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NONNEGATIVITY OF FUNCTIONALS CORRESPONDING TO THE SECOND ORDER HALF–LINEAR DIFFERENTIAL EQUATION

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Abstract. In this paper we study extremal properties of functional associated with the half–linear second order differential equation (E_p). Necessary and sufficient condition for nonnegativity of this functional is given in two special cases: the first case is when both points are regular and the second is the case, when one end point is singular. The obtained results extend the theory of quadratic functionals.

1. Introduction

We study the second order half-linear differential equation

$$\left(r(t)\Phi\big(y'(t)\big)\right)' + q(t)\Phi\big(y(t)\big) = 0, \tag{E_p}$$

where $\Phi(y) = y|y|^{p-2}$, $p > 1$, is a real constant and $r(t)$, $q(t)$ are real-valued continuous functions defined on a given non-degenerate interval I, $r(t) > 0$ on I. The domain of the operator on the left hand side is defined to be the set of all continuous real-valued functions y defined on I such that y and $r\Phi(y')$ are continuously differentiable on I. Let I_0 be a subinterval of I. Equation (E_p) is said to be *disconjugate* on I_0 if every non-trivial solution has at most one zero on I_0.

Equation (E_p) has been investigated e.g. by Bihari [1], Elbert [4], Jaroš, Kusano [6], Li, Yeh [9]. Elbert in [4] proved the existence and uniqueness of solution of (E_p) with given initial conditions. He also proved that zeros of two nontrivial solutions of (E_p) either separate each other, or these solutions differ only by a constant multiple and some another facts, that show, that the equation (E_p) has some properties similar to those of linear differential equation.

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A point T is said to be *singular point* of equation (E_p) if $r(T) = 0$ or the one-sided limit $\lim_{t\to T+} q(t)$ resp. $\lim_{t\to T-} q(t)$ are not bounded.

We will introduce two cases

(1) regular case with $I = [a, b]$
(2) singular case with $I = (0, b]$, and singular point $T = 0$.

In the regular case we study the functional

$$J_p(\eta)\Big|_a^b = \int_a^b \Big[r(t)|\eta'(t)|^p - q(t)|\eta(t)|^p\Big]\,\mathrm{d}t$$

on the class of functions

$$U = \{\eta \in AC[a,b]\colon\ \eta(a) = \eta(b) = 0,\ \eta' \in L^p[a,b]\}.$$

In case of singular point zero we study the functional

$$\mathcal{J}_p(\eta)\Big|_0^b = \liminf_{e\to 0+} \int_e^b \Big[r(t)|\eta'(t)|^p - q(t)|\eta(t)|^p\Big]\,\mathrm{d}t$$

on the class of functions

$$\begin{aligned}\mathcal{U} = \{\eta \in C[0,b] : \eta(0) = \eta(b) = 0, \eta \in AC \\ \text{and } \eta'(t) \in L^p \text{ on each closed subinterval of } (0,b]\}.\end{aligned}$$

Any function η of class U resp. $\mathcal{U}$ will be termed an *admissible function* of given class.

When $p = 2$ the equation (E_2) reduces to the usual Sturm–Liouville linear equation and the functionals $J_2(\eta)$, $\mathcal{J}_2(\eta)$ are quadratic functionals.

Jaroš and Kusano in [6] proved a Picone–type identity for (E_p) and for functional $J_p(\eta)\big|_a^b$ over the class of functions

$$\widehat{U} = \{\eta \in C^1[a,b] :\ \eta(a) = \eta(b) = 0,\ \eta(t) \neq 0 \text{ on } (a,b)\}.$$

By this identity they proved the following theorem.

Theorem A (Jaroš–Kusano [6], Li-Yeh [9])**.** *If there exists a solution $y(t)$ of* (E_p) *on $[a,b]$ such that $y(t) \neq 0$ on (a,b), then for all $\eta \in \widehat{U}$ we have $J_p(\eta)\big|_a^b \geq 0$, where the equality holds if and only if η is a constant multiple of y.*

The aim of this paper is the following:

(i) To generalize Theorem A to the class of functions U and to prove equivalency between disconjugacy of (E_p) and nonnegativity of functional $J_p(\eta)$.
(ii) To establish the necessary and sufficient condition for nonnegativity of the singular functional $\mathcal{J}_p(\eta)$.

