ARCHIVUM MATHEMATICUM (BRNO)
Tomus 35 (1999), 203 – 214

SOME RESULTS REGARDING AN EQUATION OF HAMILTON-JACOBI TYPE

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Abstract. This paper is a contribution to the mathematical modelling of the hump effect. We present a mathematical study (existence, homogenization) of a Hamilton-Jacobi problem which represents the propagation of a front flame in a striated media.

Introduction

The physical problem consists in an anomaly of overvelocity observed in the combustion room of propellers during the combustion of some solid propellant blocks. This anomaly, called 'Hump effect', attains its maximum in the middle of the burning block. The reduced mathematical model of this phenomenon is the following Hamilton-Jacobi problem:

$$P_\xi \quad \begin{cases} \dfrac{\partial \xi}{\partial t} + R_0(\xi, s_2)\sqrt{1+\left(\dfrac{\partial \xi}{\partial s_2}\right)^2} = 0 & \forall t > 0\ , \quad s_2 \in \mathbb{R} \\ \xi(s_2, 0) = \xi_0(s_2) & s_2 \in \mathbb{R} \end{cases}$$

where the unknown $s_1 = \xi(s_2, t)$ is the position of the flame front. We will show in this paper that the anomaly results from the heterogeneity of the propellant blocks. The blocks are effectively striated (with the 'linner'); and we will prove that the combustion velocity of the flame front is an increasing function of the angle between the striations (which are supposed to be straight lines here) and the flame front. Thus, we consider 3 cases: vertical striations ($\alpha = 0$), horizontal striations ($\alpha = \pi/2$), and oblique striations ($0 < \alpha < \pi/2$). We define parameters: $L_0 > 0$, $L_1 = L_0/\cos(\alpha)$ and $L_2 = L_0/\sin(\alpha)$ (see Fig.1). $R_0(s_1, s_2)$ is a positive, periodic function in s_1 with period L_1 and in s_2 with period L_2. When $\alpha = 0$ (resp. $\alpha = \pi/2$), R_0 depends periodically only on s_1 (resp. s_2) with period L_0. The parts of the couch formed by the striations are called 'linner' and 'charge'. L_0 is the sum of the thickness of the 'linner' and the 'charge'.

1991 *Mathematics Subject Classification*: 35B27.
Key words and phrases: Hump effect, striated media, homogenization, viscosity solution.
Received October 2, 1997.

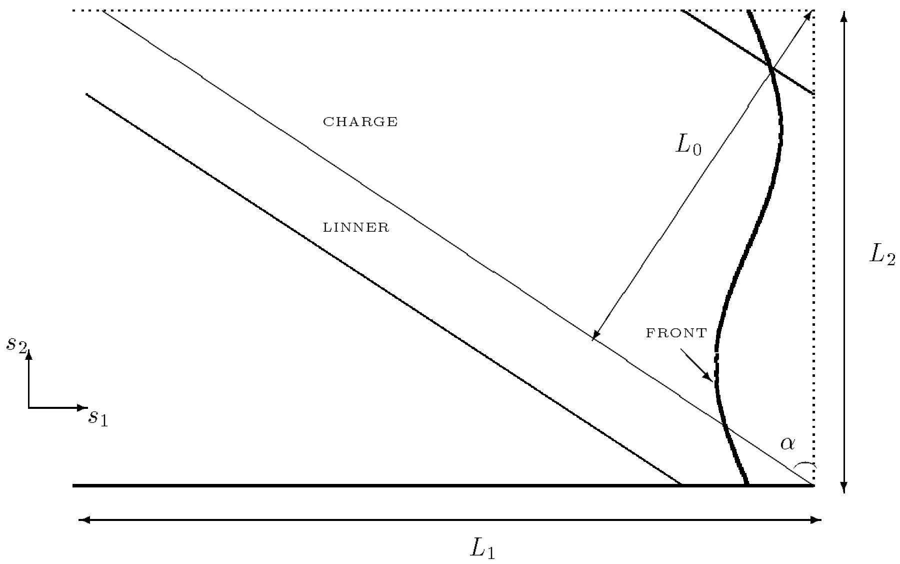

FIGURE 1. Domain of study (One period)

1. EXISTENCE AND UNIQUENESS

1.1. Vertical case. In this case, we have $R_0 = R_0(\xi)$ and the flame front can be reduced to a point. The problem becomes an ordinary differential equation of the form:

$$P_\xi^V \quad \begin{cases} \dfrac{d\xi}{dt} = -R_0(\xi) & t > 0 \\ \xi(0) = \xi_0 \end{cases}$$

One knows that P_ξ^V has a unique solution $\xi \in W^{k+1,\infty}(0,T)$ $\forall k > 0$ and $T > 0$ provided $R_0 \in W^{k,\infty}(\mathbb{R})$.

Proposition 1. *Let T be real, defined by: $\xi(T) - \xi(0) = -L_0$ where L_0 is the period of R_0. Then the unique solution of P_ξ^V verifies: $\dfrac{d\xi}{dt}$ is a periodic function of t with period T.*

Remark 1. T is well defined; it is the time necessary for the front to cover the spacial period L_0.

