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ON–LINE PACKING REGULAR BOXES IN THE UNIT CUBE

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Abstract. We describe a class of boxes such that every sequence of boxes from this class of total volume smaller than or equal to 1 can be on-line packed in the unit cube.

Let C be a subset of Euclidean d-space E^d and let (C_n) be a finite or infinite sequence of d-dimensional convex bodies. We say that (C_n) *can be packed* in C if there exist rigid motions σ_i such that sets $\sigma_i C_i$, where $i = 1, 2, \ldots$, have pairwise disjoint interiors and are subsets of C. By an *on-line packing* we mean a packing in which we are given every C_i, where $i > 1$, only after the motion σ_{i-1} has been provided. We are given C_1 at the beginning. In other words, in the on-line packing each set must be irreversibly put before the next set appears. A survey of results about packing (respectively: on-line packing) sequences of convex bodies is given in [1] (respectively: in [5]).

By a *box* we understand any set of the form

$$\{(x_1, \ldots, x_d);\ t_j \le x_j \le u_j \text{ for } j = 1, \ldots, d\},$$

where $t_j < u_j$ for $j = 1, \ldots, d$. The number $w_j = u_j - t_j$ is called the *j-th width* of this box. By the *unit cube I^d* we mean the set

$$\{(x_1, \ldots, x_d);\ 0 \le x_j \le 1 \text{ for } j = 1, \ldots, d\}.$$

The aim of this paper is to present a class of boxes such that each sequence of boxes from this class of total volume smaller than or equal to 1 can be on-line packed in the unit cube.

Let $q \ge 2$ be a positive integer. By a *q-regular box* we mean a box of the j-th widths of the form $w_j = q^{-m-1}$ for $j \le k$ and $w_j = q^{-m}$ for $j = k+1, \ldots, d$, where $k \in \{0, \ldots, d-1\}$, and where $m \in \{0, 1, \ldots\}$. If $k = 0$ in this formula, then such a q-regular box is called a *q-regular cube.*

In the paper [4] it is shown that every sequence of q-regular cubes of total volume not greater than 1 can be on-line packed in the unit cube. In Theorems 1 and 2 we generalize this result.

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Theorem 1. *Let $q \geq 2$ be a fixed integer. Every sequence of q-regular boxes of total volume smaller than or equal to* 1 *can be on-line packed in the unit cube* I^d.

Proof. Let (R_n) be a sequence of q-regular boxes of total volume not greater than 1. Let m be a non-negative integer and let $k \in \{0, \dots, d-1\}$. By a *subbox of type* (m, k) (or by a *subbox*, for short) we mean the set

$$\{(x_1, \dots, x_d);\ a_j q^{-m-1} \leq x_j \leq (a_j + 1)q^{-m-1} \ \textit{for}\ j \leq k \\ \text{and}\ \ a_j q^{-m} \leq x_j \leq (a_j + 1)q^{-m} \ \text{for}\ \ j = k+1, \dots d\},$$

where $a_j \in \{0, \dots, q^{m+1} - 1\}$ for $j \leq k$ and $a_j \in \{0, \dots, q^m - 1\}$ for $j = k+1, \dots, d$ (see Fig. 1, where $d = 3$, $q = 3$). Obviously, each subbox is a q-regular box.

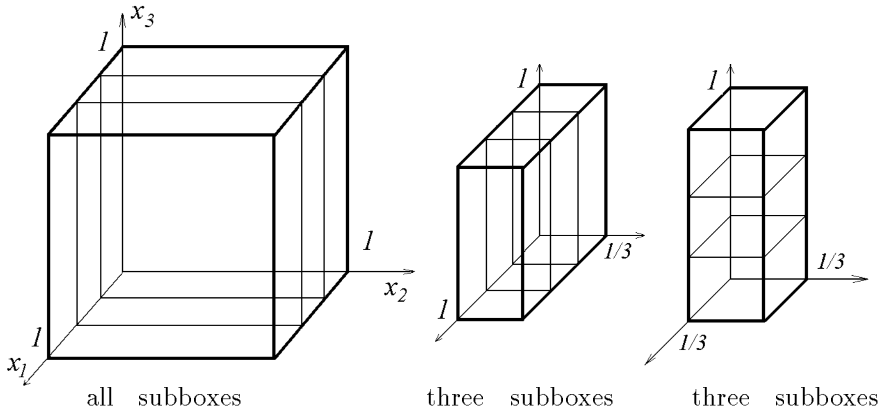

Fig. 1

We enumerate all the subboxes of type (m, k) by integers $1, \dots, q^{md+k}$ in such a way that:

(i) for $k \in \{1, \dots, d-1\}$ the integers $(\lambda - 1)q + 1, \dots, (\lambda - 1)q + q$ are given to the subboxes of type (m, k) being subsets of the subbox of type $(m, k-1)$ whose number is λ,

(ii) for $m \geq 1$ the integers $(\mu - 1)q + 1, \dots, (\mu - 1)q + q$ are given to the subboxes of type $(m, 0)$ being subsets of the subbox of type $(m-1, d-1)$ whose number is μ.