As concerns claim (i), if $q(t) \geq 0$ on $[a,b]$, then the result implicitly follows from the paper Elbert [4], where a Raleigh quotient for the half-linear differential equation is investigated.

The linear case and the singular quadratic functionals $\mathcal{J}_2(\eta)$ were studied by Došlá–Došlý [2], Kaňovský [7], Leighton–Morse [8]. The following theorem is due to Leighton–Morse.

Theorem B (Leighton–Morse [8]). *Singular quadratic functional $\mathcal{J}_2(\eta)\big|_0^b$ is nonnegative for every admissible function η if and only if equation* (E$_2$) *is disconjugate on* $(0,b)$ *and* singularity condition *is satisfied, i.e. if $w(t)$ is solution of Riccati equation*

$$w'(t)+q(t)+r(t)^{-1}w^2(t)=0$$

on $(0,b)$ such that $\lim_{t\to b^-}w(t)=-\infty$, then

$$\liminf_{t\to 0+} -w(t)\eta^2(t)\ge 0$$

holds for every admissible function η, such that $\mathcal{J}_2(\eta)\big|_0^b$ is finite.

We shall seek for analogy of Leighton–Morse's singularity condition in the case of half-linear equation. Next example shows the fact that disconjugacy of (E$_p$) is not sufficient for nonnegativity of functional $\mathcal{J}_p(\eta)$. It extends an example given in the case $p=2$, see e.g. Leighton–Morse [8].

Example 1. Let us consider half-linear equation

$$\Big(\Phi(y')\Big)'+\Big(\frac{p-1}{p}\Big)^p\frac{1}{t^p}\Phi(y)=0.$$

It is disconjugate on $(0,\infty)$ and $y(t)=t^{\frac{p-1}{p}}$ is its positive solution. Functional $\mathcal{J}_p(\eta)\big|_0^b$ takes then the form

$$\mathcal{J}_p(\eta)\big|_0^b=\liminf_{e\to 0+}\int_e^b\Big[|\eta'(t)|^p-\Big(\frac{p-1}{p}\Big)^p t^{-p}|\eta(t)|^p\Big]\,\mathrm{d}t.$$

Consider admissible function $\eta(t)=t^{\frac{p-1}{p}}(1-\big(\frac{t}{b}\big)^{1/p})$, whereby $p\ge 2$. We have $\eta'(t)=\frac{p-1}{p}t^{-1/p}\Big(1-\frac{p}{p-1}\big(\frac{t}{b}\big)^{1/p}\Big)$, and a direct computation shows that

$$\mathcal{J}_p(\eta)\big|_0^b=\Big(\frac{p-1}{p}\Big)^p\liminf_{e\to 0+}\int_e^1\Big[\Big|1-\frac{p}{p-1}\Big(\frac{t}{b}\Big)^{1/p}\Big|^p-\Big|1-\Big(\frac{t}{b}\Big)^{1/p}\Big|^p\Big]\frac{\mathrm{d}t}{t}.$$

Substituting $\big(\frac{t}{b}\big)^{1/p}=x$ we obtain

$$\mathcal{J}_p(\eta)\big|_0^b=p\Big(\frac{p-1}{p}\Big)^p\liminf_{e\to 0+}\int_e^1\Big[\Big|1-\frac{p}{p-1}x\Big|^p-\Big|1-x\Big|^p\Big]\frac{\mathrm{d}x}{x}.$$

Denote the integrand by $f(x)$. For $p\ge 2$ and $x\in[0,1]$ the following inequality holds

$$f(x)\le g(x):=\begin{cases}\frac{p}{p-1}\Big(\frac{xp^2}{2}-1\Big) & x\in[0,\frac{2}{p^2})\\ 0 & x\in[\frac{2}{p^2},1-\frac{1}{2p-1})\\ \frac{1}{p-1} & x\in[1-\frac{1}{2p-1},1].\end{cases}$$

An easy computation now shows that $\int_0^1 g(x)\,\mathrm{d}x=-\frac{1}{p(2p-1)}<0$ and hence the functional $\mathcal{J}_p(\eta)\big|_0^b$ is negative. Note that $\eta'\approx t^{-1/p}$ near zero, and hence it is not in $L^p[0,b]$, what is the most significant difference between regular and singular case.

