ARCHIVUM MATHEMATICUM (BRNO)
Tomus 34 (1998), 59 – 95

DOUBLE VECTOR BUNDLES AND DUALITY

Katarzyna Konieczna and Paweł Urbański

Abstract. The notions of the dual double vector bundle and the dual double vector bundle morphism are defined. Theorems on canonical isomorphisms are formulated and proved. Several examples are given.

1. Introduction

The notion of a double vector bundle was introduced by Pradines in [3]. The most important examples of double vector bundles are given by iterated tangent and cotangent functors applied to a manifold M: $\mathsf{TT}M$, $\mathsf{T}^*\mathsf{T}M$, TT^*M, and $\mathsf{T}^*\mathsf{T}^*M$. The double vector bundle structure on $\mathsf{TT}M$ makes possible the Lagrangian formulation of the dynamics in classical mechanics ([5]). It appears that the framework of double vector bundles is very convenient for many important constructions like linear 1-forms, linear Poisson structures, special symplectic manifolds, linear connections etc.

The paper is organized as follows. In Sections 2 and 3 we present in detail the definitions of a double vector bundle following Mackenzie [2], and morphisms of double vector bundles. The tangent $\mathsf{T}\mathbf{E}$ to a vector bundle $\mathbf{E}$ is an example of a double vector bundle. In Section 4 we show that the bundle dual to a double vector bundle with respect to the right (or left) vector bundle structure is, in a natural way, a double vector bundle. In particular, the cotangent functor applied to a vector bundle $\mathbf{E}$ provides the double vector bundle T^*E. Also the mapping dual to a morphism of double vector bundles is a morphism of double vector bundles (Section 5). The main result of the paper is contained in Section 6. We show that the third right dual to a double vector bundle $\mathbf{K}$ is canonically isomorphic to $\mathbf{K}$. In the case of $\mathbf{K} = \mathsf{T}^*\mathbf{E}$, the third dual can be identified with $\mathsf{T}^*\mathbf{E}^*$. The isomorphism $\mathcal{R}_K \colon \mathbf{K} \to \mathbf{K}^{*_r *_r *_r}$ of Theorem 17 gives in this case the canonical isomorphism of $\mathsf{T}^*\mathbf{E}$ and $\mathsf{T}^*\mathbf{E}^*$. The graph of this isomorphism is a Lagrangian submanifold of $\mathsf{T}^*(E \times E^*) \simeq \mathsf{T}^*E \times \mathsf{T}^*E^*$ generated by the pairing between $\mathbf{E}$ and $\mathbf{E}^*$.

1991 *Mathematics Subject Classification*: 53C15, 53C80.
Key words and phrases: double vector bundles, duality.
Supported by KBN, grant No 2 PO3A 074 10.
Received February 1998.

With the canonical identifications of $\mathbf{K}$ with $\mathbf{K}^{*_r *_r *_r}$ and of $\mathbf{K}'$ with $\mathbf{K}'^{*_r *_r *_r}$, the third dual to a double vector bundle morphism $\mathbf{\Phi}\colon \mathbf{K} \to \mathbf{K}'$ is equal $\mathbf{\Phi}^{-1}$.

In Section 7 we show that the framework of double vector bundles is suitable for concepts like linear vector fields and forms, linear Poisson and symplectic structures, vertical and complete lifts, and linear connections. We discuss in detail the case of linear connections, compatible with a metric on $\mathbf{E}$, and symmetric connections on the tangent and cotangent bundles.

The main results of this paper were presented at the conference in Vietri sul Mare, October 97, and one can find their formulations in the proceedings of this conference ([8]).

2. Double vector bundles

Let $\mathbf{K}$ be a system $(\mathbf{K}_r, \mathbf{K}_l, \mathbf{E}, \mathbf{F})$ of vector bundles, where $\mathbf{K}_r = (K, \tau_r, E)$, $\mathbf{K}_l = (K, \tau_l, F)$, $\mathbf{E} = (E, \bar{\tau}_l, M)$, and $\mathbf{F} = (F, \bar{\tau}_r, M)$, such that the diagram

$$
\begin{tikzcd}
 & K \arrow[dl, "\tau_l"'] \arrow[dr, "\tau_r"] & \\
F \arrow[dr, "\bar{\tau}_r"'] & & E \arrow[dl, "\bar{\tau}_l"] \\
 & M &
\end{tikzcd}
\tag{1}
$$

is commutative.

We introduce the following notation:

(1) m_r, m_l, $\bar{m}_r$, and $\bar{m}_l$ will denote the operation of addition in $\mathbf{K}_r$, $\mathbf{K}_l$, $\mathbf{E}$, and $\mathbf{F}$ respectively.

(2) we use also $v +_r w$ for $m_r(v, w)$, $v +_l w$ for $m_l(v, w)$, and simply $+$ for all other additions,

(3) $\mathbf{0}_r$, $\mathbf{0}_l$, $\bar{\mathbf{0}}_r$, $\bar{\mathbf{0}}_l$ will denote the zero sections of τ_r, τ_l, $\bar{\tau}_r$, and $\bar{\tau}_l$ respectively.

Let us suppose that the pair $(\tau_r, \bar{\tau}_r)$ is a vector bundle morphism $\mathbf{K}_l \to \mathbf{E}$. It follows that $K \times_E K$ is a vector subbundle of $\mathbf{K}_l \oplus \mathbf{K}_l$ with $(\tau_l \times \tau_l)(K \times_E K) = F \times_M F$. We denote this subbundle by $\mathbf{K}_l \oplus_E \mathbf{K}_l$. Moreover, the addition $m_r\colon K \times_E K \to K$ is a fiber bundle morphism which projects to $\bar{m}_r\colon F \times_M F \to F$.

Definition 1. A double vector bundle $\mathbf{K}$ is a system $(\mathbf{K}_r, \mathbf{K}_l, \mathbf{E}, \mathbf{F})$ of vector bundles $\mathbf{K}_r = (K, \tau_r, E)$, $\mathbf{K}_l = (K, \tau_l, F)$, $\mathbf{E} = (E, \bar{\tau}_l, M)$ and $\mathbf{F} = (F, \bar{\tau}_r, M)$ such that the diagram (1) is commutative and the following conditions are satisfied:

(1) pairs $(\tau_l, \bar{\tau}_l)$, $(\tau_r, \bar{\tau}_r)$ are vector bundle morphisms,

(2) pairs of additions $(m_l, \bar{m}_l)$ and $(m_r, \bar{m}_r)$ are vector bundle morphisms $\mathbf{K}_r \times_F \mathbf{K}_r \to \mathbf{K}_r$ and $\mathbf{K}_l \times_E \mathbf{K}_l \to \mathbf{K}_l$ respectively,

(3) zero sections $\mathbf{0}_r\colon \mathbf{E} \to \mathbf{K}_l$, $\mathbf{0}_l\colon \mathbf{F} \to \mathbf{K}_r$ are vector bundle morphisms.

In the following we use the diagram (1) to represent the double vector bundle $\mathbf{K}$.

Proposition 2. *The vector bundle structures of* $\mathbf{K}_r$ *and* $\mathbf{K}_l$ *coincide on the intersection* C *of* $\ker \tau_l$ *and* $\ker \tau_r$.

Proof. Since τ_r and τ_l are linear in fibers it follows that for each $m \in M$, we have

$$\mathbf{0}_r \circ \bar{\mathbf{0}}_l(m) \in C, \quad \mathbf{0}_l \circ \bar{\mathbf{0}}_r(m) \in C.$$

Let us denote $0_r(m) = \mathbf{0}_r \circ \bar{\mathbf{0}}_l(m)$, $0_l(m) = \mathbf{0}_l \circ \bar{\mathbf{0}}_r(m)$. Since $m_l\colon \mathbf{K}_l \times_F \mathbf{K}_l \to \mathbf{K}_l$ is a vector bundle morphism, we have

$$m_l(0_r(m), 0_r(m)) = 0_r(m).$$

It follows that $0_r(m)$ is a neutral element of m_l, i.e, $0_l = 0_r$. For $k, k' \in C$, such that $\bar{\tau}_l(\tau_r(k)) = \bar{\tau}_l(\tau_r(k')) = m$, we have

$$(k, k') = (0_r(m), k') +_r (k, 0_r(m)) = (0_l(m), k') +_r (k, 0_l(m))$$

and, consequently,

$$m_l(k, k') = m_l(k, 0_l(m)) +_r m_l(0_l(m), k') = k +_r k' . \qquad \square$$

Thus, we have a vector bundle $\mathbf{C} = (C, \tau, M)$, where $\tau = \bar{\tau}_l \circ \tau_r = \bar{\tau}_r \circ \tau_l$. This vector bundle is called *the core* of $\mathbf{K}$.

Proposition 3.

(1) $\ker \tau_r$ *with the vector bundle structure induced from* $\mathbf{K}_r$ *is canonically isomorphic to the Whitney sum* $\mathbf{F} \oplus_M \mathbf{C}$.

(2) $\ker \tau_r$ *with the vector bundle structure induced from* $\mathbf{K}_l$ *is canonically isomorphic to the vector bundle* $F \times_M \mathbf{C}$, *i.e., to the pull-back of* $\mathbf{C}$ *by the projection* $\bar{\tau}_r$.

(3) $\ker \tau_l$ *with the vector bundle structure induced from* $\mathbf{K}_l$ *is canonically isomorphic to the Whitney sum* $\mathbf{E} \oplus_M \mathbf{C}$.

(4) $\ker \tau_l$ *with the vector bundle structure induced from* $\mathbf{K}_r$ *is canonically isomorphic to the vector bundle* $E \times_M \mathbf{C}$, *i.e., to the pull-back of* $\mathbf{C}$ *by the projection* $\bar{\tau}_l$.

Proof.

(1) Since the zero section $\mathbf{0}_l$ is a vector bundle morphism

$$\mathbf{0}_l\colon \mathbf{F} \to \mathbf{K}_r\,,$$

its image $\mathbf{0}_l(F)$ is a vector subbundle of $\mathbf{K}_r$, contained in $\ker \tau_r$ and isomorphic to $\mathbf{F}$.

Let $v \in \ker \tau_r$ and let $m = \bar{\tau}_l \circ \tau_r(v) = \bar{\tau}_r \circ \tau_l(v)$. We have $\tau_r(v) = \bar{\mathbf{0}}_l$ and, since $v \in \ker \tau_r$,

$$\tau_r(\mathbf{0}_l(\tau_l(v))) = \bar{\mathbf{0}}_l(\bar{\tau}_r \circ \tau_l(v)) = \bar{\mathbf{0}}_l(m)\,.$$

It follows, that the pair $(v, \mathbf{0}_l(\tau_l(v)))$ is in the domain of m_r. Now, we can define two mappings

$$¶_C,\ ¶_F\colon \ker \tau_r \to \ker \tau_r$$

by the formulae

$$\begin{aligned} ¶_F(v) &= \mathbf{0}_l(\tau_l(v))\,, \quad m = (\bar{\tau}_r \circ \tau_l)(v)\,, \\ ¶_C(v) &= v -_r \mathbf{0}_l(\tau_l(v))\,. \end{aligned} \tag{2}$$

It is evident that $¶_F^2 = ¶_F$, $¶_F ¶_C = 0$, $¶_C + ¶_F = id$ and

$$\begin{aligned} ¶_C^2(v) &= (v -_r \mathbf{0}_l(\tau_l(v))) -_r \mathbf{0}_l(\tau_l(v -_r \mathbf{0}_l(\tau_l(v)))) \\ &= (v -_r \mathbf{0}_l(\tau_l(v))) -_r 0_r(m) = v -_r \mathbf{0}_l(\tau_l(v)) \\ &= ¶_C(v)\,. \end{aligned}$$

It follows that $¶_F, ¶_C$ are projectors, which define a splitting of $\ker \tau_r$. We have that $¶_C(v) \in \ker \tau_r$ and

$$\tau_l(¶_C(v)) = \tau_l(v) - \tau_l(\mathbf{0}_l(\tau_l(v))) = \bar{\mathbf{0}}_l(m)\,,$$

and, consequently, $¶_C(v) \in \mathbf{C}$.

