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AN EXAMPLE FOR MAPPINGS RELATED TO CONFLUENCE

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Abstract. Confluence of a mapping between topological spaces can be defined by several ways. J.J. Charatonik asked if two definitions of the confluence using the components and quasi-components are equivalent for surjective mappings with compact point inverses. We give the negative answer to this question in Example 2.1.

1. Introduction.

We recall from [1] the notation. All *mappings* considered in this paper are continuous. A mapping $f : X \to Y$ between metric continua X and Y is called *confluent* provided for each subcontinuum Q of Y each component of the inverse image $f^{-1}(Q)$ is mapped by f onto Q.

The general topological space admits several definitions of confluent mappings. We present here three definitions using connectedness.

A topological space X is said to be *connected between two its subsets A and B* provided there is no closed and open subset in X that contains A and is disjoint with B. Clearly, connectedness of a space X between points is an equivalence relation on X. The equivalence classes of this relation are called *quasi-components.* In other words, a *quasi-component* of a space X containing a point $p \in X$ is the intersection of all closed and open subsets of X containing p.

Confluence of a mapping $f : X \to Y$ between topological spaces X and Y can be defined by the following conditions :

(C1) For each connected closed nonempty subset Q of Y each *component* of the inverse image $f^{-1}(Q)$ is mapped onto Q under f.

(C2) For each connected closed nonempty subset Q of Y each *quasi-component* of the inverse image $f^{-1}(Q)$ is mapped onto Q under f.

(C3) For each connected closed nonempty subset Q of Y and points $x \in f^{-1}(Q)$ and $y \in Q$ the set $f^{-1}(Q)$ is *connected between* $\{x\}$ and $f^{-1}(y)$.

J.J. Charatonik proved in [1], p.89 the following results.

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Theorem 1.1. *Let $f : X \to Y$ be a surjective mapping between topological spaces X and Y. Then:*

(a) (C1) $\Rightarrow$ (C2) $\Rightarrow$ (C3).

(b) *conditions* (C1) , (C2) *and* (C3) *are equivalent for compact Hausdorff spaces X and Y.*

(b) *each condition* (C1) , (C2) *and* (C3) *is equivalent to confluence for continua X and Y.*

The following result is known (see [2], Corollary 1.4, p. 1337).

Theorem 1.2. *Let $f : X \to Y$ be a surjective mapping between topological spaces X and Y such that $f^{-1}(y)$ is compact for each $y \in Y$. Then conditions* (C2) *and* (C3) *are equivalent.*

J.J. Charatonik asked in [1], Question 2.5 the following question.

Question 1.3. *Does* (C2) *imply* (C1) *for surjective $f : X \to Y$ with compact point inverses between topological spaces X and Y?*

We give the negative answer in Example 2.1.

2. Counterexample.

Example 2.1. *There exist topological spaces X and Y and a surjective $f : X \to Y$ with compact point inverses such that* (C2) *is satisfied and* (C1) *is not satisfied.*

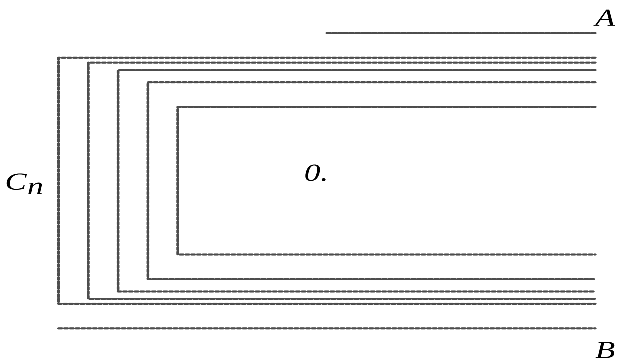

FIGURE 1 (EXAMPLE 2.1).