Proof. It is enough to prove that ξ verifies: $\xi(t) = \xi(t+T) + L_0$ $\forall t > 0$. This is immediate with the uniqueness of ξ because the function $\phi(t) = \xi(t+T) + L_0$ is a solution of P_ξ^V. Thus $\phi = \xi$ and the time-periodicity of $\dfrac{d\xi}{dt}$ (front velocity) follows. □

1.2. Horizontal case. In this section, one looks for periodic or quasi-periodic solutions. R_0 is a regular periodic function of s_2 with period L_0. So we have the following problem:

$$P_\xi^H \quad \begin{cases} \dfrac{\partial \xi}{\partial t} + R_0(s_2)\sqrt{1+\left(\dfrac{\partial \xi}{\partial s_2}\right)^2} = 0 & \forall t > 0\ , \quad s_2 \in \mathbb{R} \\ \xi(s_2, 0) = \xi_0(s_2) & s_2 \in \mathbb{R} \end{cases}$$

where R_0 verifies: $R_0 \in C^2(\mathbb{R})$, $\min\limits_{x\in\mathbb{R}} R_0(x) = R_{0l} \leq R_0(x) \leq R_{0c} = \max\limits_{x\in\mathbb{R}} R_0(x)$ $\forall x \in \mathbb{R}$ with $R_{0c} > R_{0l} > 0$.

Let Ω be an open bounded subset of $\mathbb{R}$. We denote $H(s_2, u) = R_0(s_2)\sqrt{1+u^2}$. Then we have the following theorem due to Crandall-Lions (see CL83):

Theorem 1. *If $\xi_0 \in C(\Omega)$, then the problem P_ξ^H has a unique viscosity solution $\xi \in C(\Omega\times]0,T[)$, i.e. satisfying: if (x_0, t_0) is a local maximum (resp. minimum) point of $\xi - u$, then $\frac{\partial u}{\partial t}(x_0, t_0) + H[x_0, \nabla u(x_0, t_0)] \leq 0$ (resp. ≥ 0). In addition, we have the following inequalities: if $\xi_0 \in W^{1,\infty}(\mathbb{R})$, then the viscosity solution $\xi \in W^{1,\infty}(\mathbb{R}\times]0,T[)$ verifies:*

$$\left\| \frac{\partial \xi}{\partial t} \right\|_{L^\infty(\mathbb{R}\times]0,\infty[)} \leq c_1 \qquad \textit{and} \qquad \left\| \frac{\partial \xi}{\partial s_2} \right\|_{L^\infty(\mathbb{R}\times]0,\infty[)} \leq c_2$$

where c_1 and c_2 are constants depending only on $\nabla \xi_0$.

Proposition 2. *The viscosity solution of P_ξ^H is periodic in s_2 with period L_0 as long as ξ_0 and R_0 are periodic with the same period.*

Let us formally define $\psi = \dfrac{\partial \xi}{\partial s_2}$ and study the problem Q_ψ^H which follows:

$$Q_\xi^H \quad \begin{cases} \dfrac{\partial \psi}{\partial t} + \dfrac{\partial}{\partial s_2}\left[R_0(s_2)\sqrt{1+\psi^2}\right] = 0 \quad \forall t > 0\ , & s_2 \in \mathbb{R} \\ \psi(s_2, 0) = \psi_0(s_2) & s_2 \in \mathbb{R}\,. \end{cases}$$

One remarks that if ξ is a viscosity solution of P_ξ^H, then ψ is an entropic solution (in the Kruzkov sense) of Q_ψ^H.

1.2.1. *Stationary Solutions - physical solution.* These are the solutions which verify $\dfrac{\partial \psi}{\partial t} = 0$, i.e. $\psi(s_2) = \pm\sqrt{\left[\dfrac{c}{R_0(s_2)}\right]^2 - 1}$ where c is a positive constant $c \geq R_{0c}$.

We denote ψ_c as the corresponding solution of ψ. So we have the sequence of the stationary solutions $(\psi_c)_{c\geq R_{0c}}$.