It is obvious that, for each $m \in M$, the intersection of fibers over m of $\mathbf{0}_l(F)$ and $\mathbf{C}$ is trival and equal to $\{0_r(m)\}$. Since the image of $¶_F$ is canonically isomorphic to $\mathbf{F}$, we conclude, finally, that $\ker \tau_r$ is canonically isomorphic to $\mathbf{F} \oplus_M \mathbf{C}$.

(2) From (1) we have that $\ker \tau_r$ can be identified (as a manifold) with $F \times_M C$. With this identification τ_l is the canonical projection on F and the zero section $\mathbf{0}_l$ is given by

$$\mathbf{0}_l\colon F \to F \times_M C\colon f \mapsto (f, 0)\,.$$

The addition m_l defines a vector bundle morphism

$$m_l\colon \mathbf{K}_r \times_F \mathbf{K}_r \supset \ker \tau_r \times_F \ker \tau_r = \mathbf{F} \oplus \mathbf{C} \oplus \mathbf{C} \to \mathbf{F} \oplus \mathbf{C} \subset \mathbf{K}_r\,,$$

hence

$$\begin{aligned} (f, k) +_l (f, k') = m_l((f, k, k')) &= m_l((f, 0, 0) +_r (0, k, k')) \\ &= (f, 0) +_r (0, k + k') = (f, k + k')\,. \end{aligned}$$

(3,4) The proof is analogous. □

The following two propositions will be useful in the next section.

Proposition 4. *Let $v, w \in K$ and let*

$$\tau_r(v) = \tau_r(w) = e \in E\,,$$
$$\tau_l(v) = \tau_l(w) = f \in F\,.$$

There exists $k \in C$ such that $\tau(k) = \bar{\tau}_l(e) = \bar{\tau}_r(f)$ and, using the identifications of Proposition 3,

$$v = w +_r (e,k),$$
$$v = w +_l (f,k)\,.$$

Proof. We have $\tau_l(v -_r w) = f - f = 0$ and $\tau_r(v -_l w) = e - e = 0$. It follows from Proposition 3 that there exist $k, k' \in C$ such that

$$v -_r w = (e,k), \quad v -_l w = (f,k')\,.$$

We show that $k = k'$. Indeed, since $(v,w) \in K \times_E K$ and

$$(v,w) = (v -_l w, w -_l v) +_l (w,v)\,,$$

we have (linearity of m_r)

$$v -_r w = ((v -_l w) -_r (w -_l v)) +_l (w -_r v)\,.$$

Hence (Proposition 3)

$$(e,k) = ((f,k') -_r (f,-k')) +_l (e,k) = (0,2k') +_l (e,-k)$$

and

$$(0,2k') = (e,k) -_l (e,-k) = (0,2k)\,.$$

□

Proposition 5. *Let $v, v', w, w' \in K$ be such that*

$$\tau_r(v) = \tau_r(w) = e \in E\,,$$
$$\tau_r(v') = \tau_r(w') = e' \in E\,,$$
$$\tau_l(v) = \tau_l(w) = \tau_l(v') = \tau_l(w') = f \in F,$$
$$v +_l v' = w +_l w'\,.$$

Then

$$v -_r w = (e,k) \in \ker \tau_l\,,$$
$$v' -_r w' = (e',-k) \in \ker \tau_l\,.$$

Proof. Since $v -_r w \in \ker \tau_l$ and $v' -_r w' \in \ker \tau_l$, there exist $k, k' \in C$ such that

$$v -_r w = (e,k), \quad v' -_r w' = (e',k')\,.$$

From Proposition 4 we have also

$$v -_l w = (f,k), \quad v' -_l w' = (f,k')\,.$$

The equality $v +_l v' = w +_l w'$ implies

$$(f,k) = v -_l w = w' -_l v' = (f,-k')$$

and $k' = -k$. □

Local coordinates. Let $(x^i)_{i=1}^n$ be a coordinate system on M. In the bundles $\mathbf{E}, \mathbf{F}$, we introduce coordinate systems $((x^i)_{i=1}^n, (e^a)_{a=1}^{n_E})$ and $((x^i)_{i=1}^n, (f^A)_{A=1}^{n_F})$. We denote also by x^i, e^a, f^A their pull-backs to the coordinates on K. It follows from Proposition 4 that we can introduce coordinates $(c^\alpha)_{\alpha=1}^{n_C}$ such that $(x^i, e^a, f^A, c^\alpha)$ is a local coordinate system on K and

$$\begin{aligned}
&c^\alpha(v +_r w) = c^\alpha(v) + c^\alpha(w),\\
&c^\alpha(v +_l w) = c^\alpha(v) + c^\alpha(w),\\
&c^\alpha \circ \mathbf{0}_r = 0,\\
&c^\alpha \circ \mathbf{0}_l = 0.
\end{aligned}$$

It follows that (x^i, c^α) is a vector bundle coordinate system in $\mathbf{C}$. The operation of addition $+_r$ is characterized by the following equalities

$$\begin{aligned}
c^\alpha(v +_r w) &= c^\alpha(v) + c^\alpha(w)\,,\\
f^A(v +_r w) &= f^A(v) + f^A(w)\,,\\
e^a(v +_r w) &= e^a(v) = e^a(w)\,,\\
x^i(v +_r w) &= x^i(v) = x^i(w)\,.
\end{aligned}$$

The operation of addition $+_l$ is characterized by

$$\begin{aligned}
c^\alpha(v +_l w) &= c^\alpha(v) + c^\alpha(w)\,,\\
f^A(v +_l w) &= f^A(v) = f^A(w)\,,\\
e^a(v +_l w) &= e^a(v) + e^a(w)\,,\\
x^i(v +_l w) &= x^i(v) = x^i(w)\,.
\end{aligned}$$

Examples.

1. Let $\mathbf{E} = (E, \bar{\tau}_l, M)$, $\mathbf{F} = (F, \bar{\tau}_r, M)$, $\mathbf{C} = (C, \tau, M)$ be vector bundles and let $K = F \times_M C \times_M E$. By $\mathbf{K}(\mathbf{F}, \mathbf{C}, \mathbf{E})$ we denote a double vector bundle represented by the diagram

(3)
$$\begin{tikzcd}
& K \arrow[dl, "\tau_l"'] \arrow[dr, "\tau_r"] & \\
F \arrow[dr, "\bar{\tau}_r"'] & & E \arrow[dl, "\bar{\tau}_l"] \\
& M &
\end{tikzcd}$$

where $\tau_r \colon K = F \times_M C \times_M E \to E$ and $\tau_l \colon K = F \times_M C \times_M E \to F$ are the canonical projections. The right and left vector bundle structures are obvious:

$$\begin{aligned}
(f, k, e) +_r (f', k', e) &= (f + f', k + k', e)\,,\\
(f, k, e) +_l (f, k', e') &= (f, k + k', e + e')\,.
\end{aligned}$$

The core of $\mathbf{K}(\mathbf{F}, \mathbf{C}, \mathbf{E})$ can be identified with $\mathbf{C}$.

2. Let $\mathbf{E} = (E, \tau, M)$ be a vector bundle. The tangent manifold $\mathsf{T}E$ has two vector bundle structures ([5]):

> the canonical vector bundle structure of the tangent bundle on the canonical fibration $\tau_E \colon \mathsf{T}E \to E$,
> the tangent vector bundle structure on the tangent fibration $\mathsf{T}\tau \colon \mathsf{T}E \to \mathsf{T}M$.

It is easy task to verify that the diagram

(4)
$$\begin{tikzcd}
& \mathsf{T}EK \arrow[dl, "\mathsf{T}\tau"'] \arrow[dr, "\tau_E"] & \\
\mathsf{T}M \arrow[dr, "\tau_M"] & & E \arrow[dl, "\tau"'] \\
& M &
\end{tikzcd}$$

represents a double vector bundle. We denote this double vector bundle by $\mathsf{T}\mathbf{E}$. The core consists of vertical tangent vectors at the zero section of $\mathbf{E}$. Thus, it can be identified, in an obvious way, with $\mathbf{E}$.

3. Let $\mathbf{K}$ be a double vector bundle represented by the diagram

(5)
$$\begin{tikzcd}
& K \arrow[dl, "\tau_l"'] \arrow[dr, "\tau_r"] & \\
F \arrow[dr, "\bar{\tau}_r"] & & E \arrow[dl, "\bar{\tau}_l"'] \\
& M &
\end{tikzcd}$$

then also the diagram

(6)
$$\begin{tikzcd}
& K \arrow[dl, "\tau_r"'] \arrow[dr, "\tau_l"] & \\
E \arrow[dr, "\bar{\tau}_l"] & & F \arrow[dl, "\bar{\tau}_r"'] \\
& M &
\end{tikzcd}$$

represents a double vector bundle. We denote it by $\mathsf{J}(\mathbf{K})$.

In examples (1) and (2), we identified canonically the core with a certain vector bundle. We shall write in the following a diagram

$$
\begin{tikzcd}
& K \arrow[dl, "\tau_l"'] \arrow[dr, "\tau_r"] & \\
F \arrow[dr, "\bar{\tau}_r"'] & C \arrow[u, hook] \arrow[d, "\tau"'] & E \arrow[dl, "\bar{\tau}_l"] \\
& M &
\end{tikzcd}
\tag{7}
$$

instead of the diagram (1), if we identify the core of (1) with the vector bundle $\mathbf{C}$. In the case of $\mathbf{K} = \mathsf{T}\mathbf{E}$ we write then the diagram

$$
\begin{tikzcd}
& \mathsf{T}E \arrow[dl, "\mathsf{T}\tau"'] \arrow[dr, "\tau_E"] & \\
\mathsf{T}M \arrow[dr, "\tau_M"'] & E \arrow[u, hook] \arrow[d, "\tau"'] & E \arrow[dl, "\tau"] \\
& M &
\end{tikzcd}
\tag{8}
$$

3. Morphisms of double vector bundles

Let $\mathbf{K} = (\mathbf{K}_r, \mathbf{K}_l, \mathbf{E}, \mathbf{F})$ and $\mathbf{K}' = (\mathbf{K}'_r, \mathbf{K}'_l, \mathbf{E}', \mathbf{F}')$ be double vector bundles with cores $\mathbf{C}$ and $\mathbf{C}'$ respectively. A morphism $\mathbf{\Phi}\colon \mathbf{K} \to \mathbf{K}'$ of double vector bundles is a family $\mathbf{\Phi} = (\Phi, \Phi_r, \Phi_l, \bar{\Phi})$ of mappings

$$\Phi\colon K \to K', \qquad \Phi_r\colon E \to E', \qquad \Phi_l\colon F \to F', \qquad \text{and} \quad \bar{\Phi}\colon M \to M'$$

such that $\mathbf{\Phi}_r = (\Phi, \Phi_r)$, $\mathbf{\Phi}_l = (\Phi, \Phi_l)$, $\bar{\mathbf{\Phi}}_r = (\Phi_r, \bar{\Phi})$, and $\bar{\mathbf{\Phi}}_l = (\Phi_l, \bar{\Phi})$ are morphisms of vector bundles

$$\mathbf{\Phi}_r\colon \mathbf{K}_r \to \mathbf{K}'_r, \qquad \mathbf{\Phi}_l\colon \mathbf{K}_l \to \mathbf{K}'_l, \qquad \bar{\mathbf{\Phi}}_r\colon \mathbf{E} \to \mathbf{E}', \qquad \text{and} \quad \bar{\mathbf{\Phi}}_l\colon \mathbf{F} \to \mathbf{F}'.$$

We have thus a commutative diagram

$$
\begin{tikzcd}
& K \arrow[rrr, "\Phi"] \arrow[ddl, "\tau_l"'] \arrow[dr, "\tau_r"] & & & K' \arrow[ddl, "\tau'_l"'] \arrow[dr, "\tau'_r"] & \\
& & E \arrow[rrr, "\Phi_r"] \arrow[ddl, "\bar{\tau}_l"] & & & E' \arrow[ddl, "\bar{\tau}'_l"'] \\
F \arrow[rrr, "\Phi_l"] \arrow[dr, "\bar{\tau}_r"'] & & & F' \arrow[dr, "\bar{\tau}'_r"] & & \\
& M \arrow[rrr, "\bar{\Phi}"] & & & M' &
\end{tikzcd}
\tag{9}
$$

Proposition 6. *Let $\mathbf{\Phi}: \mathbf{K} \to \mathbf{K}'$ be a morphism of double vector bundles. Then*

(1) $\Phi(\ker \tau_r) \subset \ker \tau_r'$,
(2) $\Phi(\ker \tau_l) \subset \ker \tau_l'$,
(3) $\Phi(C) \subset C'$.