2. Auxiliary results

Our method is based on Riccati equation and Picone identity. If $y(t)$ is a solution of (E_p) which has no zero on I, then the function $w(t) = r(t)\Phi(y'(t)/y(t))$ is defined on I and satisfies the *Riccati-type equation*

$$w'(t) + q(t) + (p-1)r(t)^{1-k}|w(t)|^k = 0 \tag{1}$$

where $k > 0$ is such that $1/p + 1/k = 1$ holds. This fact can be proved by a direct computation.

Following lemmas will be used in proving our main results.

Lemma 2.1. *Let (E_p) be disconjugate on (a, b). Then there exists solution $y(t)$ of (E_p) such that $y(b) = 0$ and $y(t) \neq 0$ on (a, b). Function $w(t) = r(t)\Phi(y'(t)/y(t))$ is a solution of (1) defined on (a, b) such that $\lim_{t\to b^-} w(t) = -\infty$.*

Proof. Let $y(t)$ be the solution given by the initial conditions $y(b) = 0$ and $y'(b) = 1$. Suppose that there exist a zero t_0 of $y(t)$ in (a, b). Let $e \in (a, t_0)$. Equation (E_p) is disconjugate, hence $y(e) \neq 0$. From the Sturmian theorem it follows, that the solution $y_1(t)$ given by the initial conditions $y_1(e) = 0$ and $y_1'(e) = 1$ has a zero on (t_0, b) (see Elbert [4]), which is a contradiction with disconjugacy of (E_p) on (a, b). Now the rest of the proof is obvious. □

The following two lemmas play an important role in our later considerations. The first lemma is due to Á. Elbert [3] and the second one is an integral form of the Picone identity for the functional $J_p(\eta)\big|_a^b$ over the class of functions U.

Lemma 2.2 (Elbert)**.** *Let $\eta \in AC[a, b]$, $\eta' \in L^p[a, b]$ and let w be a solution of (1) defined on (a, b). If $\eta(a) = 0$ resp. $\eta(b) = 0$ then it holds $\lim_{t\to a^+} w(t)|\eta(t)|^p = 0$ resp. $\lim_{t\to b^-} w(t)|\eta(t)|^p = 0$.*

Lemma 2.3 (Picone-type identity)**.** *Let (E_p) be disconjugate on (a, b), η be an admissible function of the class U, $y(t)$ be a solution of (E_p) which has no zero on (a, b) and $w(t)$ be the corresponding solution of (1). Then*

$$J_p(\eta)\big|_a^b = \int_a^b P_p(\eta, w, r, t)\,\mathrm{d}t \tag{2}$$

where

$$P_p(\eta, w, r, t) = r(t)|\eta'(t)|^p + (p-1)r^{1-k}(t)|w(t)|^k|\eta(t)|^p - pw(t)\Phi(\eta(t))\eta'(t). \tag{3}$$

The function $P_p(\eta, w, r, t)$ satisfies $P_p(\eta, w, r, t) \geq 0$ and $P_p(\eta, w, r, t) = 0$ if and only if η is a constant multiple of y.

Proof. First, from Lemma 2.1 it follows that there exists a solution $y(t)$ such that $y(b) = 0$ and $y(t) \neq 0$ on (a, b). Hence $w(t)$ is defined on (a, b). In the next we follow method from [3] resp. [5].