Lemma 1. *The stationary solutions $(\psi_c)_{c\geq R_{0c}}$ are discontinuous.*

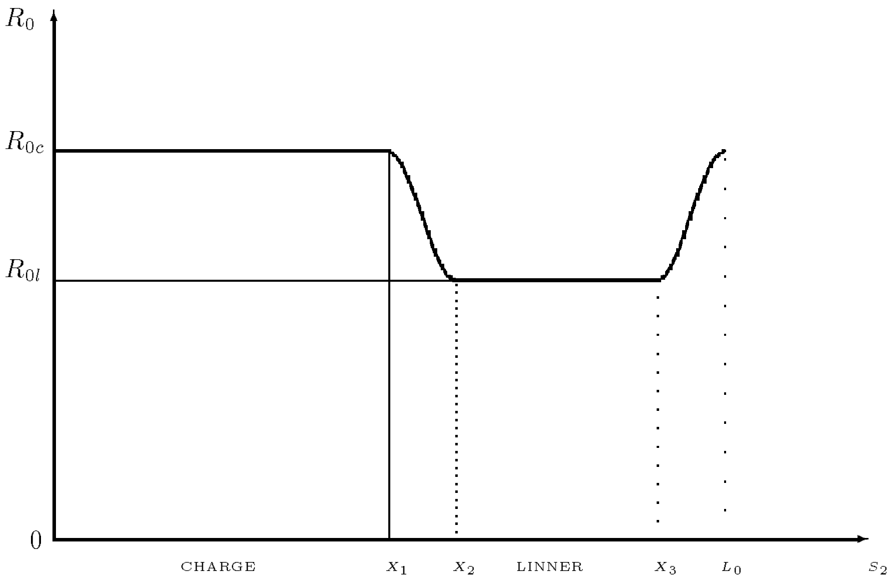

FIGURE 2. Function R_0

Proof. One has the following two properties:

$$[P1]: \quad \exists y^* \in \mathbb{R}; \;\; \psi_c(y^*) = 0, \qquad [P2]: \quad \int_0^{L_0} \psi_c(s_2) ds_2 = 0\,.$$

Applying $[P1]$ we find $c = R_{0c} \equiv c^*$. The relation $[P2]$ implies that ψ_c is negative as well as positive. As $c = c^*$, from the definition of R_0 (see FIG.2), we have: $\psi_c(s_2) = 0 \Longleftrightarrow R_0(s_2) = R_{oc}$, i.e. $s_2 \in$ 'charge'. If ψ_c was not discontinuous, one could find y $\notin$ 'charge' with $\psi_c(y) = 0$. Then since $R_0(y) < R_{oc}$, we have $c^* = R_0(y) < R_{0c} = c^*$ which is absurd. We conclude that ψ_c is not continuous.

1.2.2. *Construction of the physical solution.* The physical solution ξ is the one which verifies $R_{0l} \leq |\frac{\partial \xi}{\partial t}| \leq R_{0c}$. Under these conditions, c^* is the unique value of c which satisfies this inequality. From the curve of R_0 and the value of c^*, $\psi_{c^*}(s_2) = 0 \;\; \forall s_2 \in [0, X_1]$ (see FIG.2), i.e. ψ_{c^*} is continuous on this interval. Since ψ_{c^*} is discontinuous, it exists $x^* \in [X_1, L_0]$ so that $\forall s_2 \in [X_1, L_0]$, we have:

$$\psi_{c^*}(s_2) \;=\; \begin{cases} \sqrt{\left(\dfrac{c^*}{R_0(s_2)}\right)^2 - 1} & \text{if } X_1 \leq s_2 < x^* \\[2ex] -\sqrt{\left(\dfrac{c^*}{R_0(s_2)}\right)^2 - 1} & \text{if } x^* < s_2 \leq L_0\,. \end{cases}$$

The inverse is not possible. In fact, under these conditions, the discontinuity in x^* will be increasing, thus inadmissible, i.e. the solution ψ_{c^*} will not be entropic because H is convex in $\nabla\xi$, $\forall s_2 \in \mathbb{R}$. One easily verifies that ψ_{c^*} has a unique point of discontinuity on $[0, L_0]$ equal to $x^* = \dfrac{X_2 + X_3}{2}$. Then the function ψ_{c^*} is defined as follow:

$$\psi_{c^*}(s_2) = \begin{cases} 0 & \text{if } 0 \le s_2 \le X_1 \\ \sqrt{\left(\dfrac{c^*}{R_0(s_2)}\right)^2 - 1} & \text{if } X_1 \le s_2 < x^* \\ -\sqrt{\left(\dfrac{c^*}{R_0(s_2)}\right)^2 - 1} & \text{if } x^* < s_2 \le L_0\,. \end{cases}$$

Proposition 3. *ψ_{c^*} is the only stationary solution, periodic and with average null of $\psi_t + \left[R_0(s_2)\sqrt{1+\psi^2}\right]_{s_2} = 0$.*

1.2.3. *The evolution problem of ξ.* Let's define the following functional spaces. We note $\mathbb{R}_T = \mathbb{R}\times]0,T[$.

$$\mathcal{W}_0 = \left\{\omega \in W^{1,\infty}(\mathbb{R}); \omega \text{ periodic in } s_2 \text{ with period } L_0\right\}$$

$$\mathcal{W}_T = \left\{\omega \in W^{1,\infty}(\mathbb{R}_T); \frac{\partial\omega}{\partial s_2}, \frac{\partial\omega}{\partial t} \in L^\infty(\mathbb{R}_\infty), \omega \text{ periodic in } s_2 \text{ with period } L_0\right\}.$$

The theorem below follows:

Theorem 2. *Let $R_0 \in C^2(\mathbb{R})$, be positive, periodic on $\mathbb{R}$ with period L_0 and $\xi_0 \in \mathcal{W}_0$. Then the problem P_ξ^H admits a unique viscosity solution $\xi \in \mathcal{W}_T$. In addition, P_ξ^H has a unique wave and explicit solution $\xi_{c^*} \in \mathcal{W}_T$ of the form:*

$$\xi_{c^*}(s_2, t) = -c^*.t + \int_0^{s_2} \psi_{c^*}(x)dx\,.$$

Remark 2. By considering the quasi-periodic solutions we prove that P_ξ^H has a unique wave and explicit solution verifying $\xi_{c^*}(s_2) = \xi_{c^*}(s_2 + L_0) \pm D$ where D is the gap (to the right or left: see section 2.3). The corresponding solution ψ_{c^*} is periodic and of the form:

$$\psi_{c^*}^D(s_2) = \begin{cases} 0 & if \quad 0 \le s_2 \le X_1 \\ \pm\sqrt{\left[\dfrac{c^*}{R_0(s_2)}\right]^2 - 1} & if \quad X_1 \le s_2 < L_0 \end{cases}$$

Proof of the theorem. If $\xi_0 \in \mathcal{W}_0$ then $\dfrac{\partial\xi_0}{\partial s_2} \in L^\infty(\mathbb{R})$. From the theorem 1, P_ξ^H has a unique vicosity solution $\xi \in W^{1,\infty}(\mathbb{R}\times[0,T[)$ with $\psi \in L^\infty(\mathbb{R}\times[0,+\infty[)$ entropic solution of Q_ψ^H. As $\dfrac{\partial\xi}{\partial t} = -R_0(s_2)\sqrt{1+\psi^2}$, we have $\dfrac{\partial\xi}{\partial t}$ and $\dfrac{\partial\xi}{\partial s_2} \in L^\infty(\mathbb{R}\times[0,+\infty[)$. From the periodicity of ψ and ξ in s_2, with period L_0, we have

$\xi_0 \in \mathcal{W}_0$ implies $\xi \in \mathcal{W}_T$. ψ_{c^*} is the stationary solution of Q^H_ψ, implies that ξ_{c^*} is the corresponding wave solution of P^H_ψ.
$\dfrac{\partial \xi_{c^*}}{\partial t} = -R_0(s_2)\sqrt{1+\psi^2_{c^*}} = -c^*$ with $\dfrac{\partial \xi_{c^*}}{\partial s_2} = \psi_{c^*}$ and we deduce that: $\xi_{c^*}(s_2,t) = -c^*.t + \displaystyle\int_0^{s_2} \psi_{c^*}(x)dx$.

2. HOMOGENIZATION

2.1. Vertical case. Let ε be a positive parameter tied up to the dimension of the period and destinated to tighten to 0. We define R_0^ε by: $R_0^\varepsilon(s_1) = R_0\left(\frac{s_1}{\varepsilon}\right)$ and look for $\xi^\varepsilon(t)$ verifying the problem:

$$P^V_{\xi^\varepsilon} \quad \begin{cases} \dfrac{d\xi^\varepsilon}{dt} + R_0^\varepsilon(\xi^\varepsilon) = 0 \quad \forall t > 0 \\ \xi^\varepsilon(0) = \xi_0 \end{cases}$$

From the existence result in the vertical case, we know that for fixed ε there exists a unique $\xi^\varepsilon \in W^{k+1,\infty}(0,T)$ since $R_0 \in W^{k,\infty}(\mathbb{R})$. As R_0 is periodic with period L_0, we have R_0^ε periodic in s_1/ε with period εL_0.

For $\varepsilon \longrightarrow 0$, we have $R_0^\varepsilon \longrightarrow \dfrac{1}{L_0}\displaystyle\int_0^{L_0} R_0(s_1)ds_1 \stackrel{\text{def}}{\equiv} \mathcal{M}_{L_0}(R_0)$ which is the average of R_0. Let ϕ be a test function on [0,T]. We have $\displaystyle\int_0^T \frac{1}{R_0^\varepsilon(\xi^\varepsilon)}\phi(t)d\xi^\varepsilon = -\int_0^T \phi(t)dt$. Let $\tau = \xi^\varepsilon(t)$ and $\xi^\varepsilon(0) = 0$ to simplify then we obtain:

$$\int_0^{\xi^\varepsilon(T)} \frac{1}{R_0^\varepsilon(\tau)}\phi\left[(\xi^\varepsilon)^{-1}(\tau)\right]d\tau = -\int_0^T \phi(t)dt.$$

We also have:

$$\frac{1}{R_0^\varepsilon(\tau)} \quad \stackrel{\varepsilon \to 0}{\longrightarrow} \quad \mathcal{M}_{L_0}\left(\frac{1}{R_0}\right) \quad L^\infty(\mathbb{R}) \text{ weak star}$$

$$\xi^\varepsilon \quad \stackrel{\varepsilon \to 0}{\longrightarrow} \quad \xi \quad \text{uniformly on [0,T].}$$

So for $\varepsilon \longrightarrow 0$, we obtain: $\displaystyle\int_0^{\xi(T)} \mathcal{M}_{L_0}\left(\frac{1}{R_0}\right)\phi\left[\xi^{-1}(\tau)\right]d\tau = -\int_0^T \phi(t)dt$.

