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SUBALGEBRAS OF FINITE CODIMENSION IN SYMPLECTIC LIE ALGEBRA

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Abstract. Subalgebras of germs of vector fields leaving 0 fixed in R^{2n}, of finite codimension in symplectic Lie algebra contain the ideal of germs infinitely flat at 0. We give an application.

1. Introduction

In this paper, we characterize subalgebras of germs of vector fields X with $X(0) = 0$ in R^{2n}, of finite codimension in symplectic Lie algebra. Let $\omega = \sum_{i=1}^{n} dx_i \wedge dx_{n+i}$ denote the canonical symplectic form on R^{2n}, and let χ_ω denote the symplectic Lie algebra of vector fields leaving the origin 0 fixed. Also, denote by χ_ω^∞ the ideal of χ_ω, of infinitely flat germs at 0. Finally, let C^1 and $\Im$ denote, respectively, the space of germs of closed 1-forms and the space of germs of smooth functions. The main theorem in this note is:

Theorem 1. *Let A be a finite codimension subalgebra of χ_ω. Then A contains χ_ω^∞.*

As a consequence of this result we obtain a reduction of infinitesimal action of the group of germs of origin preserving symplectic diffeomorphisms to the action of the group of infinite jets (at the origin 0) on smooth manifold. We encounter such action when we deal with the theory of natural fiber bundles (see [1] , [2] [5]). In [2] and [5], the authors used Borel's lemma and Whitney's extension theorem to prove the reduction of the above action. In the category of manifolds endowed with geometric structures, Borel's lemma and Whitney's extension theorem do not work. The interest of finding appropriate methods for this cases was raised in [2] . Our result applies in the category of symplectic manifolds.

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2. Fundamental lemma

The proof of theorem1 is essentially based on the following lemma.

Lemma 2. *Let V be a subspace of finite codimension of real linear space E and ψ be an endomorphism of E such that:*

1) $\psi(V) \subset V$.
2) $\forall\, b \in R$, $\psi + bI$ *is onto in* E.
3) $\forall\ b\ ,\ c \in R$ *with* $b^2 - 4c < 0$, $\psi^2 + b\psi + cI$ *is onto in* E.

Then $V = E$.

Proof. Suppose that $E \neq V$ and $m = codimV > 0$. Thus there exists a vector $X_o \in E - V, X_o \neq 0$ and real numbers λ_o, $\lambda_1, \cdots \lambda_m$ such that:

$$P_m(\psi)(X_o) = \lambda_o X + \lambda_1 \psi(X_o) + \cdots + \lambda_m \psi^m(X_o) \in V\ . \tag{1}$$

The decomposition of $P_m(\psi)$, in the polynomial ring $R(\psi)$ of one undetermined ψ, in factor product, contains binomials of first degree and trinomials of second degree with negative discriminants. It follows from the assumptions that $P_m(\psi)$ is onto. Since $\psi(V) \subset V$, we obtain that $P_m(\psi)(V) \subset V$. Hence, there exists an epimorphism $\bar{\psi}$: $E/V \to E/V$ such that the following diagrams commute

$$\begin{array}{ccc} E & \overset{\psi}{\to} & E \\ \downarrow \pi & & \downarrow \pi \\ E/V & \overset{\bar{\psi}}{\to} & E/V \end{array} \qquad\qquad \begin{array}{ccc} E & \overset{P_m(\psi)}{\longrightarrow} & E \\ \downarrow \pi & & \downarrow \pi \\ E/V & \overset{P_m(\bar{\psi})}{\longrightarrow} & E/V \end{array}$$

Since E/V has finite codimension, the epimorphism $P_m(\bar{\psi})$ is also one to one, hence $KerP_m(\bar{\psi}) = V$. Now (1) yields to $P_m(\bar{\psi})([X_o]) = 0$. Hence $X_o \in V$. This is a contradiction.

3. Some bounds to flows

Let ϕ be the flow of a vector field on a manifold M. An equilibrium point a of X (i.e. $X(a) = 0$) is ω–stable (resp. α–stable) in the sense of Liapunov if, for every neighbourhood U of a, there exists another neighbourhood V of a, $V \subset U$ such that for every point $x \in V$, $\phi(t, x)$ is defined for all $t \geq 0$ (resp. for all $t \leq 0$), and lies in U. The point a is stable (in the sense of Liapunov) if it is simultaneously ω–stable and α–stable. Let f be a differentiable function, a singular point of f is a point $a \in M$ such that $Df(a) = 0$.