Let us compute $(w|\eta|^p)'$. Differentiating and using (1) we get

$$\Big(w(t)|\eta(t)|^p\Big)' = \Big(-q(t) - (p-1)r^{1-k}(t)|w(t)|^k\Big)|\eta(t)|^p + pw(t)\Phi(\eta(t))\eta'(t).$$

From here

$$\begin{aligned}(4)\quad r(t)|\eta'(t)|^p - q|\eta(t)|^p &= \Big(w(t)|\eta(t)|^p\Big)' + r(t)|\eta'(t)|^p + \\ &+ (p-1)r^{1-k}(t)|w(t)|^k|\eta(t)|^p - pw(t)\Phi(\eta(t))\eta'(t) = \\ &= \Big(w(t)|\eta(t)|^p\Big)' + P_p(\eta, w, r, t)\end{aligned}$$

for all t for which $\eta'(t)$ exists. This is a special case of Picone–type identity from Jaroš–Kusano [6]. Integrating (4) over the interval $[a+\varepsilon, b-\varepsilon]$, letting $\varepsilon \to 0^+$ and using Lemma 2.2 we get (2), where the function $P_p(\eta, w, r, t)$ is given by (3).

To complete the proof we use the well-known inequality

$$|X|^p/p + |Y|^k/k - XY \geq 0,$$

where the equality holds if and only if $|X|^p = |Y|^k$ and $XY \geq 0$. Here $p > 1$ and $1/p + 1/k = 1$. Remind that $r(t) > 0$ and let us write P_p in the form

$$P_p(\eta, w, r, t) = p\, r(t)\left(\frac{|\eta'(t)|^p}{p} + \frac{|\Phi(\eta(t))\, w(t)\, r^{-1}(t)|^k}{k} - \eta'(t)\Phi(\eta(t))\frac{w(t)}{r(t)}\right).$$

Hence $P_p(\eta, w, r, t) \geq 0$. Moreover $P_p = 0$ if and only if $|\eta'|^p = |\Phi(\eta)w/r|^k$ and $\eta'\Phi(\eta)w/r \geq 0$. From here and from relations

$$\left|\Phi(\eta)\frac{w}{r}\right|^k = \left|\eta^{p-1}\frac{y'^{p-1}}{y^{p-1}}\right|^{\frac{p}{p-1}} = \left|\eta\frac{y'}{y}\right|^p \quad \text{and} \quad \operatorname{sgn}\eta\frac{y'}{y} = \operatorname{sgn}\Phi(\eta)\frac{w}{r}$$

it follows that $P_p = 0$ holds if and only if $\eta' = \eta\, y'/y$, or, equivalently, $\eta(t)$ is a constant multiple of $y(t)$. The proof of lemma is complete. □

Remark. Relation (2) is not valid in the singular case. In this case, we must modify method of the proof of Lemma 2.3. Integrating (4) over the interval $[e, b-\varepsilon]$ we see that following lemma is true.

Lemma 2.4. *Let* (E_p) *be disconjugate on* $(0, b)$, $e \in (0, b)$, η *be an admissible function of class* $\mathcal{U}$, $y(t)$ *and* $w(t)$ *be the same functions as in Lemma* 2.3. *It holds*

$$(5)\qquad J_p(\eta)\Big|_e^b = -w(e)|\eta(e)|^p + \int_e^b P_p(\eta, w, r, t)\,\mathrm{d}t,$$

where the function $P_p(\eta, w, r, t)$ *is given by* (3).

3. Main results

In the linear case, if (E_2) is disconjugate, then the quadratic functional $J_2(\eta)$ considered on the class of AC functions with fixed end points attains its minimal value for the function, which is a solution of (E_2). This property can be partially generalized to the half–linear case. More precisely, the following theorem holds.

Theorem 3.1. *Let (E_p) be disconjugate on (t_1,t_2), $t_1,t_2 \in I$. Let u be a solution of (E_p) which has no zero on (t_1,t_2), η be a function such that $\eta \in AC[t_1,t_2]$, $\eta' \in L^p[t_1,t_2]$ and $\eta(t_1)=u(t_1)$, $\eta(t_2)=u(t_2)$. Then*

$$J(\eta)\Big|_{t_1}^{t_2} \geq J(u)\Big|_{t_1}^{t_2}.$$

Moreover, if (E_p) is disconjugate on $[t_1,t_2]$, then equality holds if and only if $\eta(t)=u(t)$ on $[t_1,t_2]$.

