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A POINTWISE INEQUALITY IN SUBMANIFOLD THEORY

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Abstract. We obtain a pointwise inequality valid for all submanifolds M^n of all real space forms $N^{n+2}(c)$ with $n \geq 2$ and with codimension two, relating its main scalar invariants, namely, its **scalar curvature** from the intrinsic geometry of M^n, and its **squared mean curvature** and its **scalar normal curvature** from the extrinsic geometry of M^n in $N^m(c)$.

1. Introduction

Aiming for an answer to a question of S.S. Chern [Ch] concerning intrinsic obstructions for a Riemannian manifold M^n to allow a minimal immersion in the Euclidean ambient space (besides the well-known compactness and the positivity of the Ricci tensor), B. Y. Chen [C2] proved a basic pointwise inequality for all submanifolds M^n in all real space forms $N^m(c)$ of constant curvature c,

$$\delta_M \leq \tfrac{n^2(n-2)}{2(n-1)}\|H\|^2 + \tfrac{1}{2}(n+1)(n-2)c, \tag{1.1}$$

involving, besides c, the square length $\|H\|^2$ of the mean curvature vector of M^n in $N^m(c)$ and a new intrinsic invariant, $\delta_M = n(n-1)/2\,\rho - \inf K$, where ρ and K are respectively the normalized scalar curvature and the sectional curvature function on M^n. In the same paper, Chen also obtained a neat characterization in terms of the second fundamental form of the case when the equality is satisfied in (1.1). Then this inequality was extended to, for instance, all totally real submanifolds in complex space forms and several interesting classes of submanifolds in these ambient spaces could be characterized as those actually satisfying Chen's equality ([CDVV1], [CDVV2], [CDVV3], [CDVV4], [CY], [Da1], [Da2],[DV]).

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Later, B. Y. Chen [C3] obtained a second general and optimal pointwise inequality,

$$\|H\|^2 \geq \rho - c, \tag{1.2}$$

this time involving, besides $\|H\|^2$, only the normalized scalar curvature.

In the light of the study related to those two inequalities of B. Y. Chen, we aimed for a new inequality, also to be pointwise, and to contain as main scalar invariants for the submanifolds under consideration, as intrinsic ones: the scalar curvature ρ and as extrinsic ones: the square $\|H\|^2$ of the mean curvature and also the normal scalar curvature function $\rho^\perp$, as defined below. In this respect, at the geometry meeting at Nordfjordeid in 1995, B. Y. Chen pointed out to us the work of Wintgen [W] in this direction for surfaces in Euclidean 4-space. Wintgen's inequality is extended to surfaces in real space forms by I.V. Guadalupe and L. Rodriguez [GR]. We could however also make progress in the higher dimensional case. Our main aim here is to prove the following theorem.

Main Theorem. *Let $\phi : M^n \to N^{n+2}(c)$ be an isometric immersion. Then at every point p, we have*

$$\|H\|^2 \geq \rho + \rho^\perp - c. \tag{1.3}$$

We remark that for $n = 2$, the inequality (1.3) is Wintgen's inequality. Further, we give examples of several classes of submanifolds realizing the equality in (1.3) and we will also obtain classification theorems about various classes of submanifolds realizing the equality in (1.3). In view of the above main theorem, we would like to conjecture the following.

Conjecture. *Let $\phi : M^n \to N^m(c)$ be an isometric immersion. Then at every point p, we have*

$$\|H\|^2 \geq \rho + \rho^\perp - c. \tag{1.4}$$

As said above, this conjecture is proved for $n = 2$, $m = 4$ and $c = 0$ by Wintgen [W]; for $n = 2$ and $m \geq 4$ by Guadalupe and Rodriguez [GR]; in both cases equality is realized in (1.4) at a point p if and only if the ellipse of curvature at p is a circle. In case of trivial normal connection, (1.4) reduces to Chen's inequality (1.2). For 3-dimensional totally real submanifolds in the nearly Kaehler 6-sphere, the conjecture is also true [DDVV].

2. Preliminaries

We will use the following three trivial inequalities.

$$\begin{aligned} 2ab &\leq a^2 + b^2 \\ &= \text{ if and only if } a = b \end{aligned} \tag{2.1}$$

$$\begin{aligned} (a+b)^2 &\geq 4ab \\ &= \text{ if and only if } a = b \end{aligned} \tag{2.2}$$

$$\begin{aligned} (a+b)^2 &\leq 2(a^2 + b^2) \\ &= \text{ if and only if } a = b. \end{aligned} \tag{2.3}$$

Putting $a = A - B$ and $b = B - C$ in (2.3), we obtain that

$$(2.4) \qquad (A-C)^2 \le 2\left((A-B)^2 + (B-C)^2\right)$$
$$= \text{ if and only if } A - B = B - C, \text{ i.e. } B = \tfrac{1}{2}(A+C)$$

Lemma 2.1. *Let $a_1, \dots, a_n \in \mathbb{R}$ and define $A = \sum_{i<j}(a_i - a_j)^2$. Then*

(1) *$A \ge \frac{n}{2}(a_1 - a_2)^2$ and equality holds if and only if $\frac{1}{2}(a_1 + a_2) = a_3 = a_4 = \cdots = a_n$.*
(2) *Let k, ℓ be integers such that $1 \le k < \ell \le n$ and $(k, \ell) \ne (1, 2)$. If*

$$A = \tfrac{n}{2}(a_1 - a_2)^2 = \tfrac{n}{2}(a_k - a_\ell)^2,$$

then $a_1 = a_2 = a_3 = \cdots = a_n$.