Proof. Let

$$X = A \cup B \cup \bigcup_{n=1}^{\infty} C_n \quad , \quad Y = A \quad ,$$

where $A = \{(x, 1) \in \mathbb{R}^2 : x \geq 0\}$, $B = \{(x, -1) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ and $C_n = \{(x, y) \in \mathbb{R}^2 : |y| = 1 - 1/n, x \geq -n\} \cup \{(x, y) \in \mathbb{R}^2 : |y| \leq 1 - 1/n, x = -n\}$ as in Figure

1. (X is a simple modification of the *nested rectangles* topological space in [3], p.137.) Let both X and Y inherit the topology from the Euclidean plane.

For the definition of the mapping $f : X \to Y$ we need some notation:

For any $(x, 1) \in A$ denote a_x the plane segment joining $(x, 1)$ with the origin and b_x the plane segment joining $(\log x, -1) \in B$ with the origin.

If $x \geq 0$ denote $x_n^A = a_x \cap C_n$ and $x_0^A = a_x \cap A$ (we have $x_0^A = (x, 1)$). For convenience let $A_n = a_0 \cap C_n$.

If $x > 0$ denote $x_0^B = b_x \cap B$. If $x \geq 1/n$ denote $x_n^B = b_x \cap C_n$. For convenience let $B_n = b_{1/n} \cap C_n$.

For any two points $S, T \in C_n$ denote ST the arc in C_n joining S with T and $d(S, T)$ the linear measure of the arc ST.

If $0 < x < 1/n$ denote x_n^B the point of $A_n B_n$ such that $d(A_n, x_n^B)/d(A_n, B_n) = nx$.

Now we define $f : X \to Y$ by the formula $f(x_j^A) = f(x_j^B) = (x, 1) \in Y = A$ for $j \geq 0$. See Figure 2.

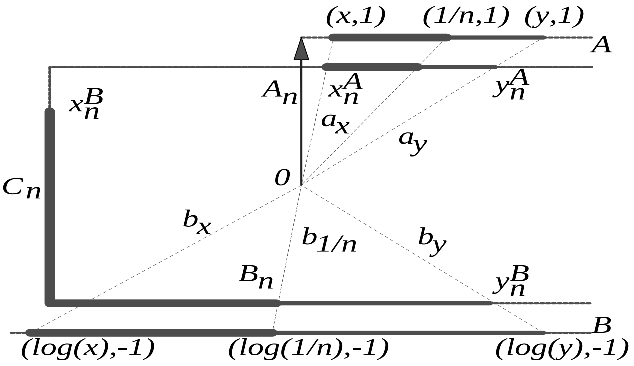

FIGURE 2 (PROOF OF EXAMPLE 2.1).

We claim the following properties of f :

(i) *f is continuous on X.*

Proof. f is a projection from the origin onto A on the set $X \cap \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$. Moreover f is "exp" projection/reflection from B thru the origin onto $A \setminus \{(0, 1)\}$. Given $(\log(x), -1) \in B$, the same is true on some neighborhood in X of $(\log(x), -1)$; in fact if $x > 1/n$ then f is locally again the "exp" projection/reflection from C_n thru the origin into $A \setminus \{(0, 1)\}$. We see that f is " piecewise linear" on the arc $A_n B_n \subset C_n$. Finally f is continuous at $(0, 1) \in X$ because the arcs $A_n B_n \subset C_n$ are mapped onto the segment joining $(0, 1)$ with $(1/n, 1)$ in A. $\square$

(ii) *f has compact point inverses.*

Proof. Clearly point inverse of $(0, 1) \in Y$ is the sequence $\{A_j\}_0^\infty$ converging to A_0 in X, hence compact. Given the point $(x, 1) \in Y$, $x > 0$, we see that the point inverse set is the set $\{x_j^A\}_0^\infty \cup \{x_j^B\}_0^\infty$. Notice that the points x_j^A belong to the segment a_x and all but finite of points x_j^B belong to the segment b_x. The sequence $\{x_j^A\}_0^\infty$ is converging to $x_0^A = (x, 1) \in A$ and the sequence $\{x_j^B\}_0^\infty$ is converging to $x_0^B = (\log(x), -1) \in B$. Hence the point inverse set of $(x, 1) \in Y$, $x > 0$, is again compact in X. □

(iii) *For* $f : X \to Y$ *the condition* (C2) *holds.*

Proof. Let Q be closed nonempty connected subset of Y. We have 2 cases :

Case (1) : When $(0, 1) \notin Q$.