By $t = \xi^{-1}(\tau)$, we find: $\displaystyle\int_0^T \mathcal{M}_{L_0}\left(\frac{1}{R_0}\right)\frac{d\xi}{dt}\phi(t)dt = -\int_0^T \phi(t)dt$, i.e. $\dfrac{d\xi}{dt} = -R_0^h$ where R_0^h is the harmonic average of R_0. The following theorem is then proved.

Theorem 3. *The solution ξ^ε of the problem $P^V_{\xi^\varepsilon}$ converges when $\varepsilon \longrightarrow 0$ to ξ verifying: $\xi(t) = -R_0^h t + \xi_0$. It is a progressive wave with velocity $-R_0^h$.*

Remark 3. $-R_0^h$ is exactly the average velocity of the front in one period.

2.2. Horizontal case. As in the vertical case, let's have $R_0^\varepsilon(s_2) = R_0\left(\frac{s_2}{\varepsilon}\right)$ and the following Cauchy problem which is to find ξ^ε verifying :

$$P^H_{\xi^\varepsilon}\quad \begin{cases} \dfrac{\partial \xi^\varepsilon}{\partial t} + R_0^\varepsilon(s_2)\sqrt{1+\left(\dfrac{\partial \xi^\varepsilon}{\partial s_2}\right)^2} = 0 & (s_2,t)\in \mathbb{R}\times]0,T[\\ \xi^\varepsilon(s_2,0) = \xi_0(s_2) & s_2\in\mathbb{R} \end{cases}$$

We look for periodic solutions in s_2 with period L_0. For fixed ε the problem $P^H_{\xi^\varepsilon}$ has a unique viscosity solution $\xi^\varepsilon \in W^{1,\infty}(\mathbb{R}\times]0,T[)$.

We write the asymptotic development of ξ^ε in the form: $\xi^\varepsilon(s_2,t) = \xi^0(s_2,t,y) + \sum_{i\geq 1}\varepsilon^i\xi^i(s_2,t,y)$ where $y = s_2/\varepsilon$. Let $Y =]0, L_0[$; then R_0 is Y-periodic in y and εY-periodic in s_2. For $i \geq 1$, the functions ξ^i are Y-periodic in y. The differenciations with regards to t and s_2 become:

$$\frac{\partial}{\partial t} \longrightarrow \frac{\partial}{\partial t} \qquad \text{and} \qquad \frac{\partial}{\partial s_2} \longrightarrow \frac{\partial}{\partial s_2} + \frac{1}{\varepsilon}\frac{\partial}{\partial y}.$$

So we obtain:

$$\frac{\partial \xi^\varepsilon}{\partial t} = \frac{\partial \xi^0}{\partial t} + \sum_{i\geq 1}\varepsilon^i\frac{\partial \xi^i}{\partial t} \qquad \text{and} \qquad \frac{\partial \xi^\varepsilon}{\partial s_2} = \frac{1}{\varepsilon}\frac{\partial \xi^0}{\partial y} + \sum_{i\geq 0}\epsilon^i\left(\frac{\partial \xi^i}{\partial s_2} + \frac{\partial \xi^{i+1}}{\partial y}\right)$$

We take the square of the equality $\frac{\partial \xi^\varepsilon}{\partial t} = -R_0^\varepsilon(s_2)\sqrt{1+\left(\frac{\partial \xi^\varepsilon}{\partial s_2}\right)^2}$ after replacing $\frac{\partial \xi^\varepsilon}{\partial t}$ and $\frac{\partial \xi^\varepsilon}{\partial s_2}$ by their development. We have after calculations and identification by the power of ε the following equations:

$$[R_0(y)]^2\left(\frac{\partial \xi^0}{\partial y}\right)^2 = 0 \tag{1}$$

$$[R_0(y)]^2\frac{\partial \xi^0}{\partial y}\left(\frac{\partial \xi^0}{\partial s_2} + \frac{\partial \xi^1}{\partial y}\right) = 0 \tag{2}$$

$$\left(\frac{\partial \xi^0}{\partial t}\right)^2 - [R_0(y)]^2\left[1 + \left(\frac{\partial \xi^0}{\partial s_2} + \frac{\partial \xi^1}{\partial y}\right)^2 + 2\left(\frac{\partial \xi^0}{\partial y}\right)\left(\frac{\partial \xi^1}{\partial s_2} + \frac{\partial \xi^2}{\partial y}\right)\right] = 0 \tag{3}$$

We deduce from equation (1) that $\frac{\partial \xi^0}{\partial y} = 0$, i.e. ξ^0 doesn't depend on y. Under these conditions, the equation (3) gives:

$$\left(\frac{\partial \xi^0}{\partial t}\right)^2 = [R_0(y)]^2\left[1 + \left(\frac{\partial \xi^0}{\partial s_2} + \frac{\partial \xi^1}{\partial y}\right)^2\right]$$