Proof. First, notice that if $n = 2$, then the lemma is trivial. Let us now assume that the lemma is satisfied for some integer number $n \ge 2$. Then, using (2.4), we find that

$$\begin{aligned}\sum_{i<j=1}^{n+1}(a_i - a_j)^2 &= \sum_{i<j=1}^{n}(a_i - a_j)^2 + \sum_{i=1}^{n}(a_i - a_{n+1})^2 \\ &\ge \tfrac{n}{2}(a_1 - a_2)^2 + (a_1 - a_{n+1})^2 + (a_2 - a_{n+1})^2 \\ &\ge \tfrac{n}{2}(a_1 - a_2)^2 + \tfrac{1}{2}(a_1 - a_2)^2 = \tfrac{n+1}{2}(a_1 - a_2)^2,\end{aligned}$$

and if the equality is realized, then $a_3 = a_4 = \cdots = a_n = a_{n+1}$ and $a_{n+1} = \frac{1}{2}(a_1 + a_2)$. The converse follows by a straightforward computation.

In order to prove (2), we first remark that if $k, \ell > 2$ it follows from (1) that $a_k = a_\ell$; then $A = 0$ and the conclusion follows. Therefore, we may assume that $\ell > 2$ and that $k = 2$. Applying then (1) for both the indices $(1, 2)$ and $(2, \ell)$, we find that

$$\begin{aligned}a_\ell &= \tfrac{1}{2}(a_1 + a_2) \\ a_1 &= \tfrac{1}{2}(a_\ell + a_2).\end{aligned}$$

From these two equations, we find that $a_\ell = a_1 = a_2$, so again $A = 0$. □

3. The normal scalar curvature

From now on we will assume that $\phi : M^n \to N^m(c)$ is an isometric immersion of M^n into a real space form of constant sectional curvature c. To avoid confusion we recall that the normalized scalar curvature ρ of M^n is defined by

$$\rho = \tfrac{2}{n(n-1)} \sum_{i<j=1}^{n} \langle R(e_i, e_j)e_j, e_i \rangle,$$

where $\{e_1, \dots, e_n\}$ is any orthonormal basis. Denoting the connection on $N^m(c)$ by D, we have the standard formulas of Gauss and Weingarten which state for tangent vector fields X, Y and a normal vector field ξ that

$$D_X Y = \nabla_X Y + h(X, Y),$$
$$D_X \xi = -A_\xi X + \nabla^\perp_X \xi.$$

We denote the curvature tensor of $\nabla^\perp$ by $R^\perp$. The equations of Gauss, Codazzi and Ricci respectively state that

$$\begin{aligned}\langle R(X,Y)Z, W\rangle &= c(\langle Y, Z\rangle \langle X, W\rangle - \langle X, Z\rangle \langle Y, W\rangle)\\ &\quad + \langle h(Y,Z), h(X,W)\rangle - \langle h(X,Z), h(Y,W)\rangle,\\ (\nabla h)(X,Y,Z) &= (\nabla h)(Y,X,Z),\\ \left\langle R^\perp(X,Y)\xi, \eta\right\rangle &= \langle [A_\xi, A_\eta]X, Y\rangle .\end{aligned}$$

We now propose the following notion of normal scalar curvature $\rho^\perp$. We define for all $p \in M^n$

$$\rho^\perp(p) = \tfrac{2}{n(n-1)} \sqrt{\sum_{i<j=1}^{n} \sum_{r<s=1}^{(m-n)} \langle R^\perp(e_i, e_j)\xi_r, \xi_s\rangle^2}, \tag{3.1}$$

where $\{e_1, \dots, e_n\}$ (resp. $\{\xi_1, \dots, \xi_{m-n}\}$) is an orthonormal basis of the tangent space (resp. normal space) at the point p. For $n = 2$, this definition is compatible with the definition of normal curvature for surfaces as introduced by Wintgen (if $m = 4$) and by Guadalupe and Rodriguez. Remark that up to a constant factor, $\rho^\perp$ corresponds to the square length of the normal curvature tensor, from which we observe that the normal connection of M^n is flat if and only if $\rho^\perp = 0$, and by a result of Cartan, this is equivalent to the simultaneous diagonalisability of all shape operators A_ξ.

One could ask "why the normalization factor in (3.1)?". The main reason is that this normalization enables us to write the inequality (1.3) in an elegant form. Further, we do not want to have the codimension involved, since we do not want the normal scalar curvature to change when enlarging the codimension artificially. Secondly, one can also ask "why the square root factor in (3.1)?". Two reasons for that : then the definition is compatible with earlier definitions and it ensures that the inequality (1.3) does not change when applying a homothetical transformation.

4. A pointwise inequality

In this section, we restrict to the case that the codimension is two, i.e. we assume that $\phi : M^n \to N^{n+2}(c)$ is an isometric immersion in a space form of dimension $n + 2$.

