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RIEMANNIAN MANIFOLDS IN WHICH CERTAIN CURVATURE OPERATOR HAS CONSTANT EIGENVALUES ALONG EACH HELIX

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Abstract. Riemannian manifolds for which a natural skew-symmetric curvature operator has constant eigenvalues on helices are studied. A local classification in dimension three is given. In the three dimensional case one gets all locally symmetric spaces and all Riemannian manifolds with the constant principal Ricci curvatures $r_1 = r_2 = 0, r_3 \neq 0$, which are not locally homogeneous, in general.

1. Introduction

Curvature is a fundamental notion of differential geometry. A useful technique to describe the curvature along a geodesic in a Riemannian manifold is the use of the Jacobi operator. If the Jacobi operator has constant (resp. pointwise constant) eigenvalues, then the Riemannian manifold M is said to be a globally Osserman (resp. a pointwise Osserman) manifold. Osserman [19] wondered if the eigenvalues of the Jacobi operator are globally constant, need the manifold be at least locally a rank one symmetric space or flat. This question has been answered in the affirmative by Chi [5] if the dimension n is odd, if $n \equiv 2 \bmod 4$, or if $n = 4$; the case $n \equiv 0 \bmod 4$ and $n > 4$ is still open. Pointwise Osserman spaces have been studied by many authors [6, 7, 8]. Examples of 4-dimensional pointwise Osserman spaces which are not globally Osserman have been given in [8].

In [1] J. Berndt and L. Vanhecke have introduced the so called $\mathcal{C}$*-spaces* and $\mathcal{P}$*-spaces* as a generalizations of locally symmetric spaces. A $\mathcal{C}$-space (resp. $\mathcal{P}$-space) is by definition a Riemannian manifold for which the Jacobi operator has constant eigenvalues (resp. parallel eigenspaces) along every geodesic. These manifolds have been investigated in [2, 3, 4]. A local classification of 3-dimensional C-spaces

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and of 3-dimensional $\mathcal{P}$-spaces has been given in [1]. Locally conformal flat 4-dimensional $\mathcal{C}$-spaces and locally conformal flat 4-dimensional $\mathcal{P}$-spaces have been described recently in [13, 14]. Examples of non locally symmetric 4-dimensional $\mathcal{P}$-spaces are constructed in [14].

The skew-symmetric curvature operator R also encodes important information about the curvature. Riemannian manifolds with pointwise constant eigenvalues of R are described locally in dimension four in [12], in all other dimensions except 7 and 8 these spaces are locally classified recently in [9]. If the eigenvalues of R are constant along every unit circle, $\mathbf{M}$ is called an $\mathcal{O}$-space; if R admits parallel Jordanian basis along every unit circle, $\mathbf{M}$ is called a $\mathcal{T}$-space. These manifolds have been studied in [10, 11]. A local description of 3-dimensional $\mathcal{O}$-spaces and $\mathcal{T}$-spaces has been also obtained.

In the present note we describe the curvature along a helix using the curvature operator. A helix in a Riemannian manifold is a smooth curve for which the first and second curvatures are constants and all other curvatures are equal to zero. We consider Riemannian manifold for which the skew-symmetric curvature operators defined along every helix with fixed first and second curvature have constant eigenvalues along the helix. Such a Riemannian manifold is called a *$\mathcal{Q}$-space*. For example, every locally symmetric space is a $\mathcal{Q}$-space. In the same way, O. Kowalski [16] have suggested that could be a relation between 3-dimensional $\mathcal{Q}$ spaces and 3-dimensional curvature homogeneous Riemannian manifolds which are precisely the Riemannian manifolds with constant Ricci eigenvalues. It is natural to expect that the constant Ricci eigenvalues depend on the fixed first and second curvatures of the helix. Unfortunately, we show that this is not entirely true. In fact, we prove the following

Theorem 1.1. *Let $(\mathbf{M}, g)$ be a 3-dimensional $\mathcal{Q}$-space of class C^∞. Then $\mathbf{M}$ is locally almost everywhere (i.e. around the points of an open and everywhere dense set) isometric to one of the following spaces:*

- i) *A space of constant Riemannian sectional curvature.*
- ii) *A Riemannian product of the form $\mathbf{M}_2 \times \mathbf{R}$, where $\mathbf{M}_2$ is a 2-dimensional Riemannian manifold of constant sectional curvature.*
- iii) *A Riemannian manifold with constant Ricci curvatures $r_1 = r_2 = 0$, $r_3 \neq 0$.*

Conversely, any 3-dimensional Riemannian manifold of type i), ii) *or* iii) *is a $\mathcal{Q}$-space.*

Milnor has studied in [18] 3-dimensional Lie groups with left invariant Riemannian metrics with constant principal Ricci curvatures $r_1 = r_2 = 0, r_3 \neq 0$. He has found a lot of examples of such spaces which are locally homogeneous but not locally symmetric ($SU(2)$ with its special left invariant metric for example). By Theorem 1.1, these spaces are examples of $\mathcal{Q}$- spaces which are locally homogeneous but not locally symmetric.

