ARCHIVUM MATHEMATICUM (BRNO)
Tomus 35 (1999), 151 – 154

AN EXAMPLE RELATED TO STRONGLY POINTWISE SELF–HOMEOMORPHIC DENDRITES

Pavel Pyrih

Abstract. Such spaces in which a homeomorphic image of the whole space can be found in every open set are called *self-homeomorphic.* W.J. Charatonik and A. Dilks posed a problem related to strongly pointwise self-homeomorphic dendrites. We solve this problem negatively in Example 2.1.

1. Introduction.

W.J. Charatonik and A. Dilks introduced four types of self-homeomorphic spaces (see [1], p.217).

Definition 1.1. A topological space X is called *self-homeomorphic* if for any open set $U \subseteq X$ there is a set $V \subseteq U$ such that V is homeomorphic to X.

Definition 1.2. A topological space X is called *strongly self-homeomorphic* if for any open set $U \subseteq X$ there is a set $V \subseteq U$ with nonempty interior such that V is homeomorphic to X.

Definition 1.3. A topological space X is called *pointwise self-homeomorphic at a point* $x \in X$ if for any neighborhood U of x there is a set V such that $x \in V \subseteq U$ and V is homeomorphic to X. The space X is called *pointwise self-homeomorphic* if it is pointwise self-homeomorphic at each of its points.

Definition 1.4. A topological space X is called *strongly pointwise self-homeomorphic at a point* $x \in X$ if for any neighborhood U of x there is a neighborhood V of x such that $x \in V \subseteq U$ and V is homeomorphic to X. The space X is called *strongly pointwise self-homeomorphic* if it is strongly pointwise self-homeomorphic at each of its points.

1991 *Mathematics Subject Classification*: 54F15, 54C25, 54F50.
Key words and phrases: continuum, dendrite, self-homeomorphic.
Research supported by the grant No. GAUK 303-10/6196 of Charles University.
Received April 1, 1998.

Moreover they formulated two conditions (see [1], p.235 conditions 6.15.3 and 6.15.4):

1^* *for any point* p, *any neighbourhood* U *of* p, *there is an embedding* $h : X \to U$ *with* $h(p) = p$ *and* $p \in \operatorname{int} h(X)$;

2^* *for any point* p, *any neighborhood* U *of* p, *there is a neighborhood* V *of* p *with* $V \subseteq U$ *and an embedding* $h : X \to U$ *satisfying* $h|_V = \mathrm{id}_V$.

W.J. Charatonik and A. Dilks posed in [1], *p.237 in Problem 6.25 the following problem.*

Problem 1.5. *Does every strongly pointwise self-homeomorphic continuum (dendrite) satisfy* 1^* *or* 2^* *?*

We give the negative answer to this problem in Example 2.1.

By a *continuum* we mean a compact, connected metric space. By a *dendrite* we mean a locally connected continuum containing no simple closed curves. For a dendrite X, the *order of a point* $x \in X$ is the number of components of $X \setminus \{x\}$. It is denoted $\operatorname{ord}(x)$. If there are infinitely many components of $X \setminus \{x\}$ we say $\operatorname{ord}(x) = \omega$, where $\omega > n$ for every natural number n. Points of order one are called *endpoints* , and points of order three or more are called *ramification* points.

2. Counterexample.

The presented dendrite is from [1], pp. 232-233, Example 6.11.

Example 2.1. *There is a strongly pointwise self-homeomorphic dendrite where the conditions* 1^* *and* 2^* *do not hold.*

Proof. Such a dendrite is pictured in Figure 1. The dendrite X has points of order four on vertical arcs and points of order three on some horizontal arcs.

We say that an arc A in a topological space Y is *of type* k if all non endpoints of A are of order 2 or k in Y and points of A of order k in Y are dense in A.

(i) One can check that X it is a strongly pointwise self-homeomorphic dendrite (see [1], p.232).

(ii) We show that the condition 1^* (and hence 2^* as well) does not hold.

(Proof. We denote G the longest horizontal arc in X and R the longest vertical arc in X. We denote s the common point of the *ground* G and *main root* R. Any arc in X disjoint with the ground G is called an *underground arc.*

We choose p some point of order 3 in X belonging to some (horizontal) underground arc G_p touching the main root.

We choose some neighborhood U of p disjoint with the main root R and show that there is no embedding $h : X \to U$ with $h(p) = p$ and $p \in \operatorname{int} h(X)$.