Theorem 4.1. *Let $\phi : M^n \to N^{n+2}(c)$ be an isometric immersion. Then at every point p, we have*

$$\|H\|^2 \geq \rho + \rho^\perp - c. \tag{4.1}$$

Moreover, the equality in (4.1) holds at a point p of M^n if and only if there exists an orthonormal basis $\{e_1, \dots, e_n\}$ of the tangent space and an orthonormal basis ξ_1, ξ_2 of the normal space such that

$$A_{\xi_1} = \begin{pmatrix} \lambda & \mu & 0 & \dots & 0 \\ \mu & \lambda & 0 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix} \qquad A_{\xi_2} = \begin{pmatrix} \mu & 0 & 0 & \dots & 0 \\ 0 & -\mu & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Proof. Let $p \in M^n$. We choose ξ_1 in the direction of the mean curvature vector and choose ξ_2 orthogonal to it. We take an orthonormal basis $\{e_1, \dots, e_n\}$ such that A_{ξ_2} is diagonal. Putting $\langle h(e_i, e_j), \xi_k \rangle = h_{ij}^k$, the equation of Ricci states that

$$\left\langle R^\perp(e_i, e_j)\xi_1, \xi_2 \right\rangle = \langle A_{\xi_2} e_i, A_{\xi_1} e_j \rangle - \langle A_{\xi_1} e_i, A_{\xi_2} e_j \rangle = (h_{ii}^2 - h_{jj}^2) h_{ij}^1.$$

From the definition of the mean curvature vector, it follows that

$$\begin{aligned} n^2 \|H\|^2 &= (\sum_{i=1}^n h_{ii}^1)^2 + (\sum_{i=1}^n h_{ii}^2)^2 \\ &= \tfrac{1}{n-1} \left(\sum_{i<j} (h_{ii}^1 - h_{jj}^1)^2 + \sum_{i<j} (h_{ii}^2 - h_{jj}^2)^2 \right) + \tfrac{2n}{n-1} \sum_{i<j} (h_{ii}^1 h_{jj}^1 + h_{ii}^2 h_{jj}^2). \end{aligned}$$

So, by applying the Gauss equation and also using our choice of basis, we deduce that

$$n^2(n-1)\|H\|^2 = \sum_{i<j} (h_{ii}^1 - h_{jj}^1)^2 + \sum_{i<j} (h_{ii}^2 - h_{jj}^2)^2 + n^2(n-1)(\rho - c) + 2n \sum_{i<j} (h_{ij}^1)^2.$$

Applying (2.2), Lemma 2.1 and the Ricci equation to this formula gives us

$$\begin{aligned} n^2(n-1)(\|H\|^2 - \rho + c) &\geq \sum_{i<j} \left((h_{ii}^2 - h_{jj}^2)^2 + 2n (h_{ij}^1)^2 \right) \\ &\geq \left(8n \sum_{k<\ell} \sum_{i<j} (h_{ii}^2 - h_{jj}^2)^2 (h_{k\ell}^1)^2 \right)^{1/2} \geq 2n \left(\sum_{k<\ell} (h_{kk}^2 - h_{\ell\ell}^2)^2 (h_{k\ell}^1)^2 \right)^{1/2} \\ &= n^2(n-1)\rho^\perp, \end{aligned}$$

which completes the proof of the inequality.

Let us now assume that the equality in (4.1) holds at a point p. Then, all the inequalities obtained above become equalities. Hence, applying again Lemma 2.1 and (2.2), we find for $1 \leq k < \ell \leq n$ that

$$h^1_{kk} - h^1_{\ell\ell} = 0, \tag{4.2}$$

$$\sum_{i<j}(h^2_{ii} - h^2_{jj})^2 = 2n \sum_{i<j}(h^1_{ij})^2, \tag{4.3}$$

$$h^1_{kl} = 0 \quad \text{or} \quad \tfrac{1}{2}(h^2_{kk} + h^2_{\ell\ell}) = h^2_{mm} \text{ for all } m \neq k, \ell. \tag{4.4}$$

Let us first assume that the second part of (4.4) is valid for 2 pairs (k, ℓ). Then it follows from Lemma 2.1 that $h^2_{11} = h^2_{22} = \cdots = h^2_{nn}$. Since ξ_2 is orthogonal to the mean curvature vector, we have $h^2_{ii} = 0$ for all i. From (4.3) it then follows that $h^1_{ij} = 0$, for $i \neq j$. (4.2) now implies that M^n is totally umbilical.

Next assume that the second part of (4.4) is valid for only one pair (k, ℓ). Reordering the basis $\{e_1, \ldots, e_n\}$ we may assume that $k = 1$ and $\ell = 2$. Then, if we put $h^2_{11} = \mu$ and $h^2_{22} = \nu$, (4.4) reduces to $h^2_{mm} = \frac{1}{2}(\mu + \nu)$. Since ξ_1 is orthogonal to the mean curvature vector, we obtain by taking the trace that $\nu = -\mu$. Using our assumption it also follows from (4.4) that $h^1_{k\ell} = 0$ for $k < \ell$ with $(k, \ell) \neq (1, 2)$. Now (4.3) reduces to

$$2n(h^1_{12})^2 = 4\mu^2 + 2(n-2)\mu^2 = 2n\mu^2.$$

Therefore, replacing e_2 by $-e_2$, if necessary, together with (4.2) completes the proof in this case.