We have Liapunov 's theorem:

Theorem 3. *Let X be a differentiable vector field on a manifold M and a be an equilibrium point of X. We assume that there exists a function f, defined and continuous on an open neighbourhood W of a, differentiable on $W - \{a\}$, which satisfies the following conditions:*

(i) $Xf \leq 0$, *(resp.* $Xf \geq 0$*) on* $W - \{a\}$;

(ii) $f(x) > f(a)$ *for every* $x \in W - \{a\}$.

Then the equilibrium point is ω–stable (resp. α–stable) in the sense of Liapunov.

Denote by $\{.,.\}$ the Poisson bracket in $\Im$ defined by

$$\{f,g\} = \sum_{i=1}^{n}(\frac{\partial f}{\partial x_i}.\frac{\partial g}{\partial x_{n+i}} - \frac{\partial f}{\partial x_{n+i}}.\frac{\partial g}{\partial x_i}) \tag{2}$$

and $\phi(t,x)$ the flow induced by the Hamiltonian vector field

$$X_f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_{n+i}}.\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}.\frac{\partial}{\partial x_{n+i}}\,. \tag{3}$$

Without loss of generality, we can assume that X_f is complete so the range of t is the real line R. If necessary we can replace X_f by μX_f, where μ is a C^∞ function which is 1 in a neighbourhood of 0 and has a compact support in the set where X is defined.

From Liapunov 's theorem we deduce

Theorem 4. [5] *Let (M,Λ) be a Poisson manifold, $f: M \to R$ be a differentiable function, a be a point in M, and S be the symplectic leaf of M which passes through this point. The point a is an equilibrium point of the Hamiltonian vector field X_f if and only if the differential $d(f\mid_S)$ of the restriction of f to the leaf S vanishes at a, when such is the case and when, in addition the following two conditions are satisfied*

(i) *the Hessian $D^2(f\mid_S)(a)$ of $f\mid_S$ at the point a is positive-definite or negative-definite,*

(ii) *the point a has a neighbourhood W on which the rank of the Poisson structure of M is constant.*

Then the point a is stable in the sense of Liapunov.

Now, we establish:

Theorem 5. *Let f be a germ of function with 0 as a an equilibrium point. Suppose the Hessian $D^2f(0)$ of f at 0 is positive-definite or negative-definite then the derivatives $D^k_x\phi(t,x)\xi^k$, $k = 1,2,...,$ of the flow $\phi(t,x)$ generated by the Hamilton vector field X_f are stable in the sense of Liapunov.*

Proof. Consider the system of differential equations

$$x' = X(x)\,, \tag{4}$$

where X is the vector field given by (2). The variational equation to the prolongation of (4) is given by

$$(5)\qquad \begin{cases} x' = X(x) \\ \xi_1' = DX(x).\xi_1 \\ \xi_2' = D^2X(x).\xi_1\xi_1 + DX(x).\xi_2 \\ \vdots \\ \xi_k' = \sum_{s=1}^{k} D^sX(x) \sum_{\alpha_1+\cdots+\alpha_s=k} \xi_{\alpha_1}\cdots\xi_{\alpha_s}\,, \end{cases}$$

where $\xi_\alpha \in R^{2n}$ for $\alpha = 1, ..., k$, with brief notation

$$(x', \xi_1', \cdots \xi_k') = F(x, \xi_1, \cdots \xi_k)\,.$$

Now consider the function

$$h(x, \xi_1, \cdots \xi_k) = f(x) + Df(x).\xi_1 + D^2f(x).\xi_1\xi_1 + Df(x).\xi_2 + \cdots + \sum_{s=1}^{k} D^sf(x) \sum_{\substack{\alpha_i>0 \\ \alpha_1+\cdots+\alpha_s=k}} \xi_{\alpha_1}\cdots\xi_{\alpha_s}\,.$$