Proof. Since $\bar{\mathbf{\Phi}}_r: \mathbf{E} \to \mathbf{E}'$ is a morphism of vector bundles, it maps the zero section of $\mathbf{E}$ into the zero section of $\mathbf{E}'$. It follows that $\Phi(\ker \tau_r) \subset \ker \tau_r'$. Similarly, $\Phi(\ker \tau_l) \subset \ker \tau_l'$ and, consequently, $\Phi(C) \subset C'$. □

We denote by $\mathbf{\Phi}_c = (\Phi_c, \bar{\Phi})$ the morphism of vector bundles $\mathbf{C}$ and $\mathbf{C}'$ induced by Φ.

Let $(x^i, e^a, f^A, c^\alpha)$ be an adapted local coordinate system on K and let $(\bar{x}^{\bar{i}}, \bar{e}^{\bar{a}}, \bar{f}^{\bar{A}}, \bar{c}^{\bar{\alpha}})$ be an adapted coordinate system in K'. We have

$$
\begin{aligned}
\bar{x}^{\bar{i}} \circ \Phi &= \Phi^{\bar{i}}\,, \\
\bar{e}^{\bar{a}} \circ \Phi &= \Phi^{\bar{a}}_b e^b\,, \\
\bar{f}^{\bar{A}} \circ \Phi &= \Phi^{\bar{A}}_B f^B\,, \\
\bar{c}^{\bar{\alpha}} \circ \Phi &= \Phi^{\bar{\alpha}}_\beta c^\beta + \Phi^{\bar{\alpha}}_{aA} e^a f^A\,,
\end{aligned}
\tag{10}
$$

where $\Phi^{\bar{i}}$, $\Phi^{\bar{a}}_b$, $\Phi^{\bar{A}}_B$, $\Phi^{\bar{\alpha}}_\beta$, $\Phi^{\bar{\alpha}}_{aA}$ are functions on the domain of (x^i) in M.

Examples.

1. Let $\mathbf{E} = (E, \tau, M)$ and $\mathbf{E}' = (E', \tau', M')$ be vector bundles and let $\mathbf{\Phi} = (\Phi, \bar{\Phi})$ be a vector bundle morphism

$$\Phi: \mathbf{E} \to \mathbf{E}'\,.$$

The quadruple $\mathsf{T}\mathbf{\Phi} = (\mathsf{T}\Phi, \Phi, \mathsf{T}\bar{\Phi}, \bar{\Phi})$ defines a morphism of double vector bundles

$$\mathsf{T}\mathbf{\Phi}: \mathsf{T}\mathbf{E} \to \mathsf{T}\mathbf{E}'$$

and, with the identification of cores as in the diagram (8), we have $\mathbf{\Phi}_c: \mathbf{E} \to \mathbf{E}'$, $\Phi_c = \Phi$.

2. An essential role in the Lagrangian formulation of the classical mechanical system is played by the isomorphism κ_M, which relates $\mathsf{TT}M$ with $\mathsf{J}(\mathsf{TT}M)$ ([5]). Here, $\mathsf{T}M$ is the vector bundle of tangent vectors. All three vector bundle morphisms $\bar{\mathbf{\Phi}}_r, \bar{\mathbf{\Phi}}_l, \mathbf{\Phi}_c: \mathsf{T}M \to \mathsf{T}M$ are indentities.

3. Let $\mathbf{K} = \mathbf{K}(\mathbf{F}, \mathbf{C}, \mathbf{E})$ and $\mathbf{K}' = \mathbf{K}(\mathbf{F}', \mathbf{C}', \mathbf{E}')$. If $\mathbf{\Phi} = (\Phi, \Phi_r, \Phi_l, \bar{\Phi})$ is a morphism of double vector bundles,

$$\mathbf{\Phi}: \mathbf{K} \to \mathbf{K}'\,, \tag{11}$$

then

$$\Phi(f, c, e) = (\Phi_l(f), \Phi_c(c) + \Psi(f, e), \Phi_r(e))\,,$$

where the mapping

$$\Psi: \mathbf{F} \times_M \mathbf{E} \to \mathbf{C}'$$

is bilinear.

4. The right dual

Let $\mathbf{K}_r^*$ be the vector bundle dual to $\mathbf{K}_r$. We denote by K^{*_r} the total fiber bundle space and by π_l the projection

$$\pi_l \colon K^{*_r} \to E\,.$$

Let $a \in K^{*_r}$ and $k \in C$ satisfy $\tau(k) = \bar{\tau}_l(\pi_l(a))$. We can evaluate a on a vector $(\pi_l(a), k)$ of $\ker \tau_l$. We define a mapping $\pi_r \colon K^{*_r} \to C^*$ by the formula

$$\langle k, \pi_r(a)\rangle = \langle (\pi_l(a), k), a\rangle\,. \tag{12}$$

It follows directly from this construction that

Proposition 7. *The mapping $\pi_r \colon K^{*_r} \to C^*$ is a morphism of vector bundles*

$$\pi_r \colon \mathbf{K}_r^* \to \mathbf{C}^*\,.$$

We define a relation

$$P_r(m_l) \colon K^{*_r} \times K^{*_r} \to K^{*_r}$$

in the following way: $c \in P_r(m_l)(a, b)$ if

(1) $\pi_l(c) = \pi_l(a) + \pi_l(b)$,
(2) $\langle w, c\rangle = \langle v, a\rangle + \langle v', b\rangle$ for each $w, v, v' \in K$ such that $\tau_r(w) = \pi_l(c)$, $\tau_r(v) = \pi_l(a)$, $\tau_r(v') = \pi_l(b)$, and $w = m_l(v, v')$.

Proposition 8. *$P(m_l)$ is univalent and, if $c = P(m_l)(a, b)$, then $\pi_r(a) = \pi_r(b) = \pi_r(c)$.*

Proof. $P(m_l)$ is univalent, because for each w we can find v, v' such that $m_l(v, v') = w$ and, consequently, c is completely determined by the condition

$$\langle w, c\rangle = \langle v, c\rangle + \langle v', c\rangle\,.$$

Let $k \in C$ and $c = P(m_l)(a, b)$. From the definition of π_r we have, using identifications of Proposition 3,

$$\begin{aligned}\langle k, \pi_r(a)\rangle - \langle k, \pi_r(b)\rangle &= \langle (\pi_l(a), k), a\rangle + \langle (\pi_l(b), -k), b\rangle = \\ &= \langle ((\pi_l(a), k) +_l (\pi_l(b), -k)), c\rangle = \langle (\pi_l(a) + \pi_l(b), 0), c\rangle = 0\,.\end{aligned}$$

It follows that $\pi_r(a) = \pi_r(b)$. Moreover,

$$\begin{aligned}\langle k, \pi_r(c)\rangle &= \langle (\pi_l(c), k), c\rangle = \langle ((\pi_l(a), k) +_l (\pi_l(b), 0)), c\rangle = \\ &= \langle (\pi_l(a), k), a\rangle + \langle (\pi_l(b), 0), b\rangle = \langle (\pi_l(a), k), a\rangle = \langle k, \pi_r(a)\rangle\,.\end{aligned}$$

Hence, $\pi_r(a) = \pi_r(c)$. □

Proposition 9. *A pair* $(a,b)\in K^{*_r}\times K^{*_r}$ *is in the domain of* $P(m_l)$ *if and only if* $\pi_l(a)=\pi_l(b)$.

Proof. It is enough to show that if $\pi_l(a)=\pi_l(b)$ then there exists $c\in K^{*_r}$ such that $c=P(m_l)(a,b)$.

Let $\pi_r(a)=\pi_r(b)$. We define $c\in\pi_l^{-1}(\pi_l(a)+\pi_l(b))$ by

$$\langle w,c\rangle=\langle v,a\rangle+\langle v',b\rangle \tag{13}$$

where $w,v,v'\in K$ are such that $\tau_r(v)=\pi_l(a)=e$, $\tau_r(v')=\pi_l(b)=e'$ and $w=m_l(v,v')$. Let $w=m_l(u,u')$ be another representation of w, such that $\tau_r(u)=e,\tau_r(u')=e'$. We have also $\tau_l(u)=\tau_l(u')=\tau_l(w)$. It follows from Proposition 5 that

$$u-_r v=(e,k)\in\ker\tau_l\,,$$
$$u'-_r v'=(e',-k)\in\ker\tau_l$$

and, consequently,

$$\langle u,a\rangle+\langle u',b\rangle=\langle u-_r v,a\rangle+\langle u'-_r v',b\rangle+\langle v,a\rangle+\langle v',b\rangle=\langle (e,k),a\rangle+\langle (e',-k),b\rangle+$$
$$+\langle u,a\rangle+\langle u',b\rangle=\langle k,\pi_r(a)\rangle-\langle k,\pi_r(b)\rangle+\langle u,a\rangle+\langle u',b\rangle=\langle u,a\rangle+\langle u',b\rangle\,.$$

This proves that c is a well defined function on $(\tau_r)^{-1}(e+e')$. Now, we have to show that this function is linear. Let be $w=w_1+_r w_2$ and $w_1=u+_l u'$, $w_2=v+_l v'$. Since $+_r$ is linear on $\mathbf{K}_l\times_E\mathbf{K}_l$, we have

$$(u+_l u')+_r(v+_l v')=(u+_r v)+_l(u'+_r v')$$

and

$$\langle w_1+_r w_2,c\rangle=\langle u+_r v,a\rangle+\langle u'+_r v',b\rangle$$
$$=\langle u,a\rangle+\langle v,b\rangle+\langle u',b\rangle+\langle v',b\rangle=\langle w_1,c\rangle+\langle w_2,c\rangle\,.$$

The function c is additive and, consequently, linear. □

We have then the operation $P(m_l)$ in fibers of π_r. In the following, we shall use also $+_r$ instead of $P(m_l)$ to denote this operation. The multiplication $\cdot_r$ is defined by the formula

$$\langle r\cdot_l v,r\cdot_r a\rangle=r\langle v,a\rangle\,.$$

We show in the following proposition that with these operations K^{*_r} becomes a vector bundle over C^*.

Proposition 10. *On each fiber of π_r the operation $P(m_l)$ defines the structure of an Abelian group.*

Proof. Associativity. Let $a, b, c \in K^{*r}$ be such that $\pi_r(a) = \pi_r(b) = \pi_r(c)$ and let $u, v, w \in K$ be such that $\tau_l(u) = \tau_l(v) = \tau_l(w)$ and $\tau_r(u) = \pi_l(a)$, $\tau_r(v) = \pi_l(b)$, $\tau_r(w) = \pi_l(c)$. We have

$$\langle w +_l v +_l u, c +_r (b +_r a)\rangle = \langle w, c\rangle + \langle v +_l u, b +_r a\rangle = \langle w, c\rangle + \langle v, b\rangle + \langle u, a\rangle .$$

Commutativity. Obvious.

Neutral element. Let us choose an element $\alpha \in \mathbf{C}^*$. Since $\ker \tau_r$, with the vector bundle structure of $\mathbf{K}_r$, is identified with $\mathbf{F} \oplus_M \mathbf{C}$, we can define an element $e_\alpha \in K^{*r}$ by

$$\pi_l(e_\alpha) = \bar{\mathbf{0}}_l(\tau(\alpha)), \quad \langle (f, k), e_\alpha\rangle = \langle k, \alpha\rangle .$$

For each $a \in K^{*r}$ such that $\pi_r(a) = \alpha$ and for each $w \in K$ such that $\tau_r(w) = \pi_l(a)$, we have

$$\langle w, a +_r e_\alpha\rangle = \langle w +_l (f, 0), a +_r e_\alpha\rangle = \langle w, a\rangle + \langle (f, 0), e_\alpha\rangle = \langle w, a\rangle ,$$

where $f = \tau_l(w)$.