Finally, we assume that the second part of (4.4) is never valid. Hence $h^1_{k\ell} = 0$ for all indices $k < \ell$. From (4.3) it then follows that $h^2_{11} = \cdots = h^2_{nn}$, which contradicts our assumption that the second part of (4.4) is never satisfied. $\square$

Remark that, if the equality is satisfied at a point p, we have at that point that

$$\|H\|^2 = \lambda^2, \tag{4.5}$$

$$\rho^\perp = \tfrac{4}{n(n-1)}\mu^2. \tag{4.6}$$

In the remainder of this Section, we will construct several classes of examples which realize the equality in (4.1).

Example 1. Totally umbilical submanifolds of real space forms, trivially realize the equality in (4.1). For a classification of totally umbilical submanifolds in real space forms, we refer to [C1].

Example 2. If M^2 is a surface in a real space form $N^4(c)$ with ellipse of curvature a circle, then the equality is realized in (4.1).

Example 3. A special case of the surfaces in Example 2 are the superminimal (i.e. minimal and ellipse of curvature a circle) surfaces in $\mathbb{R}^4$. Also a cylinder on a superminimal surface in $\mathbb{R}^4$ satisfies equality in (4.1).

Example 4. Examples of superminimal surfaces in $\mathbb{R}^4$ are holomorphic curves in $\mathbb{C}^2$. Hence also complex cylinders on holomorphic planar curves realize the equality in (4.1). More general, any complex hypersurface of $\mathbb{C}^{n+1}$ with rank of the shape operator at most 2, realizes the equality in (4.1). Recall that any such complete hypersurface must be a complex cylinder [A].

Example 5. A further class of examples is obtained by considering any warped product decomposition of the real space form $N^{n+2}(c)$ as $N^{n+2}(c) = N^{n-2}(c) \times_\rho N^4(c')$ (it can be necessary to restrict to open subsets of the real space forms). For details and a description of all possible such decompositions, we refer to [N] or [DN]. Then taking a superminimal surface M^2 in $N^4(c)$, and considering the submanifold $N^{n-2}(c) \times_\rho M^2$ of $N^{n+2}(c)$, we again obtain a submanifold realizing equality in (4.1), as follows from the basic formulas in [N].

Example 6. It is well known that the complex structure of $\mathbb{C}^3$ induces a Sasakian structure (φ, ξ, η, g) on $S^5(1)$. We only need to know that φ is the projection of the complex structure J of $\mathbb{C}^3$ onto the tangent bundle of $S^5(1)$ and that $\xi = JN$, where N is the outer normal of $S^5(1)$. For more details, see [B]. Let $\pi : S^5(1) \to \mathbb{C}P^2(4)$ be the Hopf fibration corresponding to the complex structure J. Let $\phi : N_1 \longrightarrow \mathbb{C}P^2(4)$ be a holomorphic curve and let PN_1 be the circle bundle over N_1 induced by the Hopf fibration. Let ψ be the immersion such that the following diagram commutes:

$$\begin{array}{ccc} PN_1 & \xrightarrow{\psi} & S^5(1) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\ N_1 & \xrightarrow{\phi} & \mathbb{C}P^2(4) \end{array} \tag{2.8}$$

Then ψ is an invariant immersion (in the sense of [YI], i.e. $\psi_*(T_pPN_1)$ is φ-invariant) in the Sasakian space form $S^5(1)$ with structure vector field ξ tangent along ψ. Let h denote the second fundamental form of ψ and α the second fundamental form of ϕ. The basic formulas for such submanifolds imply that $h(X, \xi) = 0$, $h(\xi, \xi) = 0$ and $h(X, Y) = \alpha(X, Y)$, where X denotes both any tangent vector field on N_1 and its horizontal lift on PN_1. Since ϕ is holomorphic, it follows from Theorem 4.1 that ψ is minimal and realizes the equality in (4.1).

Remark that all of the above examples are either umbilical (and thus have zero normal curvature) or minimal, at least if $n > 2$. So far we do not have an example, if $n > 2$, which is not minimal and not totally umbilical.

5. Some Classifications

In this section, we want to investigate two special classes of submanifolds. First we investigate the submanifolds M^n which realize the equality in Theorem 4.1 at every point p of M^n and have constant non zero mean curvature. Next, we investigate the submanifolds M^n which realize the equality in (4.1) at every point p of M^n and have constant non zero normal curvature.

Lemma 5.1. *Assume that M realizes equality in (4.1). Let p be a non totally geodesic point of M^n. If one of the following conditions holds*

(1) $H(p) \neq 0$ *and* $\rho^\perp(p) \neq 0$,
(2) $H(p) \neq 0$ *and* $\rho^\perp = 0$ *on a neighborhood of* p,
(3) $H = 0$ *on a neighborhood of* p,

then there exist local orthonormal tangent vector fields $E_1, \dots, E_n$, orthonormal normal vector fields ξ_1 and ξ_2 and functions λ and μ such that

$$\begin{aligned} h(E_1,E_1) &= \lambda\xi_1 + \mu\xi_2, & h(E_1,E_i) &= 0,\\ h(E_1,E_2) &= \mu\xi_1, & h(E_2,E_i) &= 0,\\ h(E_2,E_2) &= \lambda\xi_1 - \mu\xi_2, & h(E_i,E_j) &= \delta_{ij}\lambda\xi_1, \end{aligned}$$

where $3 \leq i,j \leq n$.

