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FIXED POINTS OF FUZZY MONOTONE MAPS

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Abstract. The existence of fixed points for monotone maps on the fuzzy ordered sets under suitable conditions is proved.

1. Introduction and preliminaries

In his seminal paper Zadeh [11] introduced the notion of fuzzy set. During last three decades the fuzzy set theory has rapidly developed into an area which scientifically as well as from the application point of view, is recognized as a very valuable contribution to the existing knowledge (see [3, 9, 13]). Recently Heilpern [7], Hadzic [6], Fang [5], Jung, Cho and Kim [8] and many other authors have started to study fixed points in fuzzy setting. The aim of this note is to prove the existence of fixed points of fuzzy monotone maps on fuzzy ordered set.

Let X be a space of points (objects), with a generic element of X denoted by x. A fuzzy set B of X is characterized by a membership function 'b' which associated with each element in X a real number in the interval [0,1], with the value of $b(x)$ at x representing the grade of membership of x in B. For details see Zimmermann [13].

Zadeh [11] gave the definition of fuzzy ordered relations which was subsequently used by Vanugopalan [9] and Beg and Islam [2] in their recent papers. Zadeh's definition has a binary inspiration. In this paper we follow the following definition of order relation due to French school lead by Prof. Claude Ponsard (see Billot [3]).

Definition 1. Let X be a crisp set. A *fuzzy ordered relation* on X is a fuzzy subset R of $X \times X$ with the following properties

(i) for all $x \in X$, $r(x, x) \in [0, 1]$ (reflexivity);
(ii) for all $x, y \in X$, $r(x, y) + r(y, x) > 1$ implies $x = y$ (antisymmetry);
(iii) for all $(x, y, z) \in X^3$,
$[r(x, y) \geq r(y, x)$ and $r(y, z) \geq r(z, y)]$ implying $r(x, z) \geq r(z, x)$
(f-transitivity).

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A set with fuzzy order defined on it is called a *fuzzy ordered set*. A fuzzy relation is *totally reflexive* if for all $x \in X$, $r(x,x) = 1$ and *totally non-reflexive* if for all $x \in X$, $r(x,x) = 0$. Any intermediate situation between these two Boolean definition derives from the fuzzy reflexivity that describes the shades of reflexivity. Definition 1, has significant interpretation/applications in economics preference theory (see Billot [3]).

A fuzzy order is said to be *total* if for all $x \neq y$ we have either $r(x,y) > r(y,x)$ or $r(y,x) > r(x,y)$. A fuzzy ordered set on which fuzzy order is total is called *fuzzy chain*. For a subset $A \subset X$, an upper bound is an element $x \in X$ satisfying $r(y,x) \geq r(x,y)$ for all $y \in A$. An element x is called *maximal element* of A if there is no $y \neq x$ in A for which $r(x,y) \geq r(y,x)$. An $x \in A$ satisfying $r(y,x) \geq r(x,y)$ for all $y \in A$ is called *greatest element* of A. Similarly, we can define lower bound, minimal and least element of A. We denote

by $\sup A =$ least element of upper bounds
and $\inf A =$ greatest element of lower bounds.

In addition to first order fuzzy theory axioms (Za) (for details see Chapin [4]), we assume *Fuzzy Zorn's Lemma* (see Beg [1]): If every fuzzy chain in a fuzzy ordered set X has an upper bound, then X has a maximal element.

A mapping $f : X \to X$ is called fuzzy *monotone* if $r(y,x) \geq r(x,y)$ implies $r(f(y), f(x)) \geq r(f(x), f(y))$. A point $x \in X$ is called a fixed point of f if $f(x) = x$.