Considering the function vector field F as a derivation and taking into account the relation $Xf(x) = 0$, we see easily that $Fh(x, \xi_1, \cdots \xi_k) = 0$, i.e. $h(x, \xi_1, \cdots \xi_k)$ is a non trivial first integral of the vector field $F(x, \xi_1, \cdots \xi_k)$. Now, we endow the space $R^{(k+1)2n}$ with the Poisson structure Λ given in the coordinate system $(x_1, \cdots x_{2n}, \xi_1^1, \cdots, \xi_{2n}^1, \cdots, \xi_1^k, \cdots, \xi_{2n}^k)$ by

$$\{x_i, x_j\} = 0\,, \quad \{x_{n+i}, x_{n+j}\} = 0\,, \quad \{x_{n+i}, x_j\} = \delta_{ij}\,,$$
$$\{x_i, \xi_s^l\} = 0\,, \quad \{x_{n+i}, \xi_s^l\} = 0\,, \quad \{\xi_s^l, \xi_r^p\} = 0\,,$$

where

$$\Lambda(dz_i, dz_j) = \{z_i, z_j\}\,.$$

Clearly, F is a Hamiltonian vector field for the structure Λ and the Hessian $D^2(h\mid_{R^{2n}})(0) = D^2f(0)$ of the restriction of the function $h(x, \xi_1, \cdots \xi_k)$ to the Poisson leaf R^{2n} of $(R^{(k+1)2n}, \Lambda)$ is positive-definite or negative-definite. Now, for any constant vector $v \in R^{2n}$ the system $(\phi(t,x), D_x\phi(t,x)v, \cdots, D_x^k\phi(t,x)v^k))$ is a solution of (5) passing through $(x, v, 0, \cdots, 0) \in R^{2(k+1)n}$. In fact,

$$\begin{aligned} \frac{d}{dt}(D_x^k\phi(t,x)v^k) &= (\frac{d}{dt}D_x^k\phi(t,x)v^k = D_x^k\frac{d}{dt}\phi(t,x)v^k \\ &= D_x^kX(\phi(t,x))v^k \\ &= (\sum_{s=1}^{k} D^sXo\phi(t,x) \sum_{\alpha_1+\cdots+\alpha_s=k} D^{\alpha_1}\phi(t,x)\cdots D^{\alpha_k}\phi(t,x))v^k \\ &= \sum_{s=1}^{k} D^sXo\phi(t,x) \sum_{\alpha_1+\cdots+\alpha_s=k} D^{\alpha_1}\phi(t,x)v^{\alpha_1}\cdots D^{\alpha_s}\phi(t,x))v^{\alpha_s}\,. \end{aligned}$$

Therefore, it follows from theorem 4 that the system (5) is stable in the sense of Liapunov i.e. for every neighbourhood U of 0 in $R^{2(k+1)n}$ there exists another neighbourhood V of 0, $V \subset U$ such that for every $(x, v, 0, \cdots, 0) \in V$, $(\phi(t,x), D_x\phi(t,x)v, \cdots, D_x^k\phi(t,x)v^k \in U$.

4. Surjectivity of some Operators

Denote by χ_ω the symplectic Lie algebra of vector fields. Let $X \in \chi_\omega$ and $i_X\omega$ be the interior product by X. It is well known that the mapping $\theta : \chi_\omega \to C^1$ defined by $\theta(X) = i_X\omega$ is an isomorphism. Let $\varphi = \theta^{-1}$, we can define a homotopy integral operator $K : C^1 \to \Im$ such that:

(i) $d \circ K = id_{C^1}$

(ii) $J_0^\infty K\alpha = 0$ for all $\alpha \in C^1$ such that $J_0^\infty \alpha = 0$.

d being the operator of exterior differentiation.

Let $\sharp = \varphi o d$ and $b = K o \theta$. Then we have $\sharp o b = id_{\chi_\omega}$ and so the operator $\sharp$ is onto. For simplicity we shall use $f^\sharp$ instead of $\sharp(f)$; $f^\sharp$ is the Hamiltonian vector field associated to the function f which we have noted before by X_f. A straightforward computation leads us to write

$$[f^\sharp, g^\sharp] = -\{f, g\}^\sharp .$$

Therefore for a fixed $X \in \chi_\omega$, the surjectivity of the adjoint mapping $ad(X)$ implies that of $\{f, .\}$, where f is some function from $\Im$ such that $f^\sharp = X$.