Proof. We first assume that $H(p) \neq 0$ and $\rho^\perp(p) \neq 0$. We then choose ξ_1 as a unit normal vector in the direction of the mean curvature vector and take ξ_2 orthogonal to ξ_1. It then follows from Theorem 4.1 that A_{ξ_2} has three different eigenvalues, two with multiplicity 1 and one with multiplicity $n-2$. Then we can find differentiable vector fields $E_1, \dots, E_n$ such that E_1 and E_2 span the corresponding 1- dimensional eigenspaces and such that $E_3, \dots, E_n$ span the $(n-2)$-dimensional eigenspace.

If $\rho^\perp = 0$ on a neighborhood of p, then M^n is totally umbilical around p and we take an arbitrary local orthonormal basis.

If $H = 0$ on a neighborhood of p, we take for ξ_1 and ξ_2 arbitrary normal vector fields and proceed as above. □

Let p be a non totally geodesic point of M^n and assume that $n \geq 3$. Suppose that Lemma 5.1 (1), (2) or (3) holds and let $\{E_1, \dots, E_n\}$ and $\{\xi_1, \xi_2\}$ be as constructed in Lemma 5.1.

Lemma 5.2. *We have for $3 \leq i,j,k \leq n$ that*

(1) $E_k(\lambda) = 0$,
(2) $\lambda \nabla^\perp_{E_k}\xi_1 = 0$,
(3) $E_1(\lambda) = -\lambda \left\langle \nabla^\perp_{E_2}\xi_1, \xi_2 \right\rangle$,
(4) $E_2(\lambda) = \lambda \left\langle \nabla^\perp_{E_1}\xi_1, \xi_2 \right\rangle$,
(5) $\mu \left\langle \nabla_{E_i}E_j, E_1 \right\rangle = -\delta_{ij}\lambda \left\langle \nabla^\perp_{E_1}\xi_1, \xi_2 \right\rangle$,
(6) $\mu \left\langle \nabla_{E_i}E_j, E_2 \right\rangle = \delta_{ij}\lambda \left\langle \nabla^\perp_{E_2}\xi_1, \xi_2 \right\rangle$,
(7) $\left\langle \nabla_{E_1}E_2 + \nabla_{E_2}E_1, E_k \right\rangle = 0$,
(8) $\left\langle \nabla_{E_1}E_1 - \nabla_{E_2}E_2, E_k \right\rangle = 0$,
(9) $E_1(\mu) = \lambda \left\langle \nabla^\perp_{E_1}\xi_1, \xi_2 \right\rangle - \mu \left\langle \nabla^\perp_{E_2}\xi_1, \xi_2 \right\rangle + 2\mu \left\langle \nabla_{E_2}E_2, E_1 \right\rangle$,
(10) $E_2(\mu) = -\lambda \left\langle \nabla^\perp_{E_2}\xi_1, \xi_2 \right\rangle + \mu \left\langle \nabla^\perp_{E_1}\xi_1, \xi_2 \right\rangle + 2\mu \left\langle \nabla_{E_1}E_1, E_2 \right\rangle$,
(11) $E_k(\mu) = \mu \left\langle \nabla_{E_1}E_1, E_k \right\rangle = \mu \left\langle \nabla_{E_2}E_2, E_k \right\rangle$,
(12) $\mu \left\langle \nabla^\perp_{E_k}\xi_1, \xi_2 \right\rangle + 2\mu \left\langle \nabla_{E_k}E_1, E_2 \right\rangle = -\mu \left\langle \nabla_{E_1}E_2, E_k \right\rangle$.

Proof. We take the local orthonormal frames constructed in the previous lemma. We denote by $k, \ell \in \{3, \ldots, n\}$. Since

$$
\begin{aligned}
(\nabla h)(E_k, E_1, E_2) &= E_k(\mu)\xi_1 + \mu(\nabla^{\perp}_{E_k}\xi_1 - 2\langle \nabla_{E_k} E_2, E_1\rangle \xi_2),\\
(\nabla h)(E_1, E_k, E_2) &= \mu(\langle \nabla_{E_1} E_1, E_k\rangle \xi_1 - \langle \nabla_{E_1} E_2, E_k\rangle \xi_2),\\
(\nabla h)(E_2, E_k, E_1) &= \mu(\langle \nabla_{E_2} E_2, E_k\rangle \xi_1 + \langle \nabla_{E_2} E_1, E_k\rangle \xi_2),
\end{aligned}
$$

we obtain (7), (8), (11) and (12) from the Codazzi equation. Using these equations, it now follows that

$$
\begin{aligned}
(\nabla h)(E_k, E_1, E_1) &= E_k(\lambda)\xi_1 + \lambda\nabla^{\perp}_{E_k}\xi_1 + E_k(\mu)\xi_2 + \mu\nabla^{\perp}_{E_k}\xi_2 - 2\langle \nabla_{E_k} E_1, E_2\rangle \mu\xi_1,\\
(\nabla h)(E_1, E_k, E_1) &= -\langle \nabla_{E_1} E_k, E_1\rangle \mu\xi_2 - \langle \nabla_{E_1} E_k, E_2\rangle \mu\xi_1,\\
&= E_k(\mu)\xi_2 + \mu\nabla^{\perp}_{E_k}\xi_2 - 2\langle \nabla_{E_k} E_1, E_2\rangle \mu\xi_1.
\end{aligned}
$$

Again the Codazzi equation implies (1) and (2).