Inverse element. Is equal $(-1) \cdot_r a$. □

Local coordinates. Let $(x^i, e^a, f^A, c^\alpha)$ be an adapted coordinate system on K and let $(x^i, e^a, p_A, q_\alpha)$ be the adopted coordinate system on the dual bundle K^{*r}, i. e., the canonical evaluation is given by the formula

$$\langle v, a\rangle = \sum_A p_A(a) f^A(v) + \sum_\alpha q_\alpha(a) c^\alpha(v). \tag{14}$$

We use (x^i, c^α) as a coordinate system on C and (x^i, q_α) as a coordinate system on C^*. In these coordinate systems, we have

$$\begin{aligned} x_i \circ \pi_r(a) &= x_i(a) , \\ q_\alpha \circ \pi_r(a) &= q_\alpha(a) , \end{aligned} \tag{15}$$

and

$$\begin{aligned} x^i(a +_r b) &= x^i(a) = x^i(b) , \\ e^a(a +_r b) &= e^a(a) + e^a(b) , \\ p_A(a +_r b) &= p_A(a) + p_A(b) , \\ q_\alpha(a +_r b) &= q_\alpha(a) = q_\alpha(b) . \end{aligned} \tag{16}$$

It follows that (K^{*r}, π_r, C^*) is a vector bundle. We denote it by $\mathbf{K}_r^{*r}$. The vector bundle (K^{*r}, π_l, E) we denote by $\mathbf{K}_l^{*r}$.

Theorem 11. *The system* $\mathbf{K}^{*r} = (\mathbf{K}_r^{*r}, \mathbf{K}_l^{*r}, \mathbf{C}^*, \mathbf{E})$ *is a double vector bundle.*

Proof. We have to show that the projection, zero section and the operation of addition related to one vector bundle structure is linear with respect to another one.

Projections. Linearity of $\pi_r\colon \mathbf{K}_l^{*r} \to \mathbf{C}^*$ follows directly from its definition. Linearity of the projection $\pi_l\colon \mathbf{K}_r^{*r} \to \mathbf{E}$ is contained in the definition of $P(m_l)$.

Zero sections. Linearity of $\mathbf{0}_l\colon \mathbf{E} \to \mathbf{K}_r^{*r}$ follows directly from the definition of $P(m_l)$. Now, let

$$\mathbf{0}_r\colon \mathbf{C}^* \to \mathbf{K}_l^{*r} = (\mathbf{K}_r)^*$$

be the zero section. For $\alpha \in C^*$, $e_\alpha = \mathbf{0}_r(\alpha)$ is an element of the space dual to $\tau_r^{-1}(0)$, defined by

$$\pi_l(e_\alpha) = \bar{\mathbf{0}}_l(\tau(\alpha)), \quad \langle (f,k), e_\alpha\rangle = \langle k, \alpha\rangle\,.$$

Hence linearity.

Additions. First, we show that $+_l$ is a morphism of vector bundles

$$+_l\colon \mathbf{K}_r^{*r} \times_E \mathbf{K}_r^{*r} \to \mathbf{K}_r^{*r}\,. \tag{17}$$

Let $a, b, a', b' \in K^{*r}$ be such that

$$\pi_l(a) = \pi_l(b) = e,\ \pi_l(a') = \pi_l(b') = e',\ \pi_r(a) = \pi_r(a')\,,\ \text{ and } \pi_r(b) = \pi_r(b')\,.$$

Since $\pi_l\colon \mathbf{K}_r^{*r} \to \mathbf{E}$ is linear, we have

$$\pi_l(a +_r a') = \pi_l(b +_l b') = e + e'$$

and, consequently, $(a +_r a', b +_r b') \in \mathbf{K}_r^{*r} \times_E \mathbf{K}_r^{*r}$. The operation $+_l$ is additive with respect to $+_r$ and $+_r$ is additive with respect to $+_l$ if for each quadruple (a, b, a', b')

$$(a +_r a') +_l (b +_r b') = (a +_l b) +_r (a' +_l b')\,.$$

To show this equality let us take $w, v, v' \in K$ such that $\tau_r(v) = e$, $\tau_r(v) = e'$, $\tau_l(v) = \tau_l(v')$ and $w = v +_l v'$. We have

$$\begin{aligned}&\langle w, (a +_r a') +_l (b +_r b')\rangle = \langle w, a +_r a'\rangle + \langle w, b +_r b'\rangle\\ &= \langle v, a\rangle + \langle v', a'\rangle + \langle v, b\rangle + \langle v', b'\rangle = \langle v, a +_l b\rangle + \langle v', a' +_l b'\rangle\rangle = \langle w, (a +_l b) +_r (a' +_l b')\rangle\,.\end{aligned}$$

□

We identify the kernel $\ker \tau_r$ with $\mathbf{F} \oplus_M \mathbf{C}$ and, consequently, the kernel of π_l with $\mathbf{C}^* \oplus_M \mathbf{F}^*$. With this identifications, we have that

$$\langle (f, c), (c^*, f^*)\rangle = \langle f, f^*\rangle + \langle c, c^*\rangle \tag{18}$$

and that π_r is the canonical projection $\mathbf{C}^* \oplus \mathbf{F}^* \to \mathbf{C}^*$. It follows that the core of $\mathbf{K}^{*_r}$ can be identified with $\mathbf{F}^*$ and that

$$\ker \pi_r = \mathbf{E} \oplus \mathbf{F}^* \,.$$

Proposition 12. *Let $a = (e, f^*) \in \ker \pi_r$ and $b \in K^{*_r}$. Then, for $v, w \in K$ such that $\tau_r(v) = e$ and $w = (e, c) \in \ker \tau_l$, $e = \pi_l(b)$, we have*

$$\begin{aligned} \langle v, a\rangle =& \langle \tau_l(v), f^*\rangle \,, \\ \langle w, b\rangle =& \langle c, \pi_r(b)\rangle \,. \end{aligned}$$

Proof. Let $f = \tau_l(v)$. We have $(f, 0) \in F \times_M C = \ker \tau_r$, $v = (v -_l (f, 0)) +_l (f, 0)$, and $a = (e, f^*) = (e, 0) +_r (0, f^*)$. But, from the definition of $+_r$ in K^{*_r} and from (18), it follows that

$$\begin{aligned} \langle v, a\rangle = \langle (v -_l (f,0)) +_l (f,0), (e,0) +_r (0, f^*)\rangle \\ = \langle (v -_l (f,0)), (e,0)\rangle + \langle (f,0), (0,f^*)\rangle = \langle f, f^*\rangle \,. \end{aligned}$$

The proof of the second equality is analogous. □

Now, we can write the diagram for $\mathbf{K}^{*_r}$. If $\mathbf{K}$ is represented by the diagram

(19)
$$\begin{tikzcd} & K \arrow[dl, "\tau_l"'] \arrow[dr, "\tau_r"] & \\ F \arrow[dr, "\bar{\tau}_r"] & C \arrow[u, hook] \arrow[d, "\tau"'] & E \arrow[dl, "\bar{\tau}_l"'] \\ & M & \end{tikzcd}$$

then the right dual $\mathbf{K}^{*_r}$ is represented by the diagram

(20)
$$\begin{tikzcd} & K^{*_r} \arrow[dl, "\pi_l"'] \arrow[dr, "\pi_r"] & \\ E \arrow[dr, "\bar{\pi}_r"] & F^* \arrow[u, hook] \arrow[d, "\pi"'] & C^* \arrow[dl, "\bar{\pi}_r"] \\ & M & \end{tikzcd}$$

The left dual. A construction, similar to that of $\mathbf{K}^{*_r}$, can be applied to the left dual K^{*_l}. As the result, we obtain the structure of a double vector bundle on K^{*_l}. We denote it by $\mathbf{K}^{*_l} = (\mathbf{K}_r^{*_l}, \mathbf{K}_l^{*_l}, \mathbf{F}, \mathbf{C}^*)$. The core of $\mathbf{K}^{*_l}$ is $\mathbf{E}^*$. There is an obvious identity

$$\mathsf{J}(\mathbf{K}^{*_l}) = (\mathsf{J}(\mathbf{K}))^{*_r}\,. \tag{21}$$

Examples.

1. Let $\mathbf{K} = \mathbf{K}(\mathbf{F}, \mathbf{C}, \mathbf{E})$. The right dual can be canonically identified with $\mathbf{K}(\mathbf{E}, \mathbf{F}^*, \mathbf{C}^*)$ and the left dual with $\mathbf{K}(\mathbf{C}^*, \mathbf{E}^*, \mathbf{F})$.

2. If $\mathbf{E} = (E, \tau, M)$ is a vector bundle, then $\mathsf{T}\mathbf{E}$ is a double vector bundle with the diagram

$$\begin{tikzcd} & \mathsf{T}E \arrow[dl, "\mathsf{T}\tau"'] \arrow[d, hook'] \arrow[dr, "\tau_E"] & \\ \mathsf{T}M \arrow[dr, "\tau_M"] & E \arrow[d, "\tau"'] & E \arrow[dl, "\tau"'] \\ & M & \end{tikzcd} \tag{22}$$

Its right dual $(\mathsf{T}\mathbf{E})^{*_r}$ is represented by the diagram

$$\begin{tikzcd} & \mathsf{T}^*E \arrow[dl, "\pi_E"'] \arrow[from=d, hook'] \arrow[dr, "\mathsf{T}^*\tau"] & \\ E \arrow[dr, "\tau"'] & \mathsf{T}^*M \arrow[d, "\pi_M"'] & E^* \arrow[dl, "\pi"'] \\ & M & \end{tikzcd} \tag{23}$$

We see that the manifold of cotangent vectors to a vector bundle has two compatible vector bundle structures. The double vector bundle $\mathsf{J}((\mathsf{T}\mathbf{E})^{*_r})$, represented by the diagram

$$\begin{tikzcd} & \mathsf{T}^*E \arrow[dl, "\mathsf{T}^*\tau"'] \arrow[from=d, hook'] \arrow[dr, "\pi_E"] & \\ E^* \arrow[dr, "\pi"'] & \mathsf{T}^*M \arrow[d, "\pi_M"'] & E \arrow[dl, "\tau"'] \\ & M & \end{tikzcd} \tag{24}$$

will be denoted by $\mathsf{T}^*\mathbf{E}$. In particular, for $\mathbf{E} = \mathsf{T}M$, the diagram (24) assumes the form

$$
\begin{tikzcd}
 & \mathsf{T}^*\mathsf{T}M \arrow[dl, "\mathsf{T}^*\tau_M"'] \arrow[dr, "\pi_{\mathsf{T}M}"] & \\
\mathsf{T}^*M \arrow[dr, "\pi_M"'] & \mathsf{T}^*M \arrow[u, hook] \arrow[d, "\pi_M"'] & \mathsf{T}M \arrow[dl, "\tau_M"] \\
 & M &
\end{tikzcd}
\tag{25}
$$

3. In order to identify the structure of the left dual of $\mathsf{T}\mathbf{E}$ let us recall that the dual to the vector bundle $(\mathsf{T}E, \mathsf{T}\tau, \mathsf{T}M)$ can be canonically identified with $(\mathsf{T}E^*, \mathsf{T}\pi, \mathsf{T}M)$ ([5]). It follows that the left dual to $\mathsf{T}\mathbf{E}$ is canonically isomorphic to $\mathsf{J}(\mathsf{T}\mathbf{E}^*)$ with the diagram

$$
\begin{tikzcd}
 & \mathsf{T}E^* \arrow[dl, "\tau_{E^*}"'] \arrow[dr, "\mathsf{T}\pi"] & \\
E^* \arrow[dr, "\pi"] & E^* \arrow[u, hook] \arrow[d, "\pi"'] & \mathsf{T}M \arrow[dl, "\tau_M"'] \\
 & M &
\end{tikzcd}
\tag{26}
$$

5. Dual morphisms

Let $\mathbf{A} = (A, \tau, M)$ and $\mathbf{A}' = (A', \tau', M')$ be vector bundles and let $\varphi: A \to A'$ and $\overline{\varphi}: M \to M'$ be mappings such that the pair $\boldsymbol{\varphi} = (\varphi, \overline{\varphi})$ is a morphism of vector bundles. The dual morphism $\boldsymbol{\varphi}^*: \mathbf{A}' \to \mathbf{A}$ is not composed of mappings, but relations, unless it is an isomorphism.