It is well known that a 3-dimensional Riemannian space is curvature homogeneous iff it has constant principal Ricci eigenvalues. Recently, O. Kowalski has constructed in [15] an explicit example of Riemannian 3-manifold with constant principal Ricci curvatures $r_1 = r_2 = 0, r_3 \neq 0$ which is not locally homogeneous. By Theorem 1.1 this is an example of $\mathcal{Q}$-space which is curvature homogeneous but not locally homogeneous. This $\mathcal{Q}$-space is also an $\mathcal{O}$-space but it is neither a $\mathcal{C}$-space nor a $\mathcal{P}$-space [10].

$\mathcal{Q}$-spaces can be regarded as a non-trivial generalization of locally symmetric spaces since every locally symmetric space is a Q-space but, in dimension 3, there is a family of $\mathcal{Q}$-spaces which are not even locally homogeneous and depend essentially on two arbitrary functions of one variable (see [15]). Theorem 1.1 implies that every 3-dimensional $\mathcal{Q}$-space is curvature homogeneous but there is a lot of curvature homogeneous (and even locally homogeneous) 3-manifolds which are not $\mathcal{Q}$-spaces. See [15, 21, 17].

It is clear that the Riemannian manifolds for which the curvature operator R has constant eigenvalues are $\mathcal{Q}$-spaces. In dimension three the converse also holds, i.e. every three dimensional $\mathcal{Q}$-space has constant eigenvalues of the curvature operator R by Theorem 1.1 and Remark 2 of [12]. Moreover, Theorem 1.1 and Theorem 1.1 of [10] imply that 3-dimensional $\mathcal{Q}$-spaces coincide with the 3-dimensional $\mathcal{O}$-spaces. As we know, it is not clear whether $\mathcal{Q}$-spaces coincide with $\mathcal{O}$-spaces or have constant eigenvalues of the curvature operator in higher dimensions.

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2. Characterizations of Q-spaces

Let $(\mathbf{M}, g)$ be an n-dimensional Riemannian manifold and ∇ the Levi-Civita connection of the metric g. We denote by $T_p\mathbf{M}$ the tangential space at a point p of $\mathbf{M}$. The curvature operator is defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for every smooth vector fields X, Y, Z, V on $\mathbf{M}$. The curvature tensor of type (0,4) is defined by $R(X,Y,Z,V) := g(R(X,Y)Z,V)$.

A smooth curve $c(t) : I_\epsilon \to \mathbf{M}, I_\epsilon = (-\epsilon, \epsilon), \epsilon > 0$, with tangent vector field $\dot{c}$, parameterized by the arc length, is said to be a helix if its first curvature k_1 is a constant different from zero, its second curvature k_2 is also a constant different from zero and all other curvatures are zeros. For a helix with fixed k_1 and k_2 we have

$$\nabla_{\dot{c}}\dot{c} = k_1 n; \qquad \nabla_{\dot{c}} n = -k_1 \dot{c} + k_2 b \qquad \nabla_{\dot{c}} b = -k_2 n\,. \tag{2.1}$$

The unit vector field n is the first normal of $c(t)$ and the unit vector field b is its second normal. All other normals orthogonal to $\dot{c}, n$ and b are parallel along $c(t)$.

Proposition 2.1. *A smooth curve $c(t)$ with tangent vector field $\dot{c}$, parameterized by the arc length, is a helix if and only if $c(t)$ satisfies the following differential equation*

$$g(\nabla_{\dot{c}}\dot{c}, \nabla_{\dot{c}}\dot{c})\nabla^3_{\dot{c}}\dot{c} + g(\nabla^2_{\dot{c}}\dot{c}, \nabla^2_{\dot{c}}\dot{c})\nabla_{\dot{c}}\dot{c} = 0\,, \tag{2.2}$$

where ∇^2 and ∇^3 denote the second and the third covariant derivative, respectively.

Proof. We calculate from (2.1) that (2.2) holds, since k_1 and k_2 are constants.

Conversely, let (2.2) hold. We have to prove $k_1 = \text{const.}, k_2 = \text{const.}$ and that the third equality of (2.1) holds. We obtain differentiating the equality $g(\nabla_{\dot{c}}\dot{c}, \dot{c}) = 0$ twice that

$$(2.3) \qquad \dot{c}g(\nabla_{\dot{c}}\dot{c}, \nabla_{\dot{c}}\dot{c}) = 0\,, \quad \dot{c}g(\nabla^2_{\dot{c}}\dot{c}, \nabla^2_{\dot{c}}\dot{c}) = 0\,.$$