The equations (3) to (6) are obtained in a similar way from $(\nabla h)(E_1, E_\ell, E_k) = (\nabla h)(E_\ell, E_1, E_k)$ and $(\nabla h)(E_2, E_\ell, E_k) = (\nabla h)(E_\ell, E_2, E_k)$, whereas equations (9) and (10) follow from $(\nabla h)(E_1, E_2, E_2) = (\nabla h)(E_2, E_1, E_2)$. □

Theorem 5.3. *Let $\phi : M^n \to N^{n+2}(c)$ be an isometric immersion realizing at every point the equality in (4.1). If M^n has constant non zero mean curvature, then M^n is totally umbilical.*

Proof. If λ is a non zero constant, then it follows from Lemma 5.2 (2), (3) and (4) that ξ_1 is parallel. Therefore M^n has trivial normal connection implying that by (4.6) that $\mu = 0$. It now follows immediately from Theorem 4.1 that M^n is totally umbilical. □

If $\mu = 0$, then we have that M^n is totally umbilical. Therefore, from now on, we will assume that $\mu \neq 0$. We shall also assume that the normal curvature of M^n is constant, which by (4.6) is equivalent to μ being constant. Then we have the following proposition.

Proposition 5.4. *Let $\phi : M^n \to N^{n+2}(c)$, $n \geq 3$, be an isometric immersion realizing at every point p of M^n the equality in (4.1). If $\rho^{\perp}$ is a non zero constant, then ϕ is minimal.*

Proof. Suppose that $\lambda \neq 0$ and that μ is a non zero constant. Then it follows from Lemma 5.2 (2) that $\nabla^{\perp}_{E_k}\xi_1 = 0$. We can choose E_3 in such a way that $[E_1, E_2] \in \mathrm{span}\{E_1, E_2, E_3\}$. Since μ is constant it now follows from Lemma 5.2 that there exist local functions a, b, e and d such that

$$
\begin{aligned}
\nabla_{E_1} E_1 &= eE_2, & \nabla_{E_2} E_2 &= dE_1.\\
\nabla_{E_1} E_2 &= -eE_1 + aE_3, & \nabla_{E_2} E_1 &= -dE_2 - aE_3,
\end{aligned}
$$

Putting $\alpha = \left\langle \nabla^\perp_{E_1}\xi_1, \xi_2\right\rangle$, $\beta = \left\langle \nabla^\perp_{E_2}\xi_1, \xi_2\right\rangle$ and denoting by ∇^1 the $\{E_1, E_2, E_3\}$-component of ∇, we also obtain that for $k \geq 3$:

$$\begin{aligned}
&\nabla_{E_k} E_1 = -\tfrac{1}{2} a\delta_{k3} E_2 + \tfrac{\lambda}{\mu}\alpha E_k,\\
&\nabla^1_{E_1} E_3 = -aE_2, \quad \nabla^1_{E_2} E_3 = aE_1,\\
&\nabla_{E_k} E_2 = \tfrac{1}{2} a\delta_{k3} E_1 - \tfrac{\lambda}{\mu}\beta E_k,\\
&\nabla^1_{E_3} E_3 = -\tfrac{\lambda}{\mu}\alpha E_1 + \tfrac{\lambda}{\mu}\beta E_2.
\end{aligned}$$

Clearly, Lemma 5.2 implies that

$$e = \tfrac{1}{2\mu}(\lambda\beta - \mu\alpha), \quad d = \tfrac{1}{2\mu}(\mu\beta - \lambda\alpha),$$

Now, we obtain from the Ricci equation that

$$0 = R^\perp(E_1, E_3)\xi_1 = -\nabla^\perp_{E_3}\alpha\xi_2 - \nabla^\perp_{[E_1,E_3]}\xi_1 = -E_3(\alpha)\xi_2 + \tfrac{1}{2}a\beta\xi_2.$$

Hence $E_3(\alpha) = \frac{1}{2}a\beta$. Similarly, we obtain that $E_3(\beta) = -\frac{1}{2}a\alpha$. From $R^\perp(E_1, E_\ell)\xi_1 = 0 = R^\perp(E_2, E_\ell)\xi_1$, $\ell \geq 4$, it now follows that $E_\ell(\alpha) = E_\ell(\beta) = 0$. We now compute $R(E_1, E_2)E_3$.

$$\begin{aligned}
0 &= \nabla_{E_1}\nabla_{E_2} E_3 - \nabla_{E_2}\nabla_{E_1} E_3 - \nabla_{[E_1,E_2]} E_3\\
&\equiv \nabla_{E_1}(aE_1) - \nabla_{E_2}(-aE_2) - \nabla_{-eE_1+dE_2+2aE_3} E_3 \qquad \mathrm{mod}\{E_1, E_2, E_3\}^\perp\\
&\equiv E_1(a)E_1 + aeE_2 + E_2(a)E_2 + adE_1\\
&\quad - aeE_2 - adE_1 - 2a(-\tfrac{\lambda}{\mu}\alpha E_1 + \tfrac{\lambda}{\mu}\beta E_2) \qquad \mathrm{mod}\{E_1, E_2, E_3\}^\perp\\
&\equiv (E_1(a) + 2a\alpha\tfrac{\lambda}{\mu})E_1 + (E_2(a) - 2a\beta\tfrac{\lambda}{\mu})E_2 \qquad \mathrm{mod}\{E_1, E_2, E_3\}^\perp,
\end{aligned}$$

from which we deduce that

$$E_1(a) = -2a\alpha\tfrac{\lambda}{\mu}, \quad E_2(a) = 2a\beta\tfrac{\lambda}{\mu}.$$