In order to avoid the use of relations instead of mappings, we restrict further considerations to the case of isomorphism only. It follows from the formulae (10) that $\boldsymbol{\Phi}$ is an isomorphism of double vector bundles if and only if $\bar{\boldsymbol{\Phi}}_r$, $\bar{\boldsymbol{\Phi}}_l$ and $\bar{\boldsymbol{\Phi}}_c$ are isomorphisms of vector bundles.

Let $\boldsymbol{\Phi}: \mathbf{K} \to \mathbf{K}'$ be an isomorphism of double vector bundles, $\boldsymbol{\Phi} = (\Phi, \Phi_r, \Phi_l, \bar{\Phi})$. Consequently, $\boldsymbol{\Phi}_r = (\Phi, \Phi_r)$ is an isomorphism of vector bundles,

$$\boldsymbol{\Phi}_r: \mathbf{K}_r \to \mathbf{K}'_r\,.$$

The dual vector bundle morphism

$$(\boldsymbol{\Phi}_r)^*: (\mathbf{K}'_r)^* \to (\mathbf{K}_r)^*$$

is also an isomorphism. Let us denote by Φ^{*_r} the mapping of bundle spaces, $\Phi^{*_r}: K'^{*_r} \to K^{*_r}$. We have

$$(\mathbf{\Phi}_r)^* = (\Phi^{*_r}, \Phi_r^{-1}).$$

Proposition 13. *Φ^{*_r} defines an isomorphism of vector bundles $\mathbf{K'}_r^{*_r}$ and $\mathbf{K}_r^{*_r}$.*

Proof. First, we show that Φ^{*_r} respects fibrations π_r' and π_r. Let $a, b \in \mathbf{K'}^{*_r}$ be such that $\pi_r'(a) = \pi_r'(b)$. The kernel $\ker \tau_l'$ is, with the right bundle structure, isomorphic to $E' \times_M \mathbf{C}'$ and, with the left bundle structure, it is isomorphic to $\mathbf{E}' \oplus_M \mathbf{C}'$. The equality $\pi_r'(a) = \pi_r'(b)$ means that for $k' \in C$, we have

$$\langle (\pi_l'(a), k'), a\rangle = \langle (\pi_l'(b), k'), b\rangle \tag{27}$$

Since Φ is linear with respect to both, the right and left, vector bundle structures, we have

$$\Phi((e,k)) = \Phi((e,0)) + \Phi((0,k)) = \Phi_c(k)$$

for $(e,k) \in \ker \tau_l$. It follows that

$$\begin{aligned} \langle (\pi_l(\Phi^{*_r}(a)), k), \Phi^{*_r}(a)\rangle &= \langle \Phi(\bar{\Phi}^{-1}(a), k), a\rangle \\ = \langle (\bar{\Phi}^{-1}(a), \Phi_c(k)), a\rangle &= \langle (\bar{\Phi}^{-1}(b), \Phi_c(k)), b\rangle = \langle (\pi_l(\Phi^{*_r}(b)), k), \Phi^{*_r}(b)\rangle \end{aligned} \tag{28}$$

and

$$\pi_r(\Phi^{*_r}(a)) = \pi_r(\Phi^{*_r}(b)).$$

Linearity of Φ^{*_r} with respect to the right vector bundle structure follows from

$$\begin{aligned} &\langle v +_l w, \Phi^{*_r}(a +_r b)\rangle = \langle \Phi(v +_l w), a +_r b\rangle = \langle \Phi(v) +_l \Phi(w), a +_r b\rangle \\ &= \langle \Phi(v), a\rangle + \langle \Phi(w), b\rangle = \langle v, \Phi^{*_r}(a)\rangle + \langle w, \Phi^{*_r}(b)\rangle = \langle v +_l w, \Phi^{*_r}(a) +_r \Phi^{*_r}(b)\rangle. \end{aligned}$$

Corollary 14. *We have the following equalities for $\mathbf{\Psi} = \mathbf{\Phi}^{*_r}$*

$$\begin{aligned} \bar{\mathbf{\Psi}}_r &= (\bar{\mathbf{\Phi}}_c)^*, \\ \bar{\mathbf{\Psi}}_l &= (\bar{\mathbf{\Phi}}_r)^{-1}, \\ \bar{\mathbf{\Psi}}_c &= (\bar{\mathbf{\Phi}}_l)^*. \end{aligned} \tag{29}$$

Proof. $\ker \tau_r$ and $\ker \tau_r'$, with the right vector bundle structures, are isomorphic to $\mathbf{F} \oplus_M \mathbf{C}$ and $\mathbf{F}' \oplus_{M'} \mathbf{C}'$ respectively. The existence of these isomorphisms implies that the restriction of $\mathbf{\Phi}$ to $\ker \tau_r$ is identifiable with $\bar{\mathbf{\Phi}}_l \oplus \bar{\mathbf{\Phi}}_c$. It follows that, with these identifications, the dual mapping

$$\mathbf{\Phi}^{*_r}: (\mathbf{C}')^* \oplus_{M'} (\mathbf{F}')^* \to (\mathbf{C})^* \oplus_M (\mathbf{F})^*$$

is equal $(\bar{\mathbf{\Phi}}_c)^* \oplus (\bar{\mathbf{\Phi}}_l)^*$.

We have also that $\ker \pi_l$ and $\ker \pi_l'$, with the left vector bundle structure, are isomorphic to $(\mathbf{C})^* \oplus_M (\mathbf{F})^*$ and $(\mathbf{C}')^* \oplus_M (\mathbf{F}')^*$ respectively. With these isomorphisms the restriction of $\mathbf{\Phi}^{*_r}$ to $\ker \pi_l'$ is identifiable with $\bar{\mathbf{\Psi}}_r \oplus \bar{\mathbf{\Psi}}_c$. It follows that $\bar{\mathbf{\Psi}}_r = (\bar{\mathbf{\Phi}}_c)^*$, $\bar{\mathbf{\Psi}}_c = (\bar{\mathbf{\Phi}}_l)^*$. The third equality in (29) is obvious. □

Examples.

1. Let $\mathbf{K} = \mathbf{K}(\mathbf{F}, \mathbf{C}, \mathbf{E})$, $\mathbf{K}' = \mathbf{K}(\mathbf{F}', \mathbf{C}', \mathbf{E}')$, and let $\mathbf{\Phi}\colon \mathbf{K} \to \mathbf{K}'$, with

$$\Phi(f, c, e) = (\Phi_l(f), \Phi_c(c) + \Psi(f, e), \Phi_r(e)) . \tag{30}$$

Since we identify $\mathbf{K}^{*_r}$ with $\mathbf{K}(\mathbf{E}, \mathbf{F}^*, \mathbf{C}^*)$ and $(\mathbf{K}')^{*_r}$ with $\mathbf{K}(\mathbf{E}', (\mathbf{F}')^*, (\mathbf{C}')^*)$, then the morphism $\mathbf{\Phi}^{*_r}$ is identified as a morphism

$$\mathbf{\Phi}^{*_r}\colon \mathbf{K}(\mathbf{E}', (\mathbf{F}')^*, (\mathbf{C}')^*) \to \mathbf{K}(\mathbf{E}, \mathbf{F}^*, \mathbf{C}^*) .$$

One can easily verify the equality

$$\Phi^{*_r} = (\Phi_l^{*_r}, \Phi_c^{*_r} + \Psi^{*_r}, \Phi_r^{*_r}) ,$$

where

$$\begin{aligned} \Phi_r^{*_r}(q') &= \Phi_c^*(q') , \\ \Phi_l^{*_r}(e') &= \Phi_r^{-1}(e') , \\ \Phi_c^{*_r}(p') &= \Phi_l^*(p') , \\ \Psi^{*_r}(e', q') &= \Psi^*(q', \Phi_r^{-1}(e')) , \end{aligned} \tag{31}$$

and Ψ^* is the vector bundle mapping dual to Ψ with respect to the left argument, i.e., with respect to $f \in F$.

2.([5]) We have (Section 3) an isomorphism of double vector bundles

$$\kappa_M\colon \mathsf{TT}M \to \mathsf{J}(\mathsf{TT}M) .$$

The right dual

$$(\kappa_M)^{*_r}\colon \mathsf{TT}^*M \to \mathsf{T}^*\mathsf{T}M$$

is usually denoted by α_M and plays a crucial role in the Lagrangian formulation of the dynamics of mechanical systems.

6. Canonical isomorphisms

Proposition 15. *The three double vector bundles*

$$(\mathbf{K}^{*_r})^{*_l},\ (\mathbf{K}^{*_l})^{*_r},\ \textit{and}\ \ \mathbf{K} \tag{32}$$

are canonically isomorphic.

Proof. It follows from the construction in Section 4 that we can identify manifolds $(K^{*_r})^{*_l}$ and K. Also the right vector bundle structures coincide. Let $\Phi\colon K \to (K^{*_r})^{*_l}$ be the canonical diffeomorphism and let ϑ_r, ϑ_l be projections in $(\mathbf{K}^{*_r})^{*_l}$. For $f^* \in F^* = \ker \pi_r \cap \ker \pi_l$, we have

$$\langle f^*, \vartheta_l(\Phi(v))\rangle = \langle (\tau_r(v), f^*), \Phi(v)\rangle = \langle v, (e, f^*)\rangle = \langle \tau_l(v), f^*\rangle ,$$

hence $\vartheta_l(\Phi(v)) = \tau_l(v)$. Equality of the left vector bundle structures follows from

$$\langle a +_r b, \Phi(v) +_l \Phi(w)\rangle = \langle a, \Phi(v)\rangle + \langle b, \Phi(w)\rangle = \langle v, a\rangle + \langle w, b\rangle = \langle v +_l w, a +_r b\rangle\,.$$

□

In the following, we show that there is a canonical isomorphism of double vector bundles $\mathbf{K}$ and $((\mathbf{K}^{*_r})^{*_r})^{*_r}$. Through this section we denote by $(\tau_r, \tau_l), (\pi_r, \pi_l)$, (ξ_r, ξ_l), and $(\vartheta_r, \vartheta_l)$ the right and left projections in $\mathbf{K}$, $\mathbf{K}^{*_r}$, $(\mathbf{K}^{*_r})^{*_r}$, and $((\mathbf{K}^{*_r})^{*_r})^{*_r}$ respectively. Identifying vector bundles with their second duals, we have

$$\begin{aligned} \xi_r &: (K^{*_r})^{*_r} \to F & \xi_l &: (K^{*_r})^{*_r} \to C^*\,, \\ \vartheta_r &: ((K^{*_r})^{*_r})^{*_r} \to E & \vartheta_l &: ((K^{*_r})^{*_r})^{*_r} \to F\,. \end{aligned}$$

The core of $(\mathbf{K}^{*_r})^{*_r}$ is $\mathbf{E}^*$ and the core of $((\mathbf{K}^{*_r})^{*_r})^{*_r}$ is $(\mathbf{C}^*)^* = \mathbf{C}$.

We define a relation $\mathcal{R}_K \subset K \times ((K^{*_r})^{*_r})^{*_r}$ in the following way:

Let $v \in K$, $\varphi \in ((K^{*_r})^{*_r})^{*_r}$ be such that $\bar{\tau}_l(\tau_r(v)) = \bar{\vartheta}_l(\vartheta_r(\varphi))$. We say that $(v, \varphi) \in \mathcal{R}_K$ if for each $a \in K^{*_r}$ and $\alpha \in (K^{*_r})^{*_r}$ such that

$$\tau_r(v) = \pi_l(a),\ \pi_r(a) = \xi_l(\alpha),\ \xi_r(\alpha) = \vartheta_l(\varphi)\,,$$

we have

$$\langle a, \alpha\rangle = \langle v, a\rangle + \langle \alpha, \varphi\rangle\,. \tag{33}$$

Theorem 16. *The relation $\mathcal{R}_K$ is a mapping which defines an isomorphism of double vector bundles.*

Proof. Let $(v, \varphi) \in \mathcal{R}_K$, $e = \tau_r(v)$, and $f = \vartheta_l(\varphi)$. We show first that $\tau_r(v) = \vartheta_r(\varphi)$ and $\tau_l(v) = \vartheta_l(\varphi)$. For $a = (e, f^*) \in \ker \pi_r$ and $\alpha = (f, e^*) \in \ker \xi_l$ equality (33) assumes the form

$$\langle a, \alpha\rangle = \langle f, f^*\rangle + \langle e, e^*\rangle = \langle v, a\rangle + \langle \alpha, \varphi\rangle = \langle \tau_l(v), f^*\rangle + \langle \vartheta_r(\varphi), e^*\rangle\,,$$

in view of (18) and Proposition 12. It follows that

$$e = \vartheta_r(\varphi), \quad f = \tau_l(v)\,,$$

since e^* and f^* are arbitrary.

