\ each\ n \in N_0,\ \Phi_n(x)\ \ is\ acyclic\ for\ each\ \ x \in Q_n$$

$$\text{(2.41)} \quad \begin{cases} for\ each\ \ n \in N_0,\ \ if\ \ \{(x_j,\lambda_j)\}_{j=1}^{\infty}\ \ is\ a\ sequence\ in\ \ \partial Q_n \times [0,1] \\ converging\ to\ (x,\lambda)\ \ with\ \ x \in \lambda\,\Phi_n(x)\ \ and\ \ 0 \le \lambda < 1\ \ then \\ there\ exists\ \ j_0 \in \{1,2,....\} \\ with\ \ \{\lambda_j\,\Phi_n(x_j)\} \subseteq Q_n\ \ for\ each\ \ j \ge j_0 \end{cases}$$

$$\text{(2.42)} \quad \begin{cases} for\ each\ \ n \in N_0,\ \ the\ map\ \ \mathcal{K}_n : Q_n \to 2^{E_n}\ \ given\ by \\ \mathcal{K}_n\, y = \cup_{m=n}^{\infty}\, \Phi_m\, y\ \ is\ compact \end{cases}$$

and

$$\text{(2.43)} \quad \begin{cases} if\ there\ exists\ a\ \ v \in E,\ \ and\ for\ every\ \ k \in N_0\ \ there\ exists \\ a\ subsequence\ \ S \subseteq \{k+1,k+2,.....\}\ \ of\ \ N_0\ \ and\ a\ sequence \\ \{u_n\}_{n\in S}\ \ with\ \ u_n \in Q_n\ \ \ and\ \ u_n \in \Phi_n\, u_n\ \ in\ \ E_n \\ for\ \ n \in S\ \ and\ with\ \ u_n \to v\ \ in\ \ E_k\ \ as\ \ n \to \infty \\ in\ \ S,\ \ then\ \ v \in \Phi\, v\ \ in\ \ E. \end{cases}$$

Then there exists $x_0 \in \cap_{n=1}^{\infty} Q_n$ *with* $x_0 \in G(x_0)$ *and* $f(x_0,x_0) = M(x_0)$.

Proof. Fix $n \in N_0$. As in Theorem 2.2, M_n is continuous and $\Phi_n : Q_n \to CD(E_n)$ is a closed map. Now apply Theorem 2.4 to deduce the result. □

Remark 2.8. The statement (and proof) of the analogue of Theorem 2.3 is also clear in this situation. We leave the details to the reader.

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