From $P^H_{\xi^\varepsilon}$ and $\dfrac{\partial \xi^0}{\partial y} = 0$, we have: $\dfrac{\partial \xi^0}{\partial t} + R_0(y)\sqrt{1+\left(\dfrac{\partial \xi^0}{\partial s_2}+\dfrac{\partial \xi^1}{\partial y}\right)^2} = 0$. As ξ^0 doesn't depend on y, one deduces that $R_0(y)\sqrt{1+\left(\dfrac{\partial \xi^0}{\partial s_2}+\dfrac{\partial \xi^1}{\partial y}\right)^2}$ is constant (from y) which can eventually depend on $\dfrac{\partial \xi^0}{\partial s_2}$; we note it $\bar{H}(p)$ where $p = \dfrac{\partial \xi^0}{\partial s_2}$. Let $v = \xi^1$. The problem to solve is

$$P_v \quad \begin{cases} \text{Find } v \text{ viscosity solution of} \\ \\ R_0(y)\sqrt{1+\left(p+\dfrac{\partial v}{\partial y}\right)^2} = \bar{H}(p) \\ \\ v \text{ Y-periodic in } \ y;\ p \text{ is a `parameter'} \end{cases}$$

From $R_0(y)\sqrt{1+\left(p+\dfrac{\partial v}{\partial y}\right)^2} = \bar{H}(p)$, we have $\dfrac{\partial v}{\partial y} = \pm\sqrt{\left[\frac{\bar{H}(p)}{R_0(y)}\right]^2 - 1} - p$ with $\bar{H}(p) \geq R_0(y) \quad \forall y \in \mathbb{R}$.

Let $y_0 \in \mathbb{R}$ with $R_0(y_0) = R_{0c}$. We consider the function f defined by:

$$f(y) = \frac{1}{L_0}\int_{y_0}^{y}\sqrt{\left[\frac{R_{0c}}{R_0(\tau)}\right]^2 - 1}\; d\tau - \frac{1}{L_0}\int_{y}^{y_0+L_0}\sqrt{\left[\frac{R_{0c}}{R_0(\tau)}\right]^2 - 1}\; d\tau\,.$$

We have: $f(y_0) = -\dfrac{1}{L_0}\displaystyle\int_{y_0}^{y_0+L_0}\sqrt{\left[\frac{R_{0c}}{R_0(\tau)}\right]^2 - 1}\; d\tau$ and $f(y_0+L_0) = \dfrac{1}{L_0}\displaystyle\int_{y_0}^{y_0+L_0}\sqrt{\left[\frac{R_{0c}}{R_0(\tau)}\right]^2 - 1}\; d\tau$.

As f is continuous, for all p as $|p| \leq \dfrac{1}{L_0}\displaystyle\int_{y_0}^{y_0+L_0}\sqrt{\left[\frac{R_{0c}}{R_0(\tau)}\right]^2 - 1}\; d\tau$, $\exists \bar{y} \in [y_0, y_0+L_0]\,;\ f(\bar{y}) = p$ i.e.

$$\int_{y_0}^{\bar{y}}\left[\sqrt{\left(\frac{R_{0c}}{R_0(\tau)}\right)^2 - 1} - p\right] d\tau = \int_{\bar{y}}^{y_0+L_0}\left[\sqrt{\left(\frac{R_{0c}}{R_0(\tau)}\right)^2 - 1} + p\right] d\tau\,.$$

We define then a function $v(y)$ by:

$$v(y) = \begin{cases} \displaystyle\int_{y_0}^{y}\left[\sqrt{\left(\frac{R_{0c}}{R_0(\tau)}\right)^2 - 1} - p\right] d\tau & \text{if } \ y_0 \leq y \leq \bar{y} \\ \\ \displaystyle\int_{y}^{y_0+L_0}\left[\sqrt{\left(\frac{R_{0c}}{R_0(\tau)}\right)^2 - 1} + p\right] d\tau & \text{if } \ \bar{y} \leq y \leq y_0+L_0\,. \end{cases}$$

and extend v to all $\mathbb{R}$ by periodicity. One can verify that $\forall p$ with $|p| \leq \frac{1}{L_0}\int_{y_0}^{y_0+L_0} \sqrt{\left[\frac{R_{0c}}{R_0(\tau)}\right]^2 - 1}$,
the function v defined above is a viscosity solution of P_v.

Lemma 2. $\bar{H}(p) = \max\limits_{y\in\mathbb{R}} R_0(y) \equiv R_{0c}$.