In the next step we show that $\mathcal{R}_K$ restricted to either right or left fiber is a linear relation. Let $(v, \varphi), (v', \varphi') \in \mathcal{R}_K$ be such that

$$\tau_r(v) = \tau_r(v') \text{ and } \vartheta_r(\varphi) = \vartheta_r(\varphi')\,.$$

Let a, α be such that $\pi_l(a) = \tau_r(v)$, $\pi_r(a) = \xi_l(\alpha)$ and $\xi_r(\alpha) = \vartheta_l(\varphi +_r \varphi')$. We can find $\beta, \beta' \in K^{*_r *_r}$ such that $\xi_r(\beta) = \vartheta_l(\varphi)$, $\xi_r(\beta') = \vartheta_l(\varphi')$, and $\alpha = \beta +_l \beta'$. We have then

$$\langle v, a\rangle + \langle v', a\rangle = \langle v +_r v', a\rangle$$

and, by the definition of the right vector bundle structure on $(K^{*_r *_r})^{*_r}$,

$$\langle \beta +_l \beta', \varphi +_r \varphi'\rangle = \langle \beta, \varphi\rangle + \langle \beta', \varphi'\rangle .$$

It follows that

$$\begin{aligned}(34)\quad \langle a, \alpha\rangle &= \langle a, \beta +_l \beta'\rangle = \langle a, \beta\rangle + \langle a, \beta'\rangle \\ &= \langle v, a\rangle + \langle \beta, \varphi\rangle + \langle v', a\rangle + \langle \beta', \varphi'\rangle = \langle v +_r v', a\rangle + \langle \alpha, \varphi +_r \varphi'\rangle ,\end{aligned}$$

and, consequently, $(v +_r v', \varphi +_r \varphi') \in \mathcal{R}_K$. Similar arguments show that $\mathcal{R}_K$ is invariant with respect to the left addition and also with respect to the right and left multiplications by a number. The dimensions of K and $((K^{*_r})^{*_r})^{*_r}$ are equal $n + n_E + n_F + n_C$. Thus it remains to show that the kernel and cokernel of $\mathcal{R}_K$ are trivial, and that the domain of $\mathcal{R}_K$ is the whole K. It will prove that $\mathcal{R}_K$ is an injective mapping and, consequently, an isomorphism of double vector bundles.

Let $(v, \varphi) \in \mathcal{R}_K$ and $(v', \varphi) \in \mathcal{R}_K$. Since $\tau_r(v) = \vartheta_r(\varphi) = \tau_r(v')$ and $\tau_l(v) = \vartheta_l(\varphi) = \tau_l(v')$, we have

$$v -_r v' \in \ker \tau_l \quad \text{and} \quad v -_l v' \in \ker \tau_r .$$

Since we identify $\ker \tau_l$ and $\ker \vartheta_l$ with $E \times_M C$, we can write

$$v -_r v' = (e, k) \quad \text{and} \quad \varphi -_r \varphi = (e, 0) .$$

Let $a \in K^{*_r}$ and $\alpha \in K^{*_r *_r}$ be such that

$$\begin{aligned}\pi_l(a) &= \tau_r(v -_r v') = e , \\ \pi_r(a) &= \xi_l(\alpha) , \\ \xi_r(\alpha) &= \vartheta_l(\varphi -_r \varphi) .\end{aligned}$$

It follows that $\alpha \in \ker \xi_r = C^* \times_M E^*$. Let $\alpha = (k^*, e^*)$ in this representation. We have from Proposition 12 that

$$\begin{aligned}\langle a, \alpha\rangle &= \langle e, e^*\rangle , \\ \langle v -_r v', a\rangle &= \langle k, \pi_r(a)\rangle = \langle k, \xi_l(\alpha)\rangle = \langle k, k^*\rangle , \\ \langle \alpha, \varphi -_r \varphi\rangle &= \langle \vartheta_r(\varphi -_r \varphi), e^*\rangle = \langle e, e^*\rangle .\end{aligned}$$

It follows that the equality

$$\langle a, \alpha\rangle = \langle v -_r v', a\rangle + \langle \alpha, \varphi -_r \varphi\rangle$$

assumes the form

$$\langle e, e^* \rangle = \langle k, k^* \rangle + \langle e, e^* \rangle .$$

Hence $\langle k, k^* \rangle = 0$ for each k^* and, consequently, $k = 0$. The kernel of $\mathcal{R}_K$ is trivial. Similarly, we prove that the cokernel is trivial.

Now, let $v \in K$ and let $\alpha \in K^{*_r *_r}$ with $\xi_r(\alpha) = \tau_l(v)$. We have to show that the formula

$$\langle \alpha, \varphi \rangle = \langle a, \alpha \rangle - \langle v, a \rangle ,$$

where $\pi_l(a) = \tau_r(v)$ and $\pi_r(a) = \xi_l(\alpha)$, defines an element $\varphi \in K^{*_r *_r *_r}$. It is enough to prove that the right hand side of this formula does not depend on the choice of a.

Let a, a' be such that $\pi_l(a) = \pi_l(a') = \tau_r(v)$ and $\pi_r(a) = \pi_r(a') = \xi_l(\alpha)$. Then

$$a -_r a' \in \ker \pi_l = C^* \times_M F^*$$

and

$$a -_l a' \in \ker \pi_r = E \times_M F^*.$$

Let $a -_r a' = (k^*, f^*)$ and $a -_l a' = (e, f'^*)$. The linearity of the left vector bundle structure on K^{*_r}, with respect to the right vector bundle structure implies that

$$\begin{aligned}(0, f^*) = (k^*, f^*) -_l (k^*, 0) &= (a -_r a') -_l (a' -_r a') \\ &= (a -_l a') -_r (a' -_l a') = (e, f'^*) -_r (e, 0) = (0, f'^*)\end{aligned}$$

and, consequently, $f^* = f'^*$. We get from Proposition 12

$$\begin{aligned}&\langle a, \alpha \rangle - \langle v, \alpha \rangle - (\langle a', \alpha \rangle - \langle v, \alpha' \rangle) \\ &= \langle a -_r a', \alpha \rangle - \langle v, a -_l a' \rangle = \langle \xi_r(\alpha), f^* \rangle - \langle \tau_l(v), f^* \rangle = \langle \tau_l(v), f^* \rangle - \langle \tau_l(v), f^* \rangle = 0 .\end{aligned}$$

We conclude that φ is well defined and, consequently, $(v, \varphi) \in \mathcal{R}_K$.

If we replace the right-hand side of (33) by a different combination of $\langle v, a \rangle$ and $\langle \alpha, \varphi \rangle$, we obtain another isomorphism. The isomorphism corresponding to $\langle v, a \rangle - \langle \alpha, \varphi \rangle$ will be denoted by $\mathcal{R}_K^{\pm}$, the isomorphism corresponding to $-\langle v, a \rangle + \langle \alpha, \varphi \rangle$ will be denoted by $\mathcal{R}_K^{\mp}$, and the isomorphism corresponding to $-\langle v, a \rangle - \langle \alpha, \varphi \rangle$ will be denoted by $\mathcal{R}_K^{=}$.

Proposition 17.

(1) $\varphi = \mathcal{R}_K^{\pm}(v)$ *if and only if* $\varphi = \mathcal{R}_K((-1) \cdot_l v)$ *or equivalently,* $(-1) \cdot_l \varphi = \mathcal{R}_K(v)$,

(2) $\varphi = \mathcal{R}_K^{\mp}(v)$ *if and only if* $\varphi = \mathcal{R}_K((-1) \cdot_r v)$ *or equivalently,* $(-1) \cdot_r \varphi = \mathcal{R}_K(v)$,

(3) $\varphi = \mathcal{R}_K^{=}(v)$ *if and only if* $(-1) \cdot_l \varphi = \mathcal{R}_K((-1) \cdot_r v)$ *or equivalently,* $(-1) \cdot_l ((-1) \cdot_r \varphi) = \mathcal{R}_K(v)$.

Proof. The proof is an immediate consequence of the equalities

$$-\langle v, a\rangle = \langle (-1)\cdot_r v, a\rangle = \langle v, (-1)\cdot_l a\rangle\,, \tag{35}$$

$$-\langle \alpha, \varphi\rangle = \langle (-1)\cdot_r \alpha, \varphi\rangle = \langle \alpha, (-1)\cdot_l \varphi\rangle\,, \tag{36}$$

and the fact that $\mathcal{R}_K$ defines an isomorphism of double vector bundles. □

In an analogous way we introduce isomorphisms $\mathcal{L}_K, \mathcal{L}_K^{\pm}, \mathcal{L}_K^{\mp}, \mathcal{L}_K^{=}$ of $\mathbf{K}^{*_l *_l *_l}$ and K.

Examples.

1. Let $\mathbf{K} = \mathbf{K}(\mathbf{F}, \mathbf{C}, \mathbf{E})$. Using the identification $\mathbf{K}^{*_r} = \mathbf{K}(\mathbf{E}, \mathbf{F}^*, \mathbf{C}^*)$ and also the identifications $\mathbf{E}^{**} = \mathbf{E}$, $\mathbf{F}^{**} = \mathbf{F}$, and $\mathbf{C}^{**} = \mathbf{C}$, we get

$$\mathbf{K}^{*_r *_r} = \mathbf{K}(\mathbf{C}^*, \mathbf{E}^*, \mathbf{F})$$

and

$$\mathbf{K}^{*_r *_r *_r} = \mathbf{K}(\mathbf{F}, \mathbf{C}, \mathbf{E})\,.$$

Thus, we have obtained another identification of $\mathbf{K}$ and $\mathbf{K}^{*_r *_r *_r}$. With this identification the formula (33) assumes the form

$$\begin{aligned} \langle (e,\varphi,\gamma), (\gamma,\varepsilon,\bar f)\rangle &= \langle (f,c,e),(e,\varphi,\gamma)\rangle + \langle (\gamma,\varepsilon,\bar f),(\bar f,\bar c,\bar e)\rangle\,, \\ \langle e,\varepsilon\rangle + \langle \bar f,\varphi\rangle &= \langle f,\varphi\rangle + \langle c,\gamma\rangle + \langle \bar c,\gamma\rangle + \langle \bar e,\varepsilon\rangle\,, \end{aligned} \tag{37}$$

where $e, \bar e \in E, f, \bar f \in F, c, \bar c \in C, \varepsilon \in E^*, \varphi \in F^*, \psi \in C^*$. Hence, $e = \bar e, f = \bar f, c = -\bar c$, and, consequently,

$$\mathcal{R}_K(f,c,e) = (f,-c,e)\,. \tag{38}$$

Analogously,

$$\begin{aligned} \mathcal{R}_K^{\pm}(f,c,e) &= (f,c,-e)\,, \\ \mathcal{R}_K^{\mp}(f,c,e) &= (-f,c,e)\,, \\ \mathcal{R}_K^{=}(f,c,e) &= (-f,-c,-e)\,. \end{aligned} \tag{39}$$

2. Let $\mathbf{K}$ be the double vector bundle $\mathsf{T}^*\mathbf{E}$ represented by the diagram

(40)
$$\begin{tikzcd}
 & \mathsf{T}^*E \arrow[dl, "\mathsf{T}^*\tau"'] \arrow[dr, "\pi_E"] & \\
E^* \arrow[dr, "\pi"'] & \mathsf{T}^*M \arrow[u, hook] \arrow[d, "\pi_M"] & E \arrow[dl, "\tau"] \\
 & M &
\end{tikzcd}$$

Then the first, second, and third right duals can be identified with double vector bundles $\mathsf{J}(\mathsf{T}E)$, $\mathsf{T}E^*$ and $\mathsf{J}(\mathsf{T}^*E)$, represented by diagrams

(41)
$$
\begin{tikzcd}
 & \mathsf{T}E \arrow[dl, "\tau_E"'] \arrow[dr, "\mathsf{T}\tau"] & \\
E \arrow[dr, "\tau"'] & E \arrow[u, hook] \arrow[d, "\tau"] & \mathsf{T}M \arrow[dl, "\tau_M"'] \\
 & M &
\end{tikzcd}
\quad
\begin{tikzcd}
 & \mathsf{T}E^* \arrow[dl, "\mathsf{T}\pi"'] \arrow[dr, "\tau_{E^*}"] & \\
\mathsf{T}M \arrow[dr, "\tau_M"'] & E^* \arrow[u, hook] \arrow[d, "\pi"] & E^* \arrow[dl, "\pi"'] \\
 & M &
\end{tikzcd}
\quad
\begin{tikzcd}
 & \mathsf{T}^*E^* \arrow[dl, "\pi_{E^*}"'] \arrow[dr, "\mathsf{T}^*\pi"] & \\
E^* \arrow[dr, "\pi"'] & \mathsf{T}^*M \arrow[u, hook] \arrow[d, "\pi_M"] & E \arrow[dl, "\tau"'] \\
 & M &
\end{tikzcd}
$$

Canonical isomorphisms $\mathcal{R}_K$, $\mathcal{R}_K^{\pm}$, $\mathcal{R}_K^{\mp}$, $\mathcal{R}_K^{=}$ define diffeomorphisms from T^*E^* to T^*E. These diffeomorphisms are antisymplectomorphisms with respect to the canonical symplectic structure of the cotangent bundle for $\mathcal{R}_K$, $\mathcal{R}_K^{=}$ and symplectomorphisms for $\mathcal{R}_K^{\pm}$, $\mathcal{R}_K^{\mp}$.