We get from the first and the second equalities of (2.1) taking into account (2.3) that

$$k_1^2 = g(\nabla_{\dot{c}}\dot{c}, \nabla_{\dot{c}}\dot{c}) = \text{const.}\,, \quad k_1^2(k_1^2+k_2^2) = k_1^2 g(\nabla_{\dot{c}}n, \nabla_{\dot{c}}n) = g(\nabla^2_{\dot{c}}\dot{c}, \nabla^2_{\dot{c}}\dot{c}) = \text{const.}$$

Now, we obtain the third equality of (2.1) differentiating the second equality of (2.1) and taking into account the first equality of (2.1) and (2.2). □

The equation (2.2) is an ordinary differential equation describing the helices. By Proposition 2.1 and the fundamental theorem of ordinary differential equations, we obtain

Proposition 2.2. *Suppose, we are given two non zero constants k_1 and k_2. If dim $\mathbf{M} \geq 2$ then for every point $p \in \mathbf{M}$ and every three orthonormal vectors $u, v, w \in T_p\mathbf{M}$ there exists locally a unique helix $c(t)$, parameterized by the arc length with first curvature equal to k_1, second curvature equal to k_2 and satisfying the following initial conditions*

$$(2.4) \quad c(0) = p, \qquad \dot{c}(0) = u\,, \quad (\nabla_{\dot{c}}\dot{c})(0) = k_1 v\,, \quad (\nabla^2_{\dot{c}}\dot{c})(0) = -k_1^2 u + k_1 k_2 w\,.$$

Further in the paper, we will consider helices with fixed first curvature k_1 and second curvature k_2. Let $c(t)$ be a helix on $\mathbf{M}$. We consider the families of smooth skew-symmetric linear operators along $c(t)$ defined by

$$(2.5) \qquad \kappa_c(X) := R(\nabla_{\dot{c}}\dot{c}, \nabla^2_{\dot{c}}\dot{c})X\,, \qquad \kappa'_c(X) := (\nabla_{\dot{c}}R)(\nabla_{\dot{c}}\dot{c}, \nabla^2_{\dot{c}}\dot{c})X$$

for every vector field X along $c(t)$.

It follows from (2.2) and (2.5) that

$$(2.6) \qquad \kappa'_c(X) = \nabla_{\dot{c}}\kappa_c(X) - \kappa_c(\nabla_{\dot{c}}X)\,.$$

We have the following characterization of locally symmetric spaces:

Proposition 2.3. *For an n-dimensional ($n \geq 2$) Riemannian manifold $(\mathbf{M}, g)$ the following conditions are equivalent:*

i) *$(\mathbf{M}, g)$ is a Riemannian locally symmetric space;*
ii) *For every helix c on $\mathbf{M}$ we have $\kappa'_c = 0$;*
iii) *For every helix c on $\mathbf{M}$ we have $\kappa_c \circ \nabla_{\dot{c}} = \nabla_{\dot{c}} \circ \kappa_c$;*
iv) *For every helix c on $\mathbf{M}$ the operator κ_c transforms a vector field parallel along c to a vector field parallel along c;*
v) *For every helix c on $\mathbf{M}$ the matrix of κ_c with respect to a basis of vector fields parallel along c is invariant i.e. the matrix has constant entries.*

Proof. The implication i) $\to$ ii) is clear.

To prove the converse, let u, v, w be three orthonormal vectors at a point $p \in \mathbf{M}$. Let $c(t)$ be the helix on $\mathbf{M}$ determined by the initial conditions (2.4). For every $X \in T_p\mathbf{M}$, we get from $\kappa_c' = 0$ the equality

$$k_1(\nabla_u R)(u,v)X = k_2(\nabla_u R)(v,w)X\,. \tag{2.7}$$

Taking the orthonormal frame $\{-u, -v, w\}$ by the frame $\{u, v, w\}$, we get from (2.7) that $(\nabla_u R)(v,w)X = 0$. By the well known properties of the Riemannian curvature operator, we can conclude that the latter equality is valid for arbitrary $u, v, w \in T_p\mathbf{M}$ which proves i).

The equivalence of ii), iii), iv) and v) follows from (2.6). $\square$

Let A be a skew-symmetric linear operator on a Euclidean vector space E. If A has a real eigenvalue then it has to be zero. If A possesses a non-zero eigenvalue of multiplicity m then it has to be of the form $\sqrt{-1}a, a \in \mathbf{R}$. Then $-\sqrt{-1}a$ is also an eigenvalue of A of the same multiplicity. Let A has eigenvalues $0, \sqrt{-1}a_1, -\sqrt{-1}a_1, ..., \sqrt{-1}a_l, -\sqrt{-1}a_l$ of multiplicity $m, m_1, m_1, ..., m_l, m_l$, respectively. Then there exists an orthonormal basis $Z_1, ..., Z_m, X_1^{(1)}, Y_1^{(1)}, ..., X_{m_1}^{(1)}$, $Y_{m_1}^{(1)}, ..., X_1^{(l)}, Y_1^{(l)}, X_{m_l}^{(l)}, Y_{m_l}^{(l)}$ of E such that

$$A(Z_p) = 0, \qquad p = 1, ..., m\,, \tag{2.8}$$

$$A(X_s^{(j)}) = a_j Y_s^{(j)}, A(Y_s^{(j)}) = -a_j X_s^{(j)}, \quad j = 1, ..., l; s = 1, ...m_j\,.$$

Such a basis of E is known as a Jordanian basis for A.