Now, we describe the packing method. We pack R_1 in the first subbox congruent to it. Let $k > 1$. By a *k-free* subbox we mean a subbox whose interior has an empty intersection with $\sigma_1 R_1 \cup \dots \cup \sigma_{k-1} R_{k-1}$. We pack each box R_k from our sequence in the congruent k-free subbox with the smallest possible number.

We will show that (R_n) can be on-line packed in I^d by this method. Assume the opposite; let R_i be a box from (R_n) which cannot be packed in I^d. Obviously, there exists no i-free subbox congruent to R_i. Consider the family $\mathcal{R}$ of all i-free subboxes of maximal volume. (In other words, $S \in \mathcal{R}$ if and only if S is i-free and if there does not exist an i-free subbox S_1 such that $S \subset S_1$ and $S \neq S_1$.) Subboxes from $\mathcal{R}$ have the volumes of the form $q^{-1}\ \mathrm{Vol}(R_i)$, $q^{-2}\ \mathrm{Vol}(R_i), \ldots$.

We show that there are at most $q-1$ subboxes of a fixed type in $\mathcal{R}$. Assume the opposite: there are at least q subboxes of type (m,k) in $\mathcal{R}$. Let us denote these subboxes by $Q_1, \ldots, Q_z$. Consider the case when $k \geq 1$. From the description of the packing method we conclude that q subboxes from among $Q_1, \ldots, Q_z$ are contained in an i-free subbox of type $(m, k-1)$. Consider the case when $k=0$. We can assume that $m \geq 1$, because the box of type $(0,0)$ is nothing else but I^d. In this case q subboxes from among $Q_1, \ldots, Q_z$ are contained in an i-free subbox of type $(m-1, d-1)$. From the above consideration we conclude that q subboxes from among $Q_1, \ldots, Q_z$ do not belong to $\mathcal{R}$ because they are not maximal, a contradiction.

A finite number of boxes has been packed in I^d. Consequently, there exists a finite number of subboxes in $\mathcal{R}$. This means that the total volume of subboxes in $\mathcal{R}$ is smaller than

$$\mathrm{Vol}(R_i)\big[(q-1)q^{-1} + (q-1)q^{-2} + \ldots\big].$$

This value is equal to $\mathrm{Vol}(R_i)$. Hence, the total volume of boxes $R_1, \ldots, R_{i-1}$ is greater than $1 - \mathrm{Vol}(R_i)$. Consequently, the total volume of boxes in (R_n) is greater than 1, a contradiction. □

Remark. Let (S_n) be a sequence of boxes. The method of packing of (S_n) is called *q-adic* (see [2-4]), if for each positive integer n, $\sigma_n S_n$ has edges parallel to the axes of the coordinate system and if for each $j \in \{1, \ldots, d\}$ the projection of $\sigma_n S_n$ on the j-th axis is a segment whose both endpoints are multiples of the j-th width of S_n. Observe, that the packing method from Theorem 1 is q-adic.

Obviously, the estimate 1 in Theorem 1 cannot be improved. It is an open question how to extend our class of q-regular boxes. For example, we cannot add the cube of the width $\frac{1}{2}$ to the class of 4-regular boxes. The reason is that one cube of the width $\frac{1}{2}$ and three boxes of the widths $w_1 = \frac{1}{4}$ and $w_2 = \cdots = w_d = 1$ cannot be packed in I^d. We show in Propositions 1 and 2 that some extensions are possible. Probably, the class of q-regular boxes is however the best possible in the sense given in Conjecture.

Proposition 1. *Let $q \geq 2$ be an integer. Moreover, let $\mathcal{F}$ be the family of boxes such that each box $B \in \mathcal{F}$ is either q-regular or the widths of B are of the form $w_1 = nq^{-1}$, $w_2 = \cdots = w_d = 1$, where $n \in \{2, \ldots, q\}$. Then each sequence of boxes from $\mathcal{F}$ of total volume smaller than or equal to 1 can be on-line packed in I^d.*

Proof. We proceed analogously as in the proof of Theorem 1. We can regard a box B with $w_1 = nq^{-1}$, $w_2 = \cdots = w_d = 1$ as the union of n q-regular boxes with $w_1 = q^{-1}$, $w_2 = \cdots = w_d = 1$. We pack such a box B in I^d similarly like in the method from Theorem 1. Just in the first free place. If such a box

B cannot be packed, then the total volume of boxes preceding B is greater than $1 - nq^{-1} = 1 - \mathrm{Vol}(B)$. $\square$