Throughout the remainder of this paper, f will be a germ of function at the origin 0 which fulfils the assumptions of theorem 4.

Lemma 6. *For any $b \in R$ and $h^\sharp$, with $h \in \Im$ and $J_0^\infty h = 0$, there exists $g \in \Im$ with $J_0^\infty g = 0$ such that*

$$[f^\sharp, g^\sharp] + bg^\sharp = h^\sharp .$$

Proof. Since the operator $\sharp$ defined above is onto, it suffices to find $g \in \Im$ with $J_0^\infty h = 0$ solution of the equation

$$\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_{n+i}}.\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}.\frac{\partial g}{\partial x_{n+i}}\right) + bg = h . \tag{6}$$

Case $b < 0$. Consider the function given by the following integral

$$g(x) = -\int_0^\infty e^{bt} h(\phi(t,x))dt \tag{7}$$

h being infinitely flat at the origin 0, there exist constants $\delta > 0$ and $M > 0$ such that for all x with $|x| < \delta$ we have $|h(x)| \leq M.\,|x|$. Since the flow ϕ generated by the Hamiltonian vector field X_f is stable, we can choose $\eta(\delta) > 0$ such that, for any x with $|x| < \eta(\delta)$, one has $|\phi(t,x)| < \delta$ for all $t \in R$. Then

$$\left|e^{bt} h(\phi(t,x))\right| \leq \delta M e^{bt} .$$

Therefore the singular integral g given by (7), converges uniformly in a neighbourhood of 0. Now, we shall prove that the function g is smooth ($= C^\infty$) and infinitely at 0. This is a consequence of theorem 5. Indeed, for any $\epsilon > 0$ and positive integer k, we have for x in a sufficiently small neighbourhood of the origin 0

$$\begin{aligned}&\left|D_x^k e^{bt} h(\phi(t,x))\xi^k\right| \\ &\qquad \le e^{bt}\sum_{i=1}^{k}\left|D^i h(\phi(t,x))\right| . \sum_{\substack{j_s>0\\ j_1+\cdots+j_i=k}}\left|D^{j_1}\phi(t,x)\xi^{j_1}\right|\cdots\left|D^{j_i}\phi(t,x)\xi^{j_s}\right| .\end{aligned}$$

Let $\epsilon > 0$ and K be a positive constant such that the vector $\frac{\xi}{K}$ is small enough, then by theorem 5 we obtain

$$\left|D^{j_s}\phi(t,x)\left(\frac{\xi}{K}\right)^{j_s}\right| \le \epsilon$$

so

$$\left|D^{j_s}\phi(t,x)\xi^{j_s}\right| \le \epsilon K^{j_s} .$$

Consequently

$$\left|D_x^k e^{bt} h(\phi(t,x))\xi^k\right| \le e^{bt}\sum_{i=1}^{k}\epsilon^i \sum_{\substack{j_s>0\\ j_1+\cdots+j_s=k}} K^{j_1+\cdots+j_s} = Conste.e^{bt} .$$

Since b is strictly negative, it follows that for any positive integer k, $\int_0^\infty e^{bt} D_x^k h(\phi(t,x))dt$ converges uniformly in a neighbourhood of the origin 0 in R^{2n}, so the function g is smooth. Because $D_x^k h(\phi(t,0)) = 0$ and $D_x^k h(\phi(t,x))$ converges uniformly to $D_x^k h(\phi(t,0))$ with respect to x, we pass to the limit and obtain $D^k g(0) = 0$. Now, in order to show that g is a solution of equation (6), we compute first

$$\begin{aligned} g(\phi(s,x)) &= -\int_0^{+\infty} e^{bt} h(\phi(t+s,x))dt \\ &= -\int_s^{+\infty} e^{b(t-s)} h(\phi(t,x))dt \\ \frac{d}{ds} g(\phi(s,x)) &= h(\phi(s,x)) + b\int_0^{+\infty} e^{bt} h(\phi(t+s,x))dt .\end{aligned}$$

Setting $s = 0$ and knowing $\phi(0,x) = x$, we get finally that $g(x)$ is solution of equation(6).