Proof. We have:

$$\frac{\partial v}{\partial y}(y_0^+) = \sqrt{\left[\frac{\bar{H}(p)}{R_0(y_0)}\right]^2 - 1} - p \quad \text{and} \quad \frac{\partial v}{\partial y}(y_0^-) = -\sqrt{\left[\frac{\bar{H}(p)}{R_0(y_0)}\right]^2 - 1} + p\,.$$

Let $p < 0$, then we have $\frac{\partial v}{\partial y}(y_0^+) \geq \frac{\partial v}{\partial y}(y_0^-)$. In the same way, we prove that $\frac{\partial v}{\partial y}(\bar{y}^+) \leq \frac{\partial v}{\partial y}(\bar{y}^-)$. As v is a viscosity solution, the following inequalities hold:

$$\begin{aligned} R_0(y_0)\sqrt{1+(p+\eta)^2} - \bar{H}(p) &\geq 0 \qquad \forall \eta\,;\ \frac{\partial v}{\partial y}(y_0^+) \geq \eta \geq \frac{\partial v}{\partial y}(y_0^-)\,,\\ R_0(\bar{y})\sqrt{1+(p+\zeta)^2} - \bar{H}(p) &\leq 0 \qquad \forall \zeta\,;\ \frac{\partial v}{\partial y}(\bar{y}^+) \leq \zeta \leq \frac{\partial v}{\partial y}(\bar{y}^-)\,. \end{aligned}$$

We deduce that $\bar{H}(p) = R_{0c}$. So the formal homogenized problem is then:

$$P_{\xi^0}^{\bar{H}} \quad \begin{cases} \dfrac{d\xi^0}{dt} + R_{0c} = 0 & t > 0 \\ \xi^0(0) = \mathcal{M}_{L_0}(\xi_0) \end{cases}$$

and the solution ξ^0 is: $\xi^0(t) = \xi_0 - R_{0c}t \quad \forall t \geq 0$. It does not depend on s_2; the 'homogenized' front is a vertical line which velocity does not depend on the **presence** of the striations ('linner').

Remark 4. The absolute value of the velocity of the wave solution is R_{0c} and it is greater than the one in the vertical case (R_0^h).

Theorem 4. *For all $\xi_0^\varepsilon \in W^{1,\infty}(\mathbb{R})$, the solution ξ^ε of $P_{\xi^\varepsilon}^H$ converges uniformly on $\mathbb{R}\times[0,T]$ $\forall T$ $(T < +\infty)$ to the solution ξ^0 of the problem $P_{\xi^0}^{\bar{H}}$ in $C(\mathbb{R}\times[0,T])$.*

Remark 5. In this section, we aimed to calculate $\bar{H}$ explicitly. For the convergence, one can consult (LPV87) in the general case where H is regular, at least locally lipschitzian in $p^\varepsilon = \nabla\xi^\varepsilon$, uniformly in s_2.

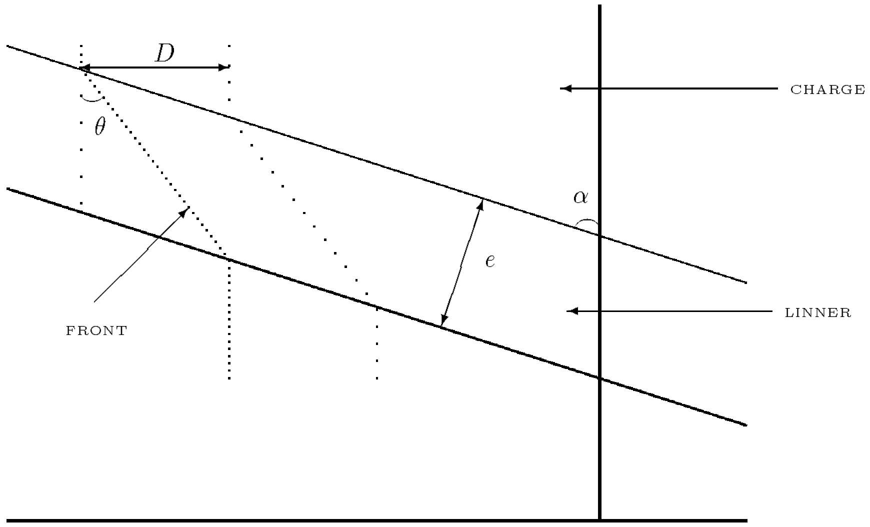

FIGURE 3. Staggered front

2.3. Oblique case. Here, we look for fronts ξ verifying the conditions below (see FIG.3):

i) θ is the angle between the front and the vertical where $R_0(s_2) = R_{0l}$,

ii) $0 \leq \theta \leq \alpha$,

iii) $\dfrac{\partial \xi}{\partial s_2} = 0$ where $R_0(s_2) = R_{0c}$,

iv) The front spreads with constant velocity in the direction of the striations.