Let $q \geq 2$ and let $p_1, \dots, p_d$ be positive integers. By a $(q, p_1, \dots, p_d)$-*regular box* we mean a box of the widths of the form $w_j = p_j^{-1} q^{-m-1}$ for $j \leq k$ and $w_j = p_j^{-1} q^{-m}$ for $j = k+1, \dots, d$, where $k \in \{0, \dots, d-1\}$, and where $m \in \{0, 1, \dots\}$. Denote by B_p the box

$$\{(x_1, \dots, x_d);\ 0 \leq x_j \leq p_j^{-1} \ \text{ for } j = 1, \dots, d\}.$$

Obviously, there is an affine image $T(I^d)$ equal to B_p. Observe that if a box B is q-regular, then the affine image $T(B)$ is $(q, p_1, \dots, p_d)$-regular. Thus, from Theorem 1 we conclude that each sequence of $(q, p_1, \dots, p_d)$-regular boxes of total volume smaller than or equal to $\prod_{i=1}^{d} p_i^{-1}$ can be on-line packed in B_p. Consequently, we obtain the following result.

Proposition 2. *Every sequence of $(q, p_1, \dots, p_d)$-regular boxes of total volume smaller than or equal to 1 can be on-line packed in the unit cube.*

Denote by $w(B)$ the greatest width of a box B.

Conjecture. Let $\mathcal{F}$ be a family of boxes such that: (i) $\mathcal{F}$ contains a cube; (ii) for each $\epsilon > 0$ and for each box $B \in \mathcal{F}$ there exists a homotetic copy $k_1 B$ of B such that $k_1 B \in \mathcal{F}$ and $w(k_1 B) < \epsilon$; (iii) for each box $B \in \mathcal{F}$ there exists a homotetic copy $k_2 B$ such that $k_2 B \in \mathcal{F}$ and $w(k_2 B) = 1$, (iv) each sequence of boxes from $\mathcal{F}$ of total volume not greater than 1 can be on-line packed in I^d. Then there exists an integer $q \geq 2$ such that all the boxes from $\mathcal{F}$ are q-regular.

Another interesting question is about the conection between usual packing and on-line packing in the unit cube sequences of boxes of total volume not greater than 1.

Problem 1. Let $\mathcal{F}$ be a family of boxes such that the conditions $(i) - (iii)$ from Conjecture are satisfied and such that each sequence of boxes from $\mathcal{F}$ of total volume not greater than 1 can be packed in I^d. Does there exist an integer $q \geq 2$ such that all the boxes from $\mathcal{F}$ are q-regular?

Problem 2. Let $\mathcal{F}$ be a family of boxes such that each sequence of boxes from $\mathcal{F}$ of total volume not greater than 1 can be packed in I^d. Let (S_n) be a sequence of boxes from $\mathcal{F}$ of total volume smaller than or equal to 1. Can (S_n) be on-line packed in I^d?

Finally, we present a theorem about packing another class of boxes. Let $p_1, \dots, p_d$ and $q_1, \dots, q_d$ be positive integers. Let $m \in \{0, 1, \dots\}$. By $(p_1, q_1, \dots, p_d, q_d)$-*regular box* we mean a box of the j-th widths of the form $w_j = p_j^{-1} q_j^{-m}$, for $j = 1, \dots, d$.

Theorem 2. *Every sequence of* $(p_1, q_1, \ldots, p_d, q_d)$*-regular boxes of total volume not greater than 1 can be on-line packed in the unit cube.*

Proof. The proof is similar to the proof of Theorem 1. We can divide the unit cube into regular subboxes. Let $m \in \{0, 1, \ldots\}$. By a *regular subbox of size* m we mean the set

$$\{(x_1, \ldots, x_d);\ a_j p_j^{-1} q_j^{-m} \le x_j \le (a_j + 1) p_j^{-1} q_j^{-m} \ \text{ for } j = 1, \ldots, d\},$$

where $a_j \in \{0, \ldots, p_j q_j^m - 1\}$. We enumerate all the subboxes. The subboxes of size 0 are enumerated from 1 to $\prod_{i=1}^d p_j$. We enumerate other subboxes in such a way that the integers $(\lambda - 1) \prod_{i=1}^d q_i + 1, \ldots, \lambda \prod_{i=1}^d q_i$ are given to the subboxes of size $m \ge 1$ being subsets of the subbox of size $m - 1$ whose number is λ.

Let (S_n) be a sequence of $(p_1, q_1, \ldots, p_d, q_d)$-regular boxes of total volume smaller than or equal to 1. We pack S_1 in the first subbox congruent to it. Let $k > 1$. By a *k-free* subbox we mean a subbox with such a property that no interior point of it is covered by $\sigma_1 S_1 \cup \cdots \cup \sigma_{k-1} S_{k-1}$. We pack each box S_k from our sequence in the congruent k-free subbox with the smallest possible number. We can now proceed analogously to the proof of Theorem 1. Consequently, (S_n) can be on-line packed in I^d. $\square$

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