Case $b = 0$. Suppose the Hessian of f at 0 is positive-definite (the same argument works in the case where the Hessian is negative-definite) then it follows from Morse's lemma that f can be written in appropriate coordinate system

$(x_1, \cdots, x_{2n})$

$$f = \sum_{i=1}^{n} (x_i^2 + x_{n+i}^2) .$$

The equation (6) becomes (with $b = 0$).

$$\sum_{i=1}^{n} (x_{n+i} \frac{\partial g}{\partial x_i} - x_i \frac{\partial g}{\partial x_{n+i}}) = h . \tag{8}$$

Now consider the following change of coordinates

$$\begin{cases} x_i = r_i \cos\theta_i \\ x_{n+i} = r_i \sin\theta_i \qquad i = 1, \cdots, n \quad ; \quad r_{i>}0 \end{cases} \tag{9}$$

The equation (8) writes

$$\sum_{i=1}^{n} \frac{\partial g}{\partial \theta_i} + h = 0$$

and admits as a solution the function

$$\begin{aligned} &g(\theta_1, \cdots \theta_n, r_1, \cdots, r_n) \\ &= -\int_0^{\theta_n} h(r_1 \cos(\theta_1 - t), \cdots, r_n \cos(\theta_n - t), r_1 \sin(\theta_1 - t), \cdots, r_n \sin(\theta_n - t))dt . \end{aligned}$$

Clearly the function g is smooth and infinitely flat at the origin.

Case $b > 0$. In this case the same argument as in case $b < 0$ show that a solution of the equation (6) is given by

$$g(x) = \int_{-\infty}^{0} e^{bt} h(\phi(t, x))dt .$$

Lemma 7. *For any b, $c \in R$ with $b^2 - 4c < 0$, and any $h^\sharp$, with $h \in \Im$ and $J_0^\infty h = 0$, there exists $g \in \Im$ with $J_0^\infty g = 0$ such that*

$$[f^\sharp, [f^\sharp, g^\sharp]] + b [f^\sharp, g^\sharp] + cg^\sharp = h^\sharp .$$

Proof. Similarly as in lemma 6, we can find $g \in \Im$ with $J_0^\infty g = 0$ such that

$$\{f, \{f, g\}\} - b \{f, g\} + cg = h . \tag{10}$$

This equation is equivalent to

$$\begin{aligned} &\sum_{i,j=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \frac{\partial^2}{\partial x_{n+j} \partial x_{n+i}} - 2 \frac{\partial f}{\partial x_j} \frac{\partial f}{\partial x_{n+i}} \frac{\partial^2}{\partial x_i \partial x_{n+j}} + \frac{\partial f}{\partial x_{n+i}} \frac{\partial f}{\partial x_{n+j}} \frac{\partial^2}{\partial x_j \partial x_i} \right) g \\ &\qquad + \sum_{j=1}^{n} \left\{ \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial^2 f}{\partial x_j \partial x_{n+i}} - \frac{\partial f}{\partial x_{n+j}} \frac{\partial^2 f}{\partial x_i \partial x_j} \right) - b \frac{\partial f}{\partial x_j} \right\} \frac{\partial g}{\partial x_{n+j}} \\ &+ \sum_{j=1}^{n} \left\{ \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_{n+i}} \frac{\partial^2 f}{\partial x_i \partial x_{n+j}} - \frac{\partial f}{\partial x_i} \frac{\partial^2 f}{\partial x_{n+j} \partial x_{n+i}} \right) + b \frac{\partial f}{\partial x_{n+j}} \right\} \frac{\partial g}{\partial x_j} + cg = h . \end{aligned}$$

Let

$$K(t)=\frac{e^{-\frac{b}{2}t}}{\sqrt{4c-b^2}}\sin\frac{\sqrt{4c-b^2}}{2}t\,. \tag{11}$$

Note that $K(t)$ is the solution of the Cauchy problem

$$\begin{cases} K''(t)+bK'(t)+cK(t)=0\,, \\ K(0)=0 \quad K'(0)=1\,. \end{cases} \tag{12}$$