Let h is such, we conclude a contradiction. We see that the points p and s are joined in X via the arc $A_4 \subset R$ and $A_3 \subset G_p$, where A_4 is of type 4 in X and A_3

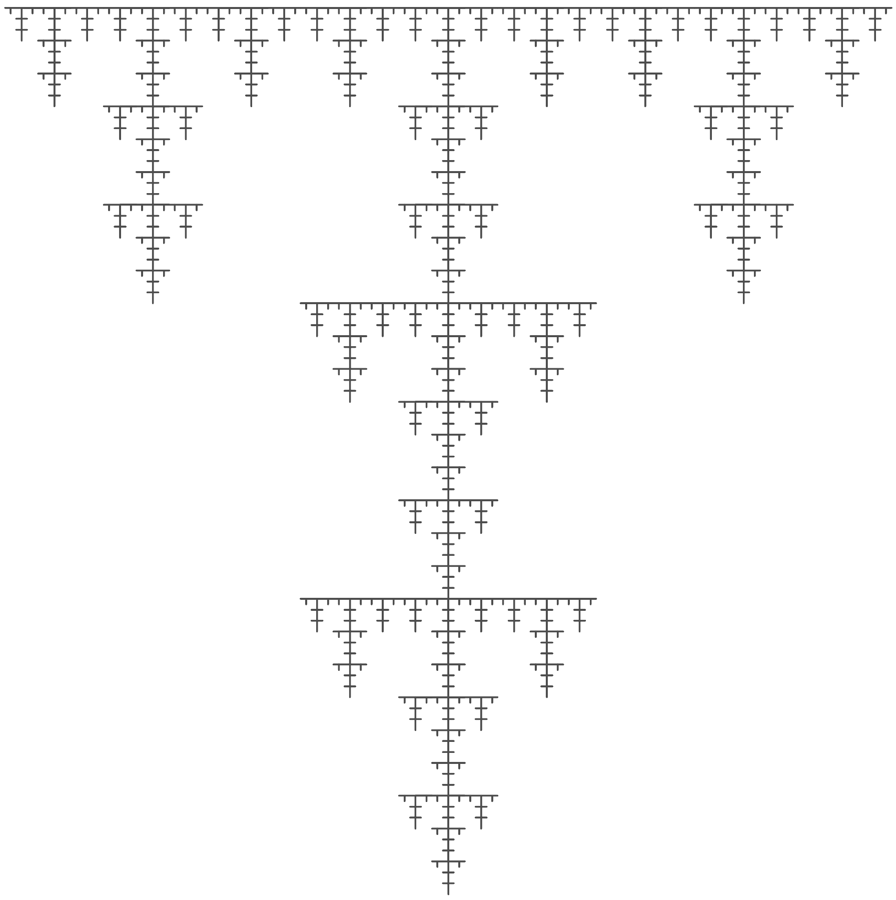

FIGURE 1 (EXAMPLE 2.1).

is of type 3 in X. Their common point q is of order 4 in X and it is an endpoint of just two maximal arcs of type 4 in X and just two maximal arcs of type 3 in X.

The point $h(q)$ must have the same property in $h(X)$. But such a point cannot be joined with $p = h(p)$ in $h(X)$ with an arc of type 3 in $h(X) \subset U$. Clearly. If $h(A_3) \subset G_p$ we cannot reach any point of the desired type. We can reach in U such a point only when we go downstairs the roots of X and cut out some branches in X to obtain the type 3 of the arc originally being of type 4. But we just enter the desired point having cut one of his 'legs' from type 4 to type 3. So we see that the point with the desired properties cannot be reached. This is a contradiction. ♡) The proof is complete. □

Remark 2.2. In fact the strongly piecewise self-homeomorphic dendrite X in Example 2.1 does not satisfy even this weaker form of 1^* and 2^* :

3^* *for any point p, any neighborhood U of p, there is an embedding $h : X \to U$ with $h(p) = p$.*

References

[1] Charatonik, W. J., Dilks, A., *On self-homeomorphic spaces*, Topology and its Applications **55** (1994), 215 – 238.

Department of Mathematical Analysis, Charles University
Sokolovská 83, CZ-186 75 Prague, CZECH REPUBLIC
E-mail: pyrih@karlin.mff.cuni.cz