In a similar way, we deduce from $\langle R(E_1, E_3)E_3, E_1\rangle = c+\lambda^2$ and $\langle R(E_2, E_3)E_3, E_2\rangle = c + \lambda^2$ that

$$E_1(\alpha) = -\tfrac{(c+\lambda^2)\mu}{\lambda} + a^2\tfrac{\mu}{\lambda} - \tfrac{\lambda}{2\mu}\beta^2 - \tfrac{\lambda}{\mu}\alpha^2 + \tfrac{3}{2}\alpha\beta, \tag{5.1}$$

$$E_2(\beta) = \tfrac{(c+\lambda^2)\mu}{\lambda} - a^2\tfrac{\mu}{\lambda} + \tfrac{\lambda}{2\mu}\alpha^2 + \tfrac{\lambda}{\mu}\beta^2 - \tfrac{3}{2}\alpha\beta. \tag{5.2}$$

Computing now $\left\langle R^\perp(E_1, E_2)\xi_1, \xi_2\right\rangle = 2\mu^2$ and $\langle R(E_1, E_2)E_2, E_1\rangle = c + \lambda^2 - 2\mu^2$, we find that

$$E_1(\beta) - E_2(\alpha) + \tfrac{1}{2\mu}(2\lambda\alpha\beta - \mu(\alpha^2 + \beta^2)) = 2\mu^2, \tag{5.3}$$

$$E_1(d) + E_2(e) - e^2 - d^2 - 2a^2 = c + \lambda^2 - 2\mu^2. \tag{5.4}$$

Expressing e and d in terms of the unknown functions α and β, using Lemma 5.2, (5.1), (5.2) and (5.3), the equation (5.4) reduces to

$$(\alpha^2+\beta^2)\lambda^2 - 6a^2\mu^2 + 6\mu^4 = 0. \tag{5.5}$$

Deriving (5.5) with respect to E_3, we find that $E_3(a)=0$. Similarly, deriving with respect to E_ℓ, $\ell > 3$ yields that $E_\ell(a)=0$. We now compute the integrability condition for a. On one hand, we have

$$\begin{aligned}[E_1,E_3]\,a &= E_1(E_3(a)) - E_3(E_1(a))\\ &= -E_3(-2a\alpha\tfrac{\lambda}{\mu}) = 2a\tfrac{\lambda}{\mu}E_3(\alpha) = a^2\beta\tfrac{\lambda}{\mu},\end{aligned}$$

while on the other hand, we have

$$[E_1,E_3]\,a = (\nabla_{E_1}E_3 - \nabla_{E_3}E_1)a = -\tfrac{1}{2}aE_2(a) = -a^2\beta\tfrac{\lambda}{\mu}.$$

Hence $a^2\lambda\beta=0$. Similarly, we also obtain that $a^2\lambda\alpha=0$. If $\alpha=\beta=0$, then λ is constant and a contradiction follows from Theorem 5.3. Therefore $a=0$ and a contradiction follows from (5.5). □

In the following theorem we prove that there is only one immersion with constant non zero normal curvature satisfying equality in (4.1). We may restrict ourselves to the cases that $c=1,0,-1$.

Theorem 5.5. *Let $\phi: M^n \to N^{n+2}(c)$, $n\geq 3$ and $c\in\{-1,0,1\}$, be an immersion realizing at every point the equality in (4.1). If M^n has constant non zero normal curvature, then $n=3$, $c=1$ and ϕ is (locally) congruent to the lift of the holomorphic curve of constant curvature 2 in $\mathbb{C}P^2(4)$, lifted as in Example 6.*

Proof. We choose E_3 in such a way that $[E_1,E_2]\in\operatorname{span}\{E_1,E_2,E_3\}$. Since μ is constant and $\lambda=0$ it now follows from Lemma 5.2 that there exist local functions a, e and d such that

$$\begin{aligned}&\nabla_{E_1}E_1 = eE_2, &&\nabla_{E_2}E_2 = dE_1.\\ &\nabla_{E_1}E_2 = -eE_1 + aE_3, &&\nabla_{E_2}E_1 = -dE_2 - aE_3,\end{aligned}$$

Denoting by ∇^1 the $\{E_1,E_2,E_3\}$-component of ∇, we also obtain that

$$\begin{aligned}&\nabla_{E_k}E_1 = b_kE_2, \quad \nabla_{E_k}E_2 = -b_kE_1,\\ &\nabla^1_{E_1}E_3 = -aE_2, \quad \nabla^1_{E_2}E_3 = aE_1, \qquad \nabla^1_{E_3}E_3 = 0,\end{aligned}$$

where $b_3,\ldots,b_n$ are local functions. We have from Lemma 5.2 that

$$\nabla^\perp_{E_1}\xi_1 = -2e\xi_2, \quad \nabla^\perp_{E_2}\xi_1 = 2d\xi_2, \quad \nabla^\perp_{E_3}\xi_1 = (-2b_3-a)\xi_2.$$