Consider the function g given by

$$g(x)=\int_0^{+\infty} e^{bt}K(t)h(\phi(t,x))dt\,. \tag{13}$$

Similarly to the proof of lemma 6, we can show that g is smooth and infinitely flat at the origin 0. Moreover we have

$$\begin{aligned} g(\phi(s,x)) &= \int_0^{+\infty} e^{bt}K(t)h(\phi(t+s,x))dt \\ &= \int_s^{+\infty} e^{b(t-s)}K(t-s)h(\phi(t,x))dt\,, \\ \frac{d}{ds}g(\phi(s,x)) &= -K(0)h(\phi(s,x)) \\ &\quad -\int_s^{+\infty} e^{b(t-s)}\left(bK(t-s)+K'(t-s)\right)h(\phi(t,x))dt \\ &= -\int_0^{+\infty} e^{bt}\left(bK(t)+K'(t)\right)h(\phi(t+s,x))dt\,. \end{aligned}$$

Then

$$\{f,g\}=-X_f(g)=-\frac{d}{ds}\,|_{s=0}\,g(\phi(s,x))=\int_0^{+\infty} e^{bt}\left(bK(t)+K'(t)\right)h(\phi(t,x))dt\,.$$

Now we compute

$$\begin{aligned} \{f,g\}\left(\phi(s,x)\right) &= \int_0^{+\infty} e^{bt}\left(bK(t)+K'(t)\right)h(\phi(t+s,x))dt \\ &= \int_s^{+\infty} e^{b(t-s)}\left(bK(t-s)+K'(t-s)\right)h(\phi(t,x))dt\,, \end{aligned}$$

$$\begin{aligned} \frac{d}{ds}\{f,g\}\left(\phi(s,x)\right) &= -(bK(0)+K'(0))h(\phi(s,x)) \\ &\quad -\int_s^{+\infty} e^{b(t-s)}\left(K''(t-s)+2bK'(t-s)+b^2K(t-s)\right)h(\phi(t,x))dt \\ &= -h(\phi(s,x))-\int_0^{+\infty} e^{bt}\left(K''(t)+2bK'(t)+b^2K(t)\right)h(\phi(t+s,x))dt\,. \end{aligned}$$

Therefore

$$\begin{aligned}\{f,\{f,g\}\} &= -\frac{d}{ds}\mid_{s=0}\{f,g\}(\phi(s,x))\\ &= h(x)+\int_0^{+\infty} e^{bt}\left(K''(t)+2bK'(t)+b^2K(t)\right)h(\phi(t,x))dt\,,\end{aligned}$$

so

$$\{f,\{f,g\}\}-b\{f,g\}+cg = h(x)+\int_0^{+\infty} e^{bt}\left(K''(t)+bK'(t)+cK(t)\right)h(\phi(t,x))dt$$

and taking account of (12) we see that the function g given by (13) is a solution of the equation(10).

Case $b=0$. Using the Morse's lemma, the equation (10) writes (in case $b=0$)

$$\text{(14)}\quad \sum_{i,j=1}^{n}\left(x_ix_j\frac{\partial^2}{\partial x_{n+j}\partial x_{n+i}}-2x_ix_{n+j}\frac{\partial^2}{\partial x_{n+i}\partial x_j}+x_{n+i}x_{n+j}\frac{\partial^2}{\partial x_j\partial x_i}\right)g- \sum_{i=1}^{n}\left(x_i\frac{\partial}{\partial x_i}+x_{n+i}\frac{\partial}{\partial x_{n+i}}\right)g+cg=h$$

Now in the coordinate system $(r_1,\cdots,r_n,\theta_1,\cdots,\theta_n)$ given by (9) the equation(14) becomes

$$\text{(15)}\qquad \sum_{i,j=1}^{n}\frac{\partial^2 g}{\partial\theta_i\partial\theta_j}+cg=h$$

and has a solution given by

$$\begin{aligned}&g(\theta_1,\cdots\theta_n,r_1,\cdots,r_n)\\ &=\int_0^{\theta_n}K(t)h(r_1\cos(\theta_1-t),\cdots,r_n\cos(\theta_n-t),r_1\sin(\theta_1-t),\cdots,r_n\sin(\theta_n-t))dt\end{aligned}$$

where

$$K(t)=\frac{1}{\sqrt{c}}\sin\sqrt{c}t$$

is solution of the Cauchy problem

$$\text{(16)}\qquad \begin{cases} K''(t)+cK(t)=0\\ K(0)=0\ ;\ K'(0)=1\end{cases}$$