Definition 2.1. A Riemannian manifold $(\mathbf{M}, g)$ is said to be *a $\mathcal{Q}$-space* if for any helix $c(t)$ the operator κ_c has constant eigenvalues along $c(t)$.

An equivalent point of view is the following. Let $c(t)$ be a helix on a Riemannian manifold $(\mathbf{M}, g)$. We may consider the "extrinsic helix flow" on the orthonormal 2-frame bundle of $\mathbf{M}$ whose orbits are defined by $t \longrightarrow (n(t), -k_1\dot{c}(t) + k_2 b(t)) \in T_{c(t)}\mathbf{M} \times T_{c(t)}\mathbf{M}$. A $\mathcal{Q}$-space is a space such that the skew-symmetric curvature operators $R(n(t), -k_1\dot{c}(t) + k_2 b(t))$ on $T_{c(t)}\mathbf{M}$ have constant eigenvalues along every orbit.

Definition 2.2. A Riemannian manifold $(\mathbf{M}, g)$ is said to be a *$\mathcal{J}$-space* if for any helix $c(t)$ the operator κ_c has Jordanian basis parallel along $c(t)$ (at least locally around almost every point of $c(t)$).

Remark 1. It follows from Proposition 2.3 that a smooth Riemannian manifold $\mathbf{M}$ is a locally symmetric Riemannian manifold iff $\mathbf{M}$ is a $\mathcal{Q}$-space and $\mathbf{M}$ is a $\mathcal{J}$-space simultaneously.

Due to this observation it is natural to study the $\mathcal{Q}$-spaces and the $\mathcal{J}$-spaces separately. This leads to two natural generalizations of locally symmetric Riemannian manifolds.

In this paper we consider the $\mathcal{Q}$-spaces.

Our further considerations are based on the following simple

Observation 1.1. If a skew-symmetric linear operator A on an Euclidean vector space E has eigenvalues $0, \sqrt{-1}a_1, -\sqrt{-1}a_1, ..., \sqrt{-1}a_l, -\sqrt{-1}a_l$ then the operator A^2 is self-adjoint linear operator and it has eigenvalues $0, -a_1^2, ..., -a_l^2$.

Further, for each point $p \in \mathbf{M}$ and for any three orthonormal tangent vectors u, v, w at p we define two skew-symmetric endomorphisms $\kappa_{u,v,w}, \kappa'_{u,v,w}$ of $T_p\mathbf{M}$ by

$$\kappa_{u,v,w} = R(k_1 v, -k_1 u + k_2 w), \qquad \kappa'_{u,v,w} = (\nabla_u R)(k_1 v, -k_1 u + k_2 w)\,.$$

The proof of the following result is similar to the proof of Theorem 2.4 in [10] and we omit it.

Theorem 2.4. *Let* $(\mathbf{M}, g)$ *be a smooth Riemannian manifold of dimension* $n(n \geq 2)$. *Then the following conditions are equivalent:*

i) $\mathbf{M}$ *is a* $\mathcal{Q}$*-space;*
ii) *For each point* $p \in \mathbf{M}$ *and for any three tangent vectors* u, v, w *at* p *there exists a skew-symmetric endomorphism* $T_{u,v,w}$ *of* $T_p\mathbf{M}$ *such that*

$$\kappa'_{u,v,w} = \kappa_{u,v,w} \circ T_{u,v,w} - T_{u,v,w} \circ \kappa_{u,v,w}\,. \tag{2.9}$$

Corollary 2.5. *Let* $\mathbf{M} = \mathbf{M}_1 \times ... \times \mathbf{M}_r$ *be a locally reducible smooth Riemannian manifold and each of* $\mathbf{M}_i$ *is smooth. Then* $\mathbf{M}$ *is a* $\mathcal{Q}$*-space iff each of* $\mathbf{M}_i$ *is a* $\mathcal{Q}$*-space.*

3. Three dimensional Q-spaces

In this section we treat the classification problem for $\mathcal{Q}$-spaces of dimension three. The key point for the three dimensional case is the following