Proof. Since $u(t)\neq 0$ on (t_1,t_2), the solution of Riccati–type equation $w(t) = r(t)\Phi\big(u'(t)/u(t)\big)$ is defined on (t_1,t_2). Integrating (4) from $t_1+\varepsilon$ to $t_2-\varepsilon$ we have

$$J(\eta)\Big|_{t_1+\varepsilon}^{t_2-\varepsilon} = w|\eta|^p\Big|_{t_1+\varepsilon}^{t_2-\varepsilon} + \int_{t_1+\varepsilon}^{t_2-\varepsilon} P_p(\eta,w,r,t)\,\mathrm{d}t$$

$$J(u)\Big|_{t_1+\varepsilon}^{t_2-\varepsilon} = w|u|^p\Big|_{t_1+\varepsilon}^{t_2-\varepsilon} + \int_{t_1+\varepsilon}^{t_2-\varepsilon} P_p(u,w,r,t)\,\mathrm{d}t.$$

Letting $\varepsilon \to 0^+$, using Lemma 2.2 in case when $w(t_1)$ or $w(t_2)$ are not bounded, i.e. u and η have zero t_1 or t_2, and due to the fact that $P_p(u,w,r,t)=0$ we get

$$J(\eta)\Big|_{t_1}^{t_2} = J(u)\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} P_p(\eta,w,r,t)\,\mathrm{d}t \geq J(u)\Big|_{t_1}^{t_2}.$$

Here the equality holds if and only if $P_p = 0$ almost everywhere on (t_1,t_2) i.e. η is a constant multiple of u. If (E_p) is disconjugate on $[t_1,t_2]$ then at least one of $u(t_1)$, $u(t_2)$ is nonzero value, hence $\eta(t)=u(t)$ on $[t_1,t_2]$. □

Remark. In the linear case we need not condition $u(t)\neq 0$ on (t_1,t_2), see e.g. Reid [10]. Method from here makes use of some properties of bilinear forms and have no direct analogy in the half–linear case.

Following theorem is an extension of Theorem A.

Theorem 3.2. *Functional $J_p(\eta)\Big|_a^b$ is nonnegative for every admissible function $\eta \in U$ if and only if equation (E_p) is disconjugate on (a,b).*

Proof. "⇒": Let $u(t)$ be a solution of (E_p) given by the initial conditions $u(a) = 0$, $u'(a)=1$. Suppose that there exists a point $c \in (a,b)$ conjugate to a. Let $\lambda \in (a,c)$ be a real number. Define an admissible function

$$\eta_\lambda(t) := \begin{cases} u(t) & t \in [a,\lambda] \\ \frac{u(\lambda)}{b-\lambda}(b-t) & t \in [\lambda,b]. \end{cases}$$

and then define the function $\varphi(\lambda) = J_p\big(\eta_\lambda(\cdot)\big)\big|_a^b$. We shall show that there exists λ_0 such that $\varphi(\lambda_0) < 0$. It holds

$$J_p(\eta_\lambda)\big|_a^b = \int_a^\lambda \Big[r(t)|u'(t)|^p - q(t)|u(t)|^p\Big]\,\mathrm{d}t + \Big|\frac{u(\lambda)}{b-\lambda}\Big|^p \int_\lambda^b \Big[r(t) - q(t)|b-t|^p\Big]\,\mathrm{d}t =$$

$$= \big(r(t)\Phi(u'(t))\,u(t)\big|_a^\lambda - \int_a^\lambda u(t)\Big[\big(r(t)\Phi(u'(t))\big)' + q(t)\Phi(u(t))\Big]\,\mathrm{d}t +$$

$$+ \Big|\frac{u(\lambda)}{b-\lambda}\Big|^p \int_\lambda^b \Big[r(t) - q(t)|b-t|^p\Big]\,\mathrm{d}t =$$

$$= |u(\lambda)|^p\Big(r(\lambda)\Phi(u'(\lambda)/u(\lambda)) - \Big|\frac{1}{b-\lambda}\Big|^p \int_a^\lambda \Big[r(t) - q(t)|b-t|^p\,\mathrm{d}t\Big]\Big),$$

where the first integral was computed by integration by parts.