Indeed, we have

$$\begin{aligned}\sum_{i=1}^{n}\frac{\partial g}{\partial\theta_i} &= \int_0^{\theta_n}K(t)\sum_{i=1}^{n}\left(-r_i\sin(\theta_i-t)\frac{\partial h}{\partial x_i}+r_i\cos(\theta_i-t)\frac{\partial h}{\partial x_{n+i}}\right)dt\\ &\quad + K(t)h(t)\mid_{t=\theta_n} = -\int_0^{\theta_n}K(t)\frac{dh}{dt}dt+K(t)h(t)\mid_{t=\theta_n}\end{aligned}$$

where

$$h(t) = h(r_1 \cos(\theta_1 - t), \cdots, r_n \cos(\theta_n - t), r_1 \sin(\theta_1 - t), \cdots, r_n \sin(\theta_n - t))$$

and

$$\sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial\theta_i \partial\theta_j} = \int_0^{\theta_n} K(t) \frac{d^2 h}{dt^2} dt - K(t) \frac{dh(t)}{dt} \mid_{t=\theta_n} + K'(t) h(t) \mid_{t=\theta_n} .$$

An integration by parts gives us

$$\begin{aligned} \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial\theta_i \partial\theta_j} &= K(t) \frac{dh(t)}{dt} \mid_0^{\theta_n} \\ &\quad - \int_0^{\theta_n} K'(t) \frac{dh}{dt} dt - K(t) \frac{dh(t)}{dt} \mid_{t=\theta_n} + K'(t) h(t) \mid_{t=\theta_n} \\ &= -K(t) \frac{dh(t)}{dt} \mid_{t=0} + K'(t) h(t) \mid_{t=0} + \int_0^{\theta_n} K''(t) h(t) dt . \end{aligned}$$

Consequently

$$\begin{aligned} \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial\theta_i \partial\theta_j} + cg = -K(t) \frac{dh(t)}{dt} \mid_{t=0} &+ K'(t) h(t) \mid_{t=0} \\ &+ \int_0^{\theta_n} \left(K''(t) + cK(t)\right) h(t) dt . \end{aligned}$$

Finally, taking into account (16) we get the result.

Case $b > 0$. Similarly to case $b < 0$, we show that a solution of (10) is given by

$$g(x) = -\int_{-\infty}^{0} e^{bt} K(t) h(\phi(t, x)) dt$$

with the function $K(t)$ given by the system (11).

5. Proof of the main result

We apply lemma 2 to $E = \chi_\omega^\infty$ and $A^\infty = \chi_\omega^\infty \cap A$. First observe that A^∞ is of finite codimension in χ_ω^∞; indeed, from well known facts in linear algebra, we get

$$\dim(\chi_\omega^\infty / \chi_\omega^\infty \cap A) = \dim((\chi_\omega^\infty + A)/A) \leq \dim(\chi_\omega / A) = m < +\infty .$$

Let f be the germ of a function at the origin 0 of R^{2n}, which satisfies $f(0) = 0$, $Df(0) = 0$, and in addition the Hessian $D^2 f(0)$ is positive-definite or negative-definite, $H_1, \cdots, H_m : R \to R$ be distinct functions such that their derivatives $H_i'(u) = \frac{dH}{du}(u) > 0$. Put $f_o = f$, $f_1 = H_1 \circ f, \cdots,$ $f_m = H_m \circ f$; obviously, $f_o, f_1, \cdots, f_m$ fulfil the conditions of f and satisfy for all pair $0 \leq i, j \leq m$

$$\{f_i, f_j\} = 0 . \tag{17}$$

Since A is of finite codimension subalgebra there exist not all vanishing real numbers $a_o, a_1, \cdots, a_m$ such that the Hamiltonian vector field $F^\sharp = a_o f_o^\sharp + a_1 f_1^\sharp + \cdots + a_m f_m^\sharp \in A$. Let $\phi^o, \phi^1, \cdots, \phi^m$ be the flows generated respectively by $a_o f_o^\sharp, a_1 f_1^\sharp, \cdots, a_m f_m^\sharp$.