Then $f^{-1}(Q) \cap C_n$ is the set consisting of just two disjoint arcwise connected sets $Q_n^A = \{x_n^A : (x, 1) \in Q\}$ and $Q_n^B = \{x_n^B : (x, 1) \in Q\}$, both of them being closed and open in $f^{-1}(Q)$. Hence both Q_n^A and Q_n^B are quasi-components of $f^{-1}(Q)$ both being mapped onto Q under f.

Similarly $f^{-1}(Q) \cap A$ is the set $Q^A = A \cap \{a_x : (x, 1) \in Q\} = Q$. Clearly Q^A is a quasi-component of $f^{-1}(Q)$, since it is arcwise connected and all sufficiently small ε-neighborhoods of Q^A in $f^{-1}(Q)$ are both open and closed in $f^{-1}(Q)$. Moreover Q^A is mapped onto Q under f.

Similarly $f^{-1}(Q) \cap B$ is the set $Q^B = B \cap \{b_x : (x, 1) \in Q\}$. Clearly Q^B is a quasi-component of $f^{-1}(Q)$, since it is arcwise connected and all sufficiently small ε-neighborhoods of Q^B in $f^{-1}(Q)$ are both open and closed in $f^{-1}(Q)$. Moreover Q^B is mapped onto Q under f. See Figure 2.

Case (2) : When $(0, 1) \in Q$.

The case when $(0, 1) \in Q$ is similar. The set $f^{-1}(Q) \cap C_n$ contains all arcs joining x_n^A with x_n^B in C_n for any $(x, 1) \in Q$. It is again a quasi-component of $f^{-1}(Q)$ and is mapped onto Q under f.

The set $Q^A \cup Q^B$ is a quasi-component of $f^{-1}(Q)$ because no closed and open set of $f^{-1}(Q)$ containing the arcwise connected set Q^A can be disjoint with the arcwise connected set Q^B due to the fact that all arcs joining x_n^A with x_n^B in C_n for any $(x, 1) \in Q$ are in $f^{-1}(Q)$, see Figure 3. Moreover the set $Q^A \cup Q^B$ is mapped onto Q under f. □

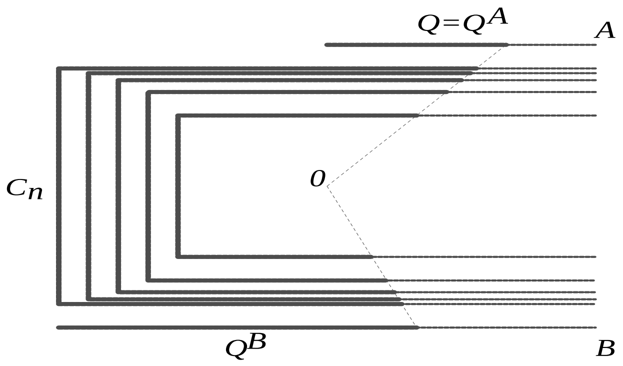

FIGURE 3 (PROOF OF (III) AND (IV) IN EXAMPLE 2.1).

(iv) *For* $f : X \to Y$ *the condition* (C1) *does not hold.*

Proof. If $(0,1) \in Q$, then the set Q^B is component of $f^{-1}(Q)$ because it is arcwise connected and no larger subset of $f^{-1}(Q)$ containing Q^B is connected. Finally we see that $(0,1) \notin f(Q^B)$. The set Q^B is a component of $f^{-1}(Q)$ which is not mapped onto Q under f. See Figure 3. □

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