Proposition 3.1. *Let* $(\mathbf{M}, g)$ *be a 3-dimensional Riemannian manifold. Then* $\mathbf{M}$ *is a* $\mathcal{Q}$*-space iff the following identities hold*

$$\begin{aligned} & k_1^2\,\{R(X,Y,Y,X)(\nabla_X R)(X,Y,Y,X) \\ & +R(X,Y,Y,Z)(\nabla_X R)(X,Y,Y,Z) + R(Y,X,X,Z)(\nabla_X R)(Y,X,X,Z)\} \\ & + k_2^2\,\{R(Y,Z,Z,Y)(\nabla_X R)(Y,Z,Z,Y) \\ & + R(X,Y,Y,Z)(\nabla_X R)(X,Y,Y,Z) + R(Y,Z,Z,X)(\nabla_X R)(Y,Z,Z,X)\} = 0\,; \end{aligned} \tag{3.10}$$

$$\begin{aligned} & k_1 k_2\,\{R(X,Y,Y,Z)(\nabla_X R)(X,Y,Y,X) \\ & + R(X,Y,Y,Z)(\nabla_X R)(Z,Y,Y,Z) + R(Y,Z,Z,X)(\nabla_X R)(Y,X,X,Z) \\ & - R(Y,X,X,Z)(\nabla_X R)(Y,Z,Z,X) + (R(Y,Z,Z,Y) \\ & +R(Y,X,X,Z))\nabla_X R(X,Y,Y,Z)\} = 0 \end{aligned} \tag{3.11}$$

for any orthonormal vectors X, Y, Z *at any point* $p \in \mathbf{M}$.

Proof. Since $\mathbf{M}$ is 3-dimensional then $\mathbf{M}$ is a $\mathcal{Q}$-space iff for any helix $c(t)$

$$\operatorname{trace}(\kappa_c^2) = \text{const.} \tag{3.12}$$

Let $c(t)$ be the helix determined by the initial conditions (2.4), n be its first normal and b be its second normal given by (2.2). The equality (3.12) is equivalent to the following equation

$$\dot{c}\,[g(\kappa_c(\dot{c}), \kappa_c(\dot{c})) + g(\kappa_c(n), \kappa_c(n)) + g(\kappa_c(b), \kappa_c(b))] = 0\,.$$

By (2.1) the latter equality can be written as

$$g(\kappa_c'(\dot{c}), \kappa_c(\dot{c})) + g(\kappa_c'(n), \kappa_c(n)) + g(\kappa_c'(b), \kappa_c(b)) = 0\,.$$

Evaluating this equality at the point p we obtain that the sum (3.10) + (3.11) is equal to zero. Replacing the orthonormal frame $\{-X, -Y, Z\}$ by the orthonormal frame $\{X, Y, Z\}$, we obtain that the difference (3.11) − (3.10) is equal to zero. Thus, we have proved Proposition 3.1. □

In every point $p \in \mathbf{M}$ we consider the Ricci operator Ric as a linear self-adjoint operator on the tangential space $T_p\mathbf{M}$. Let Ω be the subset of $\mathbf{M}$ on which the number of distinct eigenvalues of Ric is locally constant. This set is open and dense on $\mathbf{M}$. We can choose C^∞ eigenvalue functions of Ric on Ω, say r_1, r_2, r_3, in such a way such that they form the spectrum of Ric at each point of Ω (see for example [1, 20, 15]). We fix a point $p \in \Omega$. Then there exists a local orthonormal frame field E_1, E_2, E_3 on an open connected neighborhood U of p such that

$$Ric(E_i) = r_i E_i\,, \qquad i = 1, 2, 3\,. \tag{3.13}$$

We set $\quad \omega_{ij,k} = g(\nabla_{E_i} E_j, E_k), \quad i, j, k \in \{1, 2, 3\}$.
We obtain using Observation 1.1 the following technical result

Proposition 3.2. it Let $(\mathbf{M}, g)$ be a 3-dimensional Riemannian manifold. Let Ω be an open and dense set of $\mathbf{M}$ and r_1, r_2, r_3 smooth functions forming at each point of Ω the spectrum of Ric. Let $U \subset \Omega$ and let E_1, E_2, E_3 be orthonormal vector fields on U satisfying the condition (3.13). Let $\mathbf{M}$ be a $\mathcal{Q}$-space. Then

i) The Ricci eigenvalues r_1, r_2, r_3 are constants on U;
ii) For all distinct $i, j, k \in \{1, 2, 3\}$, the following formulae are valid on U:

$$r_j(r_k - r_i)\omega_{ii,k} = 0\,; \tag{3.14}$$

$$r_i(r_j - r_k)\omega_{ij,k} = 0\,. \tag{3.15}$$

Proof. For a 3-dimensional Riemannian, manifold the curvature tensor R is given by

$$\begin{aligned} &R(X,Y)Z \\ &\quad = ric(Y,Z)X - ric(X,Z)Y + g(Y,Z)Ric(X) - g(X,Z)Ric(Y) \\ &\qquad\qquad - \frac{1}{2}s\,[g(Y,Z)X - g(X,Z)Y]\,, \quad X, Y, Z \in T_p\mathbf{M}, p \in \mathbf{M}\,, \end{aligned} \tag{3.16}$$

where ric and s denote the Ricci tensor and the scalar curvature of R, respectively.