The first term in the parenthesis tends to $-\infty$ if λ tends to c from the left, the second one is bounded and $y(\lambda) \neq 0$ in some ring neighborhood of c, hence, there exists λ_0 such that $\varphi(\lambda_0) < 0$, i.e. the functional $J_p(\eta_{\lambda_0})$ is negative, a contradiction. This means that u have no zero on (a, b). Then by the Sturmian separation theorem no nontrivial solution of (E_p) can have two zeros on (a, b).

"$\Leftarrow$": Follows immediately from (2). □

From Theorem 3.2 we have also immediately necessary condition for nonnegativity of the singular functional $\mathcal{J}_p(\eta)$.

Corollary 3.1. *If $\mathcal{J}_p(\eta)\big|_0^b$ is nonnegative for every admissible function $\eta \in \mathcal{U}$, then equation (E_p) is disconjugate on $(0, b)$.*

Proof. Suppose that $\mathcal{J}_p(\eta)\big|_0^b$ is nonnegative for all $\eta \in \mathcal{U}$ and (E_p) is not disconjugate, i.e. there exists a non-trivial solution $y(t)$ with two zeros on $(0, b)$, say t_1, t_2. Let $e \in (0, t_1)$. From the nonnegativity of $\mathcal{J}_p(\eta)\big|_0^b$ on the class of functions $\mathcal{U}$ it follows nonnegativity of $J_p(\eta)\big|_e^b$ on the class of functions which are admissible on the interval $[e, b]$. From Theorem 3.2, (E_p) is disconjugate on (e, b), which is a contradiction with properties of $y(t)$. □

The opposite statement to Corollary 3.1 does not hold, as we have shown in Example 1. The next theorem gives a necessary and sufficient condition for nonnegativity of singular functional $\mathcal{J}_p(\eta)$ and it is a straightforward generalization of Theorem B.

Theorem 3.3. *Singular functional $\mathcal{J}_p(\eta)\big|_0^b$ is nonnegative for every admissible curve $\eta \in \mathcal{U}$ if and only if (E_p) is disconjugate on $(0, b)$ and the singularity condition is satisfied:*

$$\liminf_{t\to 0+}\Big[-w(t)\big|\eta(t)\big|^p\Big] \geq 0$$

for every admissible curve η for which the functional $\mathcal{J}_p(\eta)\big|_0^b$ is finite, where $w(t)$ is the solution of Riccati-type equation defined on $(0, b)$ such that $\lim_{t\to b^-} w(t) = -\infty$.

Proof. "$\Rightarrow$": Our method follows the idea used by Leighton-Morse [8]. Let $\mathcal{J}_p(\eta)\big|_0^b \geq 0$ for every admissible function of the class $\mathcal{U}$ and let $\eta \in \mathcal{U}$ be such that $\mathcal{J}_p(\eta)\big|_0^b < \infty$. There exists a decreasing sequence $\{e_n\}_{n=1}^{\infty}$ such that $e_n \to 0$ and

$$\mathcal{J}_p(\eta)\big|_0^b = \liminf_{e\to 0+} J_p(\eta)\big|_e^b = \lim_{n\to\infty} J_p(\eta)\big|_{e_n}^b.$$

Let g_n denotes the curve defined by

$$g_n(t) = \begin{cases} \eta(t) & \text{for } t \in (0, e_n] \\ \eta(e_n)y^{-1}(e_n)y(t) & \text{for } t \in (e_n, b], \end{cases}$$

where $y(t)$ is the solution of (E_p) from Lemma 2.1. The function g_n is admissible, so

$$\lim_{n\to\infty} \lim_{m\to\infty} J_p(g_n)\big|_{e_m}^b \geq 0.$$

It holds

$$\lim_{n\to\infty} \lim_{m\to\infty} J_p(g_n)\big|_{e_m}^b = \lim_{n\to\infty} \lim_{m\to\infty} J_p(\eta)\big|_{e_m}^{e_n} + \lim_{n\to\infty} J_p(g_n)\big|_{e_n}^b.$$