Identification of isomorphisms. Let $\mathbf{\Phi}\colon \mathbf{K} \to \mathbf{K}'$ be an isomorphism of double vector bundles. We have

$$\mathbf{\Phi}^{*_r *_r *_r}\colon (\mathbf{K}')^{*_r *_r *_r} \to \mathbf{K}^{*_r *_r *_r}.$$

Using one of the introduced isomorphisms of a double vector bundle and its third right dual, we can compare $\mathbf{\Phi}$ and its third right dual.

Proposition 18. *We have the following equality*

$$\mathcal{R}_K^{-1} \circ \Phi^{*_r *_r *_r} \circ \mathcal{R}_{K'} = \Phi^{-1}. \tag{42}$$

Proof. Let $\varphi \in K^{*_r *_r *_r}$, $\varphi' \in (K')^{*_r *_r *_r}$, $v \in K$, and $v' \in K'$ be such that

$$\varphi = \Phi^{*_r *_r *_r}(\varphi'), \quad \varphi = \mathcal{R}_K(v), \quad \varphi' = \mathcal{R}_{K'}(v')\,. \tag{43}$$

Then, as in (33), we have

$$\langle a', \alpha' \rangle = \langle v', a' \rangle + \langle \alpha', v \rangle$$

and, for $a = \Phi^{*_r}(a')$, $\alpha = (\Phi^{*_r *_r})^{-1}(\alpha')$, we have

$$\langle (\Phi^{*_r})^{-1}(a), \Phi^{*_r *_r}(\alpha) \rangle = \langle v', (\Phi^{*_r})^{-1}(a) \rangle + \langle \Phi^{*_r *_r}(\alpha), \varphi' \rangle\,, \tag{44}$$

$$\langle \Phi^{*_r *_r *_l} \circ (\Phi^{*_r})^{-1}(a), \alpha \rangle = \langle ((\Phi^{*_r})^{-1})^{*_l}(v'), a \rangle + \langle \alpha, \Phi^{*_r *_r *_r}(\varphi') \rangle\,. \tag{45}$$

From the formula (33) and from (45), we derive the following identities

$$\Phi^{*_r*_r*_l}\circ(\Phi^{*_r})^{-1}(a)=a,$$
$$((\Phi^{*_r})^{-1})^{*_l}(v')=\Phi^{-1}(v'),$$
$$\Phi^{*_r*_r*_r}(\varphi')=\varphi$$

which make the formula (45) equivalent to

$$\langle a,\alpha\rangle=\langle\Phi^{-1}(v'),a\rangle+\langle\alpha,\varphi\rangle.$$

It follows that

$$\mathcal{R}_K(\Phi^{-1}(v'))=\varphi=\mathcal{R}_K(v)\,.$$

Consequently,

$$\Phi^{-1}(v')=v\,. \qquad \square$$

Equalities similar to (42) hold for pairs of relations $(\mathcal{R}^{\mp}_K,\mathcal{R}^{\mp}_{K'})$, $(\mathcal{R}^{\pm}_K,\mathcal{R}^{\pm}_{K'})$, and $(\mathcal{R}^{=}_{\overline{K}},\mathcal{R}^{=}_{\overline{K}'})$ in place of $(\mathcal{R}_K,\mathcal{R}_{K'})$.

Remark. In the case of $\mathbf{K}=\mathbf{K}(\mathbf{F},\mathbf{C},\mathbf{E})$ and $\mathbf{K}'=\mathbf{K}(\mathbf{F}',\mathbf{C}',\mathbf{E}')$ we have another isomorphisms of $\mathbf{K}$ and $\mathbf{K}^{*_r*_r*_r}$, $\mathbf{K}'$ and $(\mathbf{K}')^{*_r*_r*_r}$ (see Example 1 of this section). In contrast to (42), $\Phi^{*_r*_r*_r}$ does not correspond, with respect to these isomorphisms, to Φ^{-1}.

7. Examples and applications

7.1. Vector and co-vector fields on a vector bundle. Let $\mathbf{E}=(E,\tau,M)$ be a vector bundle and let X be a vector field on E. By $\widetilde{X}$ we denote the corresponding function on T^*E.

Theorem 19. *The following three conditions are equivalent.*

(1) *For each function f on E, linear on fibers, the function $\langle X,\mathrm{d}f\rangle$ is linear on fibers.*
(2) *The mapping $X\colon E\to\mathsf{T}E$ is a vector bundle morphism from $\mathbf{E}$ to $(\mathsf{T}E,\mathsf{T}\tau,\mathsf{T}M)$.*
(3) *The function $\widetilde{X}$ is linear with respect to the right and left vector bundle structures on T^*E.*

Proof. Let f be a linear function on E and let $v,v'\in\mathsf{T}E$ be such that $\mathsf{T}\tau(v)=\mathsf{T}\tau(v')$. We can choose curves γ,γ' on E which represent v,v', respectively, and satisfy the following condition: $\tau\circ\gamma=\tau\circ\gamma'$. The curve $\gamma+\gamma'\colon t\mapsto\gamma(t)+\gamma'(t)$ represents the vector $v+_l v'$. Since f is linear, we have $f\circ(\gamma+\gamma')(t)=f\circ\gamma(t)+f\circ\gamma'(t)$ and, consequently,

$$\langle v+_l v',\mathrm{d}f(e+e')\rangle=\langle v,\mathrm{d}f(e)\rangle+\langle v',\mathrm{d}f(e')\rangle=\langle v+_l v',\mathrm{d}f(e)+_l\mathrm{d}f(e')\rangle\,,$$

i.e.,

$$\mathrm{d}f(e+e') = \mathrm{d}f(e) +_l \mathrm{d}f(e')\,. \tag{46}$$

(1 $\Rightarrow$ 2) First, we have to show that, if $\tau(e) = \tau(e')$, then $\mathsf{T}\tau(X(e)) = \mathsf{T}\tau(X(e'))$. It is enough to show that for each function g on E, which is constant on fibers,

$$\langle X(e), \mathrm{d}g(e)\rangle = \langle X(e'), \mathrm{d}g(e')\rangle\,. \tag{47}$$

Let f be a linear function on E and and let g be a function on E, constant on fibers. The function fg is linear on fibers, hence $X(fg)$ is also linear. Since

$$X(fg) = fX(g) + gX(f)$$

and $gX(f)$ is linear on fibers, it follows that $fX(g)$ is linear and, consequently, $X(g)$ is constant on fibers and

$$X(g)(e) = X(g)(e')\,.$$

Since

$$X(g)(e) = \langle X(e), \mathrm{d}g(e)\rangle\,,$$

we get (47).

Now, we show that X is linear on fibers. Let μ be a covector from $\mathsf{T}^*_{e+e'}E$, where $e+e' \neq 0$. There exists a function F_μ on E which is linear on fibers and such that $\mathrm{d}F_\mu(e+e') = \mu$.

We have from (46)

$$\begin{aligned}\langle X(e+e'), \mathrm{d}F_\mu(e+e')\rangle &= X(F_\mu)(e+e') = X(F_\mu)(e) + X(F_\mu)(e') \\ &= \langle X(e), \mathrm{d}F_\mu(e)\rangle + \langle X(e'), \mathrm{d}F_\mu(e')\rangle = \langle X(e) +_l X(e'), \mathrm{d}F_\mu(e+e')\rangle\end{aligned}$$

The above calculation gives, for every μ,

$$\langle X(e+e'), \mu\rangle = \langle X(e) +_l X(e'), \mu\rangle$$

and, consequently $X(e+e') = X(e) + X(e')$ for $e+e' \neq 0$. By the continuity argument, we get the desired equality for all e, e'.

Similar arguments show that $\langle X(\alpha e) = \alpha \cdot_l X(e)$.

(2 $\Rightarrow$ 3) Let $\tau(e) = \tau(e')$. Since X is a vector bundle morphism, we have

$$\tau(e) = \tau(e') \Rightarrow \mathsf{T}\tau(X(e)) = \mathsf{T}\tau(X(e'))$$

and

$$\begin{aligned}&X(e+e') = X(e) +_l X(e'),\\ &X(\alpha e) = \alpha \cdot_l X(e).\end{aligned}$$

Let μ and ν be two covectors on E, $\mu \in \mathsf{T}^*_e E$ and $\nu \in \mathsf{T}^*_{e'} E$, such that $\mathsf{T}^*\tau(\mu) = \mathsf{T}^*\tau(\nu)$. From the definition of the vector bundle structure on $(\mathsf{T}^*E, \mathsf{T}^*\tau, E^*)$, we have

$$\langle X(e+e'), \mu +_l \nu\rangle = \langle X(e) +_l X(e'), \mu +_l \nu\rangle = \langle X(e), \mu\rangle + \langle X(e'), \nu\rangle$$

and

$$\langle X(\alpha p), \alpha \cdot_l \mu\rangle = \langle \alpha \cdot_l X(e), \alpha \cdot_l \mu\rangle = \alpha\langle X(e), \mu\rangle$$

The above calculation shows that X, treated as a function on T^*E, is linear with respect to the vector bundle structure $(\mathsf{T}^*E, \mathsf{T}^*\tau, E^*)$. It is obviously linear with respect to the canonical vector bundle structure on $(\mathsf{T}^*E, \pi_E, E)$.

$(3 \Rightarrow 1)$ It follows from (46) that for a linear function f

$$\begin{aligned}&\langle X(e+e'), \mathrm{d}f(e+e')\rangle \\ &\qquad = \langle X(e) +_l X(e'), \mathrm{d}f(e) +_l \mathrm{d}f(e')\rangle = \langle X(e), \mathrm{d}f(e)\rangle + \langle X(e'), \mathrm{d}f(e')\rangle\,.\end{aligned}$$

□

We say that a vector field X is *of degree zero* if one of the conditions of this theorem is satisfied.

Let (X^i, e^a) be an adapted coordinate system on E. A vector field X on E is of degree zero if, in local coordinates,

$$X = X^i \frac{\partial}{\partial x^i} + X^b_a e^a \frac{\partial}{\partial e^b}\,, \tag{48}$$

where X^i, X^i_a are functions of (x^i) only.

Now, let θ be a 1-form on E and let $\widetilde{\theta}$ be the corresponding function on $\mathsf{T}E$.