2. The results

Theorem 1. *Let X be a fuzzy ordered set with the property that every fuzzy chain in X has a supremum. Let $f : X \to X$ be a fuzzy monotone map and assume that there exists some $a \in X$ with $r(a, f(a)) \geq r(f(a), a)$. Then the set of fixed points of f is nonempty and has a maximal fixed point.*

Proof. Consider the fuzzy ordered subset

$$P = \{x \in X : r(x, f(x)) \geq r(f(x), x)\}.$$

Since $a \in P$, therefore P is nonempty. Let C be a chain in P and b be its supremum in X. Then $r(c,b) \geq r(b,c)$ for every $c \in C$. Thus $r(f(c), f(b)) \geq r(f(b), f(c))$. As $r(c, f(c)) \geq r(f(c), c)$. Therefore $r(c, f(b)) \geq r(f(b), c)$ for $c \in P$. It follows that $f(b)$ is an upper bound for C. Since b is supremum of C and $f(b)$ is an upper bound for C, we have $r(b, f(b)) \geq r(f(b), b)$. Therefore $b \in P$. Thus supremum of any chain in P belongs to P. Fuzzy Zorn's Lemma further implies that P has a maximal element, $x_0(say)$. Since $x_0 \in P$,

$$r(x_0, f(x_0)) \geq r(f(x_o), x_0).$$

As f is monotone, $r(f(x_0), f(f(x_0))) \geq r(f(f(x_0)), x_0)$. It further implies that $f(x_0)$ belong to P. Since x_0 is a maximal element of P, we see that $x_0 = f(x_0)$.

Furthermore, if y is a another fixed point of f, then $y \in P$. This shows that x_0 is a maximal fixed point of f. $\square$

Definition 2. A map $f : X \to X$ is said to be *fuzzy order continuous* if for each countable fuzzy chain $\{c_i\}$ having a supremum, $f(\sup\{c_i\}) = \sup(\{f(c_i)\})$.

A fuzzy order continuous map is necessarily monotone; for if $r(x,y) \geq r(y,x)$ then $y = \sup\{x,y\}$, so, by continuity, $f(y) = \sup\{f(x), f(y)\}$. Therefore $r(f(x), f(y)) \geq r(f(y), f(x))$.

Theorem 2. *Let X be a fuzzy ordered set and $f : X \to X$ be fuzzy order continuous map. Assume that there is a $b \in X$ such that:*

(i) *$r(b, f(b)) \geq r(f(b), b)$, and*
(ii) *every countable fuzzy chain in $\{x : r(b,x) \geq r(x,b)\}$ has a supremum.*

Then the point $c = \sup_n f^n(b)$ is a fixed point of f. Moreover the point c is also the infimum of the set of fixed points of f in $\{x : r(b,x) \geq r(x,b)\}$.

Proof. Because $r(b, f(b)) \geq r(f(b), b)$ and f is monotone, we have

$$r(f(b), f^2(b)) \geq r(f^2(b), f(b))$$

and inductively,

$$r(f^n(b), f^{n+1}(b)) \geq r(f^{n+1}(b), f^n(b))$$

for each $n \geq 1$. Thus $\{f^n(b) : n \geq 1\}$ is a chain in $\{x : x \geq b\}$; so $c = \sup_n f^n(b)$ exists.

Since f is continuous,

$$f(c) = f\Big(\sup_n f^n(b)\Big) = \sup_n f\Big(f^n(b)\Big) = \sup_n f^{n+1}(b) = c\,.$$

Hence c is a fixed point of f. Let e be another fixed point of f in $\{x : r(b,x) \geq r(x,b)\}$, then $r(c,e) \geq r(e,c)$. Indeed, since $r(b,e) \geq r(e,b)$, we have $r(f(b), e) = r(f(b), f(e)) \geq r(f(e), f(b)) = r(e, f(b))$, and by induction, that $r(f^n(b), e) \geq r(e, f^n(b))$ for every $n \geq 1$. Thus e is an upper bound for $\{f^n(b) : n \geq 1\}$, so $r(c,e) \geq r(e,c)$. Thus c is the infimum of the set of fixed points of f in $\{x : r(b,x) \geq r(x,b)\}$. □

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