Let R_0 be discontinuous with two constant states R_{0c} and R_{0l}. Then we obtain the following relation: $R_{0c}\left(1 - \operatorname{cotg}\alpha \operatorname{tg}\theta\right) = R_{0l}\sqrt{1+\operatorname{tg}^2\theta}$. We deduce the equation for $\operatorname{tg}\theta$ of the form:

$$\left(R_{0l}^2 - R_{0c}^2 \operatorname{cotg}^2\alpha\right)\operatorname{tg}^2\theta + \left(2R_{0c}^2 \operatorname{cotg}\alpha\right)\operatorname{tg}\theta + \left(R_{0l}^2 - R_{0c}^2\right) = 0$$

where $\Delta' = -R_{0l}^4 + R_{0l}^2 R_{0c}^2\left(1+\operatorname{cotg}^2\alpha\right) > 0$ for all R_0 and $\alpha \neq 0$. The relation ii) implies that:

$$\theta = \operatorname{arctg}\left[\left(-R_{0c}^2 \operatorname{cotg}\alpha + \sqrt{\Delta'}\right) / \left(R_{0l}^2 - R_{0c}^2 \operatorname{cotg}^2\alpha\right)\right].$$

If the initial condition is a front with gradient null in the 'charge' and presenting an angle θ in the 'linner', one verifies that these solutions don't distort, i.e. the angle θ is preserved and the velocity in the direction of the striations is constant. These solutions are not periodic but staggered from one period to another with (see FIG.3):

$$D = e\frac{\sin\theta}{\sin(\alpha-\theta)}$$

where e is the thickness of the striations. Under these conditions, one can resolve the problem in the bounded domain $]0, \bar{Y}[$ with the following boundary conditions

$\xi(0) = \xi(\bar{Y}) - D$ for the staggering to the left. In the general case, the staggering to the right doesn't produce fronts with constant velocity in the direction of the striations. Concretely, it is to solve the Hamilton-Jacobi problem with the staggered condition. So we have:

$$P_\xi^D \quad \begin{cases} \dfrac{\partial \xi}{\partial t} + R_0(\xi, s_2)\sqrt{1 + \left(\dfrac{\partial \xi}{\partial s_2}\right)^2} = 0 & \forall (s_2, t) \in]0, \bar{Y}[\times]0, T[, \\ \xi(s_2, 0) = \xi_0(s_2) & s_2 \in]0, \bar{Y}[\\ \xi(0, t) = \xi(\bar{Y}, t) - D & t \geq 0 \end{cases}$$

with $\bar{Y}$ defined by: $\bar{Y} = L_0 + (L_0 - e/\sin\alpha)\dfrac{\operatorname{tg}\theta}{\operatorname{tg}\alpha - \operatorname{tg}\theta}$.

Remark 6. In the horizontal case, $\theta_1 = -\theta_2$. Then one can have the two staggerings, i.e. $\xi(0) = \xi(\bar{Y}) \pm D$ if we wish to stagger to the left or right.

2.3.1. *The average velocity.* We recall that $R_0(s_1, s_2)$ is periodic in s_1 and s_2 with period $L_1 = L_0/\cos\alpha$ and $L_2 = L_0/\sin\alpha$ respectively, for $0 < \alpha < \pi/2$. The average velocity is the quotient of L_1 by the time necessary for the front (or a point of the front) to cover the distance L_1. Let L_c and L_l be the lengths of the 'charge' and the 'linner' respectively on a period, T_c and T_l the corresponding times. Let r be the quotient of the thickness of the 'charge' by the one of the 'linner'. Then we have:

$$e = \frac{L_0}{1+r} \qquad L_l = \frac{L_0}{(1+r)\cos\alpha} \qquad L_c = L_1 - L_l$$

$$T_l = \frac{L_l}{R_{0l}\sqrt{1 + \operatorname{tg}^2\theta}} \qquad T_c = \frac{L_1 - L_l}{R_{0c}}$$

The velocity of the front is equal to $V_c = -R_{0c}$ in the 'charge' and $V_l = -R_{0l}\sqrt{1 + \operatorname{tg}^2\theta}$ in the 'linner'. Let V_m be the absolute value of the average velocity. It is a function of r and α with $\theta = \theta(\alpha)$, let us note it $V_m(r, \alpha)$. Then it verifies: $V_m(r, \alpha) = \dfrac{L_1}{T_c + T_l}$. By replacing L_1 , L_l , T_c , T_l... by their values, one finds after simplification:

$$V_m(r, \alpha) = \frac{1 + r}{\left(\frac{r}{R_{0c}} + \dfrac{1}{R_{0l}\sqrt{1 + \operatorname{tg}^2\theta}}\right)}.$$

- In the vertical case, we have: $\alpha = \theta = 0$ and $V_m(r, 0) = R_0^h$.
- In the horizontal case, $\alpha = \pi/2$, $R_{0c} = R_{0l}\sqrt{1 + \operatorname{tg}^2\theta}$ and $V_m(r, \pi/2) = R_{0c}$.

These values are the same as we found previously. One easily verifies that $V_m(r, \alpha)$ is an increasing function of r and α for fixed R_0.

2.3.2. *The overvelocity coefficient.* For fixed r, it is the rate of the growth of $V_m(r,\alpha)$ between 0 and $\pi/2$. We note it $G(r)$ and have:

$$G(r) = 1 - \frac{V_m(r,0)}{V_m(r,\pi/2)} = 1 - \frac{R_0^h}{R_{0c}}\,.$$

It is a decreasing function of r. For reasonable values of r which determines the lenght of the striations, we observe an overvelocity coefficient analogous to the one found experimentally.

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