Using (3.16), we see that (3.10) and (3.11) are equivalent respectively to

$$\begin{aligned}(3.17)\quad & k_1^2\{[s-2ric(Z,Z)][2(\nabla_X ric)(X,X)+2(\nabla_X ric)(Y,Y)-X(s)]\\ &+4ric(X,Z)(\nabla_X ric)(X,Z)+4ric(Y,Z)(\nabla_X ric)(Y,Z)\}\\ &+k_2^2\{[s-2ric(X,X)][2(\nabla_X ric)(Z,Z)+2(\nabla_X ric)(Y,Y)-X(s)]\\ &+4ric(X,Y)(\nabla_X ric)(X,Y)+4ric(X,Z)(\nabla_X ric)(X,Z)\}=0\,;\end{aligned}$$

$$\begin{aligned}(3.18)\quad & k_1k_2\{[(\nabla_X ric)(X,X)+(\nabla_X ric)(Z,Z)\\ &+2(\nabla_X ric)(Y,Y)-X(s)]\,ric(X,Z)-(\nabla_X ric)(X,Z)ric(Y,Y)\\ &-(\nabla_X ric)(Y,Z)ric(X,Y)-(\nabla_X ric)(X,Y)ric(Y,Z)\}=0\end{aligned}$$

for any orthonormal tangent vectors X,Y,Z.

Let $A=(a^1,a^2,a^3), B=(b^1,b^2,b^3), C=(c^1,c^2,c^3)$ be three orthonormal vectors in $\mathbf{R}^3$. We set $X=a^iE_i,\quad Y=b^iE_i,\quad Z=c^iE_i$ (the Einstein summation convention is assumed). Substituting these vectors into (3.17) and into (3.18), we obtain after some calculations

$$\begin{aligned}(3.19)\quad & k_1^2\big\{\big[r_1+(2(c^3)^2-1)r_2+(2(c^2)^2-1)r_3\big]X(r_1)\\ &+\big[r_2+(2(c^3)^2-1)r_1+(2(c^1)^2-1)r_3\big]X(r_2)\\ &+\big[r_3+(2(c^2)^2-1)r_1+(2(c^1)^2-1)r_2\big]X(r_3)\\ &-4c^1c^2r_3\big[a^1(\nabla_{E_1}ric)(E_1,E_2)+a^2(\nabla_{E_2}ric)(E_2,E_1)+a^3(\nabla_{E_3}ric)(E_1,E_2)\big]\\ &-4c^1c^3r_2\big[a^3(\nabla_{E_3}ric)(E_3,E_1)+a^1(\nabla_{E_1}ric)(E_1,E_3)+a^2(\nabla_{E_2}ric)(E_3,E_1)\big]\\ &-4c^2c^3r_1\big[a^1(\nabla_{E_1}ric)(E_2,E_3)+a^2(\nabla_{E_2}ric)(E_2,E_3)+a^3(\nabla_{E_3}ric)(E_3,E_2)\big]\big\}\\ &+k_2^2\big\{\big\{\big[r_1+(2(a^3)^2-1)r_2+(2(a^2)^2-1)r_3\big]X(r_1)\\ &+\big[r_2+(2(a^3)^2-1)r_1+(2(a^1)^2-1)r_3\big]X(r_2)\\ &+\big[r_3+(2(a^2)^2-1)r_1+(2(a^1)^2-1)r_2\big]X(r_3)\\ &-4a^1a^2r_3\big[a^1(\nabla_{E_1}ric)(E_1,E_2)+a^2(\nabla_{E_2}ric)(E_2,E_1)+a^3(\nabla_{E_3}ric)(E_1,E_2)\big]\\ &-4a^1a^3r_2\big[a^3(\nabla_{E_3}ric)(E_3,E_1)+a^1(\nabla_{E_1}ric)(E_1,E_3)+a^2(\nabla_{E_2}ric)(E_3,E_1)\big]\\ &-4a^2a^3r_1\big[a^1(\nabla_{E_1}ric)(E_2,E_3)+a^2(\nabla_{E_2}ric)(E_2,E_3)+a^3(\nabla_{E_3}ric)(E_3,E_2)\big]\big\}=0\,;\end{aligned}$$

$$\begin{aligned}(3.20)\quad & k_1k_2\big\{-\big[a^3c^3r_2+a^2c^2r_3\big]X(r_1)\\ &-\big[a^3c^3r_1+a^1c^1r_3\big]X(r_2)-\big[a^2c^2r_1+a^1c^1r_2\big]X(r_3)+(a^1c^2+a^2c^1)r_3\\ &\big[a^1(\nabla_{E_1}ric)(E_1,E_2)+a^2(\nabla_{E_2}ric)(E_2,E_1)+a^3(\nabla_{E_3}ric)(E_1,E_2)\big]\\ &+(a^1c^3+a^3c^1)r_2\big[a^3(\nabla_{E_3}ric)(E_3,E_1)+a^1(\nabla_{E_1}ric)(E_1,E_3)\\ &+a^2(\nabla_{E_2}ric)(E_3,E_1)\big]+(a^2c^3+a^3c^2)r_1\\ &\big[a^1(\nabla_{E_1}ric)(E_2,E_3)+a^2(\nabla_{E_2}ric)(E_2,E_3)+a^3(\nabla_{E_3}ric)(E_3,E_2)\big]\big\}=0\,.\end{aligned}$$