Here the first term on the right hand side tends to zero, and for the second one we can use the Picone identity (5). We get

$$J_p(g_n)\big|_{e_n}^b = -w(e_n)|\eta(e_n)|^p,$$

where $w(t)$ is the solution of (1) corresponding to $y(t)$, i.e. $\lim_{t\to b^-} w(t) = -\infty$. Hence

$$\lim_{n\to\infty} -w(e_n)|\eta(e_n)|^p \geq 0. \tag{8}$$

To prove that $\liminf_{t\to 0+} -w(t)|\eta(t)|^p \geq 0$ we use (5), setting $e = e_n$ and letting n tend to infinity. Note that $P_p \geq 0$. We find that both limits $\lim_{n\to\infty} \int_{e_n}^b P_p \,\mathrm{d}t$ and $\lim_{e\to 0+} \int_e^b P_p \,\mathrm{d}t$ exist, are finite and equal. From (5) we get relations

$$\lim_{n\to\infty} J_p(\eta)\big|_{e_n}^b = \lim_{n\to\infty} -w(e_n)|\eta(e_n)|^p + \lim_{n\to\infty} \int_{e_n}^b P_p \,\mathrm{d}t,$$
$$\liminf_{e\to 0+} J_p(\eta)\big|_e^b = \liminf_{e\to 0+} -w(e)|\eta(e)|^p + \lim_{e\to 0+} \int_e^b P_p \,\mathrm{d}t.$$

From here and from (8) it follows that

$$\liminf_{e\to 0+} -w(e)|\eta(e)|^p = \lim_{n\to\infty} -w(e_n)|\eta(e_n)|^p \geq 0.$$

We have proved that the singular condition is necessary for to be $\mathcal{J}_p(\eta)\big|_0^b \geq 0$.

"$\Leftarrow$": Let (E_p) be disconjugate and $\eta \in \mathcal{U}$ be an admissible function. If $\mathcal{J}_p(\eta)\big|_0^b = \infty$, then $\mathcal{J}_p(\eta)\big|_0^b$ is positive and theorem holds. So assume $\mathcal{J}_p(\eta)\big|_0^b < \infty$ and $\liminf_{e\to 0^+} -w|\eta|^p \geq 0$. We use formula (5). Hence

$$\liminf_{e\to 0^+} J_p(\eta)\big|_e^b \geq \liminf_{e\to 0^+} -w(e)|\eta(e)|^p + \liminf_{e\to 0^+} \int_e^b P_p \,\mathrm{d}t.$$

Both terms on the right side of the last inequality are nonnegative, hence the functional is nonnegative too. ☐

Remark. Theorem 3.3 gives a necessary and sufficient condition in the case when the left end point $T = 0$ of I is singular. Using the transformation $t = 1/s$ we can study the functional $\mathcal{J}_p(\eta)\big|_a^\infty$ with the singularity at the right end point $T = \infty$. Denote

$$\begin{aligned}\mathcal{U}_\infty = \{\eta \in C[a,\infty] : \eta(a) = \lim_{t\to\infty} \eta(t) = 0, \eta \in AC \\ \text{and } \eta'(t) \in L^p \text{ on each closed subinterval of } [a,\infty)\}.\end{aligned}$$

Next theorem is then corollary of Theorem 3.3.

Corollary 3.2. *Let $T = \infty$ be a singular point of the equation (E_p). Singular functional $\mathcal{J}_p(\eta)\big|_a^\infty$ is nonnegative for every admissible curve $\eta \in \mathcal{U}_\infty$ if and only if (E_p) is disconjugate on (a,∞) and the singularity condition is satisfied:*

$$\liminf_{t\to\infty} w(t)\big|\eta(t)\big|^p \geq 0$$

for every admissible curve η for which the functional $\mathcal{J}_p(\eta)\big|_a^\infty$ is finite, where $w(t)$ is a solution of Riccati-type equation (1) defined on (a,∞) such that $\lim_{t\to a^+} w(t) = \infty$.

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