Theorem 20. *The following three conditions are equivalent.*

(1) *For each vector field X of degree zero, the function $\langle X, \theta\rangle$ is linear on fibers.*
(2) *The mapping $\theta\colon E \to \mathsf{T}^*E$ is a vector bundle morphism from* $\mathbf{E}$ *to $(\mathsf{T}^*E, \mathsf{T}^*\tau, E^*)$.*
(3) *The function $\widetilde{\theta}$ is linear with respect to the right and left vector bundle structures on $\mathsf{T}E$.*

Proof.

$(1 \Rightarrow 2)$ Let X be a vertical vector field on E, constant on fibers. For every linear function f on E the vector field fX is polynomial of degree 0 (see (48)). The function $\langle fX, \theta\rangle$ is then linear on E and, since

$$\langle fX, \theta\rangle = f\langle X, \theta\rangle,$$

the function $\langle X, \theta\rangle(\cdot)$ is constant on fibers. It follows that $\mathsf{T}^*\tau(\theta(e))$ and $\mathsf{T}^*\tau(\theta(e'))$ are equal if $\tau(e) = \tau(e')$ and, consequently, that θ is a fiber preserving mapping from $\mathbf{E}$ to $(\mathsf{T}^*E, \mathsf{T}^*\tau, E^*)$. In order to prove the linearity of θ on fibers, i.e., that

$$\theta(e) +_l \theta(e') = \theta(e + e')$$

and

$$\alpha \cdot_l \theta(e) = \theta(\alpha e)\,,$$

it is enough to prove the first equality for e, e' such that $e + e' \neq 0$.

Let $e + e' \neq 0$. For every vector $v \in \mathsf{T}_{e+e'}E$ there exists a vector field X of degree 0, such that $X(e + e') = v$. We have then

$$\begin{aligned}\langle v, \theta(e + e')\rangle &= \langle X(e + e'), \theta(e + e')\rangle \\ &= \langle X(e), \theta(e)\rangle + \langle X(e'), \theta(e')\rangle = \langle X(e) +_l X(e'), \theta(e) +_l \theta(e')\rangle \\ &= \langle X(e + e'), \theta(e) +_l \theta(e')\rangle = \langle v, \theta(e) +_l \theta(e')\rangle\,.\end{aligned}$$

Since it holds for every v in $\mathsf{T}_{e+e'}E$, we have

$$\theta(e + e') = \theta(e) +_l \theta(e')\,.$$

$(2 \Rightarrow 3)$ Let $v \in \mathsf{T}_eE$, $w \in \mathsf{T}_{e'}E$, $\tau(e) = \tau(e')$ and $\mathsf{T}\tau(v) = \mathsf{T}\tau(w)$. We have

$$\begin{aligned}\widetilde{\theta}(v +_l w) &= \langle v +_l w, \theta(e + e')\rangle \\ &= \langle v +_l w, \theta(e) +_l \theta(e')\rangle = \langle v, \theta(e)\rangle + \langle w, \theta(e')\rangle = \widetilde{\theta}(v) + \widetilde{\theta}(w)\end{aligned}$$

and

$$\langle \alpha \cdot_l v, \theta(\alpha p)\rangle = \langle \alpha \cdot_l v, \alpha \cdot_l \theta(e)\rangle = \alpha\langle v, \theta(e)\rangle\,,$$

i.e., θ is a linear function on $\mathsf{T}E$ with respect to the tangent vector bundle structure on $\mathsf{T}E$. Linearity with respect to the canonical vector bundle structure on $(\mathsf{T}E, \tau_E, E)$ is obvious.

$(3 \Rightarrow 1)$ Let X be a vector field on E of degree 0. It follows from Theorem 19 that

$$\begin{aligned}\langle X, \theta\rangle(e{+}e') &= \langle X(e{+}e'), \theta(e{+}e')\rangle = \langle X(e){+_l}X(e'), \theta(e{+}e')\rangle = \widetilde{\theta}(X(e){+_l}X(e')) \\ &= \widetilde{\theta(X(e)}) + \widetilde{\theta(X(e')}) = \langle X(e), \theta(e)\rangle + \langle X(e'), \theta(e')\rangle = \langle X, \theta\rangle(e) + \langle X, \theta\rangle(e')\end{aligned}$$

Similarly,

$$\begin{aligned}\langle X, \theta\rangle(\alpha e) = \langle X(\alpha e), \theta(\alpha e)\rangle &= \langle \alpha \cdot_l X(e), \theta(\alpha e)\rangle \\ &= \alpha\langle X(e), \theta(e)\rangle = \alpha\langle X, \theta\rangle(e)\,.\end{aligned}$$

$\square$

We say that θ is of degree 1 (linear) if one of the equivalent conditions of the theorem above is satisfied.

In a local coordinate system, a linear 1-form θ has the following form

$$\theta = \theta_a \mathrm{d}e^a + \theta_{ia} e^a \mathrm{d}x^i\,, \tag{49}$$

where θ_a, θ_{ia} are functions of (x^i) only.

7.2. Linear Poisson structures. A Poisson structure on a vector bundle $\mathbf{E}$ is called linear if, for every two functions f, g, linear on fibers of $\mathbf{E}$, the Poisson bracket $\{f, g\}$ is also linear on fibers. It follows that for f linear on fibers and g, g' constant on fibers the bracket $\{f, g\}$ is constant on fibers and $\{g, g'\} = 0$.

Let Λ be a Poisson bivector field on $\mathbf{E}$ and let $\widetilde{\Lambda}\colon \mathsf{T}^*E \to \mathsf{T}E$ be the correspondig mapping of vector bundles.

Proposition 21. *Λ defines a linear Poisson structure on $\mathbf{E}$ if and only if $\widetilde{\Lambda}$ defines a morphism of double vector bundles $\mathsf{T}^*\mathbf{E} \to \mathsf{T}\mathbf{E}$.*

Proof. Let Λ be the bivector field of a linear Poisson structure. In the proof of Theorem 19 we have shown that for a linear function f on E the differential $\mathrm{d}f$ is linear, i.e.,

$$\mathrm{d}f(e + e') = \mathrm{d}f(e) +_l \mathrm{d}f(e')\,. \tag{50}$$

The vector field $\widetilde{\Lambda}(\mathrm{d}f)$ is a linear vector field because it satisfies the condition (1) from the theorem 19. It means that

$$\widetilde{\Lambda}(\mathrm{d}f(e) + \mathrm{d}f(e')) = \widetilde{\Lambda}(\mathrm{d}f(e + e')) = \widetilde{\Lambda}(\mathrm{d}f(e)) +_l \widetilde{\Lambda}(\mathrm{d}f(e'))\,. \tag{51}$$

On the other hand, for a function g, constant on fibers, the vector field $\widetilde{\Lambda}(\mathrm{d}g)$ is vertical and constant on fibers (we identify spaces of vertical vectors at different points in a fiber). For every pair of covectors μ, ν such that $\mu \in \mathsf{T}^*_e E$, $\nu \in \mathsf{T}^*_{e'} E$ and they have the same projection on $\mathbf{E}^*$, there exist a linear function f and a function g, costant on fibers, such that $\mathrm{d}f(e) = \mu$ and $\mathrm{d}f(e') + \mathrm{d}g(e') = \nu$. Therefore, we have, in view of (51)

$$\begin{aligned}\widetilde{\Lambda}(\mu +_l \nu) = \widetilde{\Lambda}(\mathrm{d}f(e) +_l (\mathrm{d}f(e') + \mathrm{d}g(e'))) = \widetilde{\Lambda}((\mathrm{d}f(e) +_l (\mathrm{d}f(e')) + \mathrm{d}g(e + e')))\\ = \widetilde{\Lambda}(\mathrm{d}f(e) +_l (\mathrm{d}f(e')) + \widetilde{\Lambda}(\mathrm{d}g(e' + e)) = \Big(\widetilde{\Lambda}(\mathrm{d}f(e)) +_l \widetilde{\Lambda}(\mathrm{d}f(e'))\Big) + \widetilde{\Lambda}(\mathrm{d}g(e' + e))\\ = \widetilde{\Lambda}(\mathrm{d}f(e)) +_l \Big(\widetilde{\Lambda}(\mathrm{d}f(e')) + \widetilde{\Lambda}(\mathrm{d}g(e'))\Big) = \widetilde{\Lambda}(\mu) +_l \widetilde{\Lambda}(\nu)\,.\end{aligned}$$

Now, let the Poisson bivector $\widetilde{\Lambda}$ be a morphism of double vector bundles. It follows that, for every pair of covectors μ, ν such that their left projections are equal, we have

$$\Lambda(\mu +_l \nu) = \Lambda(\mu) +_l \Lambda(\nu)\,.$$

Let f, g be linear on fibers. Consequently, $\mathrm{d}f, \mathrm{d}g$ are linear one forms. We have then

$$\begin{aligned}\{f, g\}(e + e') &= \langle \widetilde{\Lambda}(\mathrm{d}f(e + e')), \mathrm{d}g(e + e')\rangle\\ &= \langle \widetilde{\Lambda}(\mathrm{d}f(e) +_l \mathrm{d}f(e')), \mathrm{d}g(e) +_l \mathrm{d}g(e')\rangle\\ &= \langle \widetilde{\Lambda}(\mathrm{d}f(e)) +_l \widetilde{\Lambda}(\mathrm{d}f(e')), \mathrm{d}g(e) +_l \mathrm{d}g(e')\rangle\\ &= \langle \widetilde{\Lambda}(\mathrm{d}f(e)), \mathrm{d}g(e)\rangle + \langle \widetilde{\Lambda}(\mathrm{d}f(e')), \mathrm{d}g(e')\rangle\\ &= \{f, g\}(e) + \{f, g\}(e')\,.\end{aligned}$$

□

7.3. Special symplectic manifolds. Let $\mathbf{E} = (E, \tau, M)$ be a vector bundle and let ω be a 2-form on E. By $\widetilde{\omega}$ we denote the corresponding vector bundle morphism

$$\widetilde{\omega} \colon \mathsf{T}E \to \mathsf{T}^* E\,. \tag{52}$$

We say that ω is linear with respect to the vector bundle structure $\mathbf{E}$ if $\widetilde{\omega}$ is a morphism of double vector bundles

$$\widetilde{\omega} \colon \mathsf{T}\mathbf{E} \to \mathsf{T}^* \mathbf{E}\,. \tag{53}$$

If ω is linear, then there are three derived vector bundle morphisms:

$$\begin{aligned}
&\widetilde{\omega}_r \colon E \to E\,,\\
&\widetilde{\omega}_l \colon \mathsf{T}M \to E^*\,,\\
&\widetilde{\omega}_c \colon E \to \mathsf{T}^* M\,.
\end{aligned}$$

Of course, $\widetilde{\omega}_r = \mathrm{id}_E$ and, because $\widetilde{\omega}$ is skew-symmetric, we have, from (29),

$$\widetilde{\omega}_l = -\widetilde{\omega}_c^*\,.$$

Proposition 22. *ω is closed if and only if the pull-back of the canonical symplectic form ω_M on T^*M by $\widetilde{\omega}_c$ is equal ω:*

$$\omega = \widetilde{\omega}_c^* \omega_M\,. \tag{54}$$

Proof. Let (x^i, e^a) be a local coordinate system on E and let $(x^i, e^a, \dot{x}^j, \dot{e}^b)$, (x^i, e^a, p_j, f_b) be adopted coordinate systems on $\mathsf{T}E$, T^*E respectively. For a 2-form ω on E,

$$\omega = \frac{1}{2}\omega_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j + \omega_{ia}\mathrm{d}x^i \wedge \mathrm{d}e^a + \frac{1}{2}\omega_{ab}\mathrm{d}e^a \wedge \mathrm{d}e^b\,,$$

we have

$$\begin{aligned}
p_j \circ \widetilde{\omega} &= \omega_{ij}\dot{x}^i - \omega_{ja}\dot{e}^a\,,\\
f_b \circ \widetilde{\omega} &= \omega_{ib}\dot{x}^i + \omega_{ab}\dot{e}^a\,.
\end{aligned}$$

The linearity of ω implies, in view of (10),

$$\begin{aligned}
\omega_{ij}(x,e) &= \omega_{ija}(x)e^a\,,\\
\omega_{ia}(x,e) &= \omega_{ia}(x)\,,\\
\omega_{ab}(x,e) &= 0\,.
\end{aligned}$$

The exterior derivative of ω assumes