Set

$$\Phi(t,x) = \phi_t^o \circ \phi_t^1 \circ \cdots \circ \phi_t^m ,$$

where $\phi_t^i(x) = \phi^i(t,x)$.

Now (17) leads to $\left[f_i^\sharp, f_j^\sharp\right] = 0$, so Φ is a flow. Clearly, the vector field $F^\sharp = a_o f_o^\sharp + a_1 f_1^\sharp + \cdots + a_m f_m^\sharp$ is the generator of Φ and since the flows $\phi^o, \phi^1, \cdots, \phi^m$ are stable in the sense of Liapunov then so does Φ. It follows, by lemmas 6 and 7, that the endomorphism $\psi = \left[F^\sharp, .\right]$ defined on χ_ω^∞ fulfils the conditions of lemma 2, consequently $\chi_\omega^\infty = A^\infty$ i.e. $\chi_\omega^\infty \subset A$.

6. Application

Let Γ be a pseudogroup of local diffeomorphisms of R^n.

Definition 8. [1] A left action of a pseudogroup Γ on a manifold F, is a functorial assignment to each $f \in \Gamma$ of domain U a smooth map $\bar{f} : U \times F \to F$ such that the following axioms are satisfied

1) For any $\xi \in U$, $\bar{f}_\xi$, where $\bar{f}_\xi\ (y) = \bar{f}\ (\xi, y)$, is a diffeomorphism on F.
2) For any open set V of U, we have

$$\overline{f\ |_V} = \bar{f}\ |_{V \times F} \ .$$

3) For any f, g from Γ, we have

$$\overline{g \circ f}(\xi, y) = \bar{g}\ (f(\xi), \bar{f}\ (\xi, y)) \, .$$

4) Let I be an open segment from the real line R. If $f : I \times U \to R^n$ is a smooth map such that $\forall t \in I$, $f_t \in \Gamma$, where $f_t(x) = f(t,x)$, the map

$$\begin{gathered} I \times U \times F \to F \,, \\ (t, (\xi, y)) \to\ \bar{f}_t\ (\xi, y) \end{gathered}$$

is smooth.

Remark. It is obvious from the axiom 2) that the action is local i.e. it depends only on germs.

Let P be the pseudogroup of symplectic diffeomorphisms and P_o the group of germs of P fixing 0. Suppose that P acts on a closed manifold F and denote by $\gamma(T_F)$ the Lie algebra of vector fields on F and by $D(F)$ the group of global diffeomorphisms on F. It is known (see [3]) that $D(F)$ is an infinite dimensional Lie group with Lie algebra $\gamma(T_F)$.

Let $X \in \chi_\omega$ and $\phi_t = exp(tX)$ be the flow generated by X. So $\phi_t \in P_o$ for any $t \in I$ (since X leaves 0 fixed). Let $(\bar{\phi}_t)_t$ be the flow of diffeomorphisms (induced by the action of the pseudogroup P on F). Consider the vector field defined by

$$\bar{X}(y) = \frac{d}{dt} |_{t=o} (\bar{\phi}_t)_o(y) ,$$

where $(\bar{\phi}_t)_o(y) = \bar{\phi}_t \ (0, y)$.

Then, we obtain a Lie algebra homomorphism:

$$H : \chi_\omega \to \gamma(T_F), \ H(X) = \bar{X} \ .$$

Proposition 9. *The homomorphism H depends only on the infinite jet, $J_o^\infty X$, of X at 0.*

Proof. For a fixed $y \in F$, consider the linear homomorphism $H_y \ : \ \chi_\omega \ \to T_{yF}$ given by

$$H_y(X) = \bar{X} \ (y) ,$$

where T_{yF} denotes the tangent space to F at y. The kernel K_y of H_y is a finite codimension subalgebra of χ_ω, so by theorem 1 its contains χ_ω^∞ and hence H depends only on $J_o^\infty X$.

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