Next, we compute $\langle R(E_1, E_2)E_2, E_1\rangle = c - 2\mu^2$. This yields that

$$E_1(d) + E_2(e) - e^2 - d^2 - a^2 + 2ab_3 = c - 2\mu^2. \tag{5.6}$$

It then follows from $\left\langle R^{\perp}(E_1, E_2)\xi_1, \xi_2\right\rangle = 2\mu^2$ that

$$2c - 6\mu^2 + 4a^2 = 0, \tag{5.7}$$

from which we deduce that a is constant. Next, we denote by V the $\{E_1, E_2, E_3\}^{\perp}$-component of $\nabla_{E_1}E_3$ and compute

(5.8)
$$\begin{aligned} -c = \langle R(E_1, E_3)E_1, E_3\rangle &= \langle \nabla_{E_1}\nabla_{E_3}E_1 - \nabla_{E_3}\nabla_{E_1}E_1 - \nabla_{-aE_2 - b_3E_2 + V}E_1, E_3\rangle \\ &= ab_3 - (a + b_3)a = -a^2. \end{aligned}$$

So comparing (5.7) and (5.8), we obtain that $c = 1$ and $a = \pm 1$. By replacing E_3 by $-E_3$, if necessary, we may assume that $a = -1$. By replacing ξ_2 by $-\xi_2$, if necessary, it now follows from (5.7) that $\mu = 1$.

Computing now $R(E_1, E_2)E_2 = -E_1$, it follows that $\nabla_{E_1}E_3 \in \operatorname{span}\{E_1, E_2, E_3\}$. Similarly, it also follows that $\nabla_{E_2}E_3 \in \operatorname{span}\{E_1, E_2, E_3\}$. If the dimension is greater than 3, a contradiction then follows from

$$1 = \langle R(E_2, E_4)E_4, E_2\rangle = \left\langle \nabla_{E_2}\nabla_{E_4}E_4 - \nabla_{E_4}\nabla_{E_2}E_4 - \nabla_{[E_2, E_4]}E_4, E_2\right\rangle = 0.$$

Hence $n = 3$ and then (5.6) reduces to

$$E_1(d) + E_2(e) - e^2 - d^2 = 2b, \tag{5.9}$$

where $b = b_3$. Computing further Ricci equations, we still obtain the following differential equations for b, e and d:

$$E_1(b) - E_3(e) = d(b - 1) \tag{5.10}$$
$$E_2(b) + E_3(d) = e(b - 1) \tag{5.11}$$

The equations (5.9), (5.10) and (5.11) turn out to be the integrability conditions for the system of differential equations

$$E_1(\theta) = -e \quad E_2(\theta) = d, \quad E_3(\theta) = -b.$$

We now define

$$\begin{aligned} &F_1 = \cos\theta E_1 + \sin\theta E_2, \quad F_2 = -\sin\theta E_1 + \cos\theta E_2, \quad F_3 = E_3, \\ &\eta_1 = \cos 2\theta\xi_1 - \sin 2\theta\xi_2, \quad \eta_2 = \cos 2\theta\xi_2 + \sin 2\theta\xi_1, \end{aligned}$$

It then follows, after straightforward computations, that

$$\begin{array}{lll}
\nabla_{F_1}F_1 = 0, & \nabla_{F_2}F_2 = 0, & \nabla_{F_3}F_3 = 0, \\
\nabla_{F_1}F_2 = -F_3, & \nabla_{F_2}F_3 = -F_1, & \nabla_{F_3}F_1 = 0, \\
\nabla_{F_2}F_1 = F_3, & \nabla_{F_3}F_2 = 0, & \nabla_{F_1}F_3 = F_2, \\
h(F_1, F_1) = \eta_2, & h(F_2, F_2) = -\eta_2, & h(F_1, F_2) = \eta_1, \\
h(F_1, F_3) = 0, & h(F_2, F_3) = 0, & h(F_3, F_3) = 0, \\
\nabla^{\perp}_{F_1}\eta_1 = 0, & \nabla^{\perp}_{F_2}\eta_1 = 0, & \nabla^{\perp}_{F_3}\eta_1 = \eta_2.
\end{array}$$

We now consider S^5 as a hypersurface in $\mathbb{R}^6$, and denote the position vector by F. Using the above formulas, a straightforward computation shows that

$$JF_1 = F_2, \quad JF_3 = -F, \quad J\eta_1 = -\eta_2,$$

defines a parallel complex structure J on $\mathbb{R}^6$. For example, we have

$$\begin{aligned}
(D_{F_1}J)F_1 &= D_{F_1}F_2 - JD_{F_1}F_1 = -F_3 + \eta_1 - J(\eta_2 - F) = 0, \\
(D_{F_1}J)F &= D_{F_1}F_3 - JD_{F_1}F = F_2 - JF_1 = 0, \\
(D_{F_1}J)\eta_1 &= -D_{F_1}\eta_2 - JD_{F_1}\eta_1 = F_1 + JF_2 = 0.
\end{aligned}$$

Considering now the corresponding Sasakian structure on $S^5(1)$, we notice that M^3 is an invariant submanifold tangent to the structure vector field. If $\pi : S^5(1) \to \mathbb{C}P^2(4)$ is the Hopf fibration, then $p(M^3)$ is a holomorphic curve N^2 in $\mathbb{C}P^2(4)$. Since E_1 and E_2 are horizontal vector fields for the submersion $p : M^3 \to N^2$, and since $\langle R(E_1, E_2)E_2, E_1\rangle = -1$, it follows from the basic equations for submersions (see [O]) that N^2 has constant curvature 2. $\square$

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