We consider the following two-frames
$A=(1,0,0), C=(0,0,1)$; $A=(0,1,0), C=(1,0,0)$; $A=(0,0,1), C=(0,1,0)$.
Substituting these two-frames consequently into (3.20) we get (3.14). Analogously, we obtain from (3.19) that for distinct $i,j,k\in\{1,2,3\}$

$$k_1^2E_i(r_i+r_j-r_k)^2+k_2^2E_i(-r_i+r_j+r_k)^2=0\,. \tag{3.21}$$

Using the two-frames
$A=(1,0,0), C=(0,1,0)$; $A=(0,1,0), C=(0,0,1)$; $A=(0,0,1), C=(1,0,0)$ and (3.14), we get from (3.19) and (3.20 that

$$k_1^2E_i(r_i-r_j+r_k)^2+k_2^2E_i(-r_i+r_j+r_k)^2=0 \tag{3.22}$$

for distinct $i,j,k\in\{1,2,3\}$.

Subtracting (3.22) from (3.21), we obtain

$$E_i(r_i(r_j-r_k))=0 \tag{3.23}$$

for distinct $i,j,k\in\{1,2,3\}$. We get adding (3.21) to (3.22)

$$k_1^2\left\{E_i(r_i)^2+E_i(r_j-r_k)^2\right\}+k_2^2E_i(-r_i+r_j+r_k)^2=0 \tag{3.24}$$

for distinct $i,j,k\in\{1,2,3\}$. Substituting the two-frames
$A=(1,0,0), C=(0,\frac{\sqrt2}{2},\frac{\sqrt2}{2})$; $A=(0,1,0), C=(\frac{\sqrt2}{2},0,\frac{\sqrt2}{2})$; $A=(0,0,1), C=(\frac{\sqrt2}{2},\frac{\sqrt2}{2},0)$ into (3.19) and (3.20), we derive taking into account (3.14) that for distinct $i,j,k\in\{1,2,3\}$

$$\begin{aligned}k_1^2\left\{E_i(r_i)^2+E_i(r_j-r_k)^2-4r_i(r_j-r_k)\omega_{ij,k}\right\}&\\ +k_2^2E_i(-r_i+r_j+r_k)^2&=0\,.\end{aligned} \tag{3.25}$$

Similarly, the two-frames
$A=(1,0,0), C=(0,\frac{\sqrt2}{2},-\frac{\sqrt2}{2})$; $A=(0,1,0), C=(-\frac{\sqrt2}{2},0,\frac{\sqrt2}{2})$; $A=(0,0,1), C=(\frac{\sqrt2}{2},-\frac{\sqrt2}{2},0)$, (3.19), (3.20) and (3.14) imply

$$\begin{aligned}k_1^2\left\{E_i(r_i)^2+E_i(r_j-r_k)^2+4r_i(r_j-r_k)\omega_{ij,k}\right\}&\\ +k_2^2E_i(-r_i+r_j+r_k)^2&=0\end{aligned} \tag{3.26}$$

for distinct $i,j,k\in\{1,2,3\}$.

Subtracting (3.26) from (3.25), we get (3.15) which proves ii).

We shall use the following two frames:

$$A=(\frac{\sqrt2}{2},0,\frac{\sqrt2}{2}), C=(-\frac{\sqrt2}{2},0,\frac{\sqrt2}{2})\,; \tag{3.27}$$

$$A=(0,\frac{\sqrt2}{2},\frac{\sqrt2}{2}), C=(0,\frac{\sqrt2}{2},-\frac{\sqrt2}{2});\quad A=(\frac{\sqrt2}{2},\frac{\sqrt2}{2},0), C=(\frac{\sqrt2}{2},-\frac{\sqrt2}{2},0)\,;$$

$$A=(-\frac{\sqrt2}{2},0,\frac{\sqrt2}{2}), C=(\frac{\sqrt2}{2},0,\frac{\sqrt2}{2})\,; \tag{3.28}$$

$$A=(0,\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}),C=(0,\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2});\quad A=(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},0),C=(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0)\,.$$

Substituting (3.27) and (3.28) into (3.19), then adding and subtracting the obtained equalities, we obtain taking into account (3.14) that

$$(k_1^2+k_2^2)E_i\left(r_k^2+(r_i-r_j)^2\right)-4k_1k_2E_i(r_k(r_i-r_j))=0\,, \tag{3.29}$$

$$(k_1^2+k_2^2)E_i\left(r_j^2+(r_k-r_i)^2\right)-4k_1k_2E_i(r_j(r_k-r_i))=0\,. \tag{3.30}$$

for all distinct $i,j,k\in\{1,2,3\}$.

Subtracting (3.30) from (3.29) and taking into account (3.23), we get

$$E_i(r_jr_k)=E_i(r_ir_j) \tag{3.31}$$

for all distinct $i,j,k\in\{1,2,3\}$.

Combining the equalities (3.23), (3.31), (3.14) and (3.21), we obtain for all distinct $i,j,k\in\{1,2,3\}$ that

$$E_i(-r_i+r_j+r_k)=E_i(r_i-r_j+r_k)=E_i(r_i+r_j-r_k)\,.$$

The latter equalities imply that the Ricci eigenvalues are constants on U which completes the proof. □

4. Proof of Theorem 1.1

We have the following possibilities for the constants $r_1(r_2-r_3), r_2(r_3-r_1), r_3(r_1-r_2)$ on U:

Case A. All of them are identically equal to zero;

Case B. One of them is identically equal to zero and the others are non zero;

Case C. All of them are non zero.

CASE A: In this case we have (up to ordering) two possibilities for the constants r_1,r_2,r_3: $r_1=r_2=r_3 \quad \textit{and} \quad r_1=r_2=0, r_3\neq 0$.

In the first case (U,g) is an Einstein manifold and hence it is of constant sectional curvature. The second case is exactly the condition iii) of Theorem 1.1.

CASE B: In this case we have two possibilities for the constants r_1,r_2,r_3:

Case $\mathbf{B}_1$: Two of them are equal and different from the third;

Case $\mathbf{B}_2$: One of them is zero and all are distinct.

Case $\mathbf{B}_1$: We may assume $r_1=r_2\neq r_3$. We have the formula $(\nabla_{E_i}ric)(E_j,E_k)$ $=(r_k-r_j)\omega_{ij,k}$ for all $i,j,k\in\{1,2,3\}$, since the Ricci curvatures are constants. Then, the conditions of the $\mathbf{C}aseB_1$, (3.14) and (3.15) imply $(\nabla ric)=0$. Hence, (U,g) is locally symmetric space by (3.16) and it is locally isometric to i) and ii).

Case $\mathbf{B}_2$. To solve this case we rely essentially on the work of O. Kowalski and F. Prüfer [17]. We use the special coordinate system on U described in [17], p. 18–19. To fit the notations, we note that our r_i and $\omega_{ij,k}$ are denoted by ρ_i and a_{ji}^k in [17], respectively. Further, we have to consider the following two possibilities: a) $r_3=0, r_1\neq r_2$ and b) $r_1=0, r_2\neq r_3$

since the fixed special coordinate system described in [17] has a special choice of the coordinate vector X_3.

In a), the formulas (3.14) and (3.15) imply that the connection coefficients $\omega_{ij,k}$ which may not be zero are $\omega_{32,1} \neq 0, \quad \omega_{22,1} \neq 0, \quad \omega_{11,2} \neq 0$. Then, the formula (16) of [17] implies $\omega_{11,2} = \omega_{22,1} = 0$. So, the non-zero connection coefficient may only be $\omega_{32,1}$. We proceed examining the conditions described in [17]. The formulas (28)-(30) in [17] imply that all nine functions $U_i, V_i, W_i, i = 1,2,3$ are identically zero. Then the formula (26) in [17] implies $\lambda_2 = 0$ and the formula (4) in [17] leads to $\lambda_1 = 0$. Then (4) in [17] asserts $\rho_1 = r_1 = \rho_2 = r_2$ which is impossible.

Assuming b), by similar arguments as in a), we deduce consequently that: the non-zero connection coefficient may only be $\omega_{12,3}$, $\lambda_2 = \lambda_3 = 0$ and finally $\rho_2 = r_2 = \rho_3 = r_3$. So, the case $\mathbf{B}_2$ is impossible.

Case C. In this case, the formulas (3.14) and (3.15) imply that all connection coefficients $\omega_{ij,k}$ are zeros which implies that (U, g) is flat.

It is easy to verify that any manifold of type i) ii) and iii) is a Q-space. Especially, in the case iii) we check that $r_1 = r_2 = 0$ and $r_3 = const$ always satisfies the equations (3.19) and (3.20) which are equivalent to (3.10) and (3.11). This completes the proof of Theorem 1.1. $\square$

Remark 2. One may consider the operators $n_c = R(\dot{c}, \nabla_{\dot{c}}\dot{c})$ and $m_c = R(\dot{c}, \nabla^2_{\dot{c}}\dot{c})$ along a helix instead of the operator κ_c and look for the Riemannian 3-manifolds for which each of these operators has constant eigenvalues along each helix. Applying the same arguments as in section 2, one derive that all these 3-spaces are the 3-dimensional $\mathcal{Q}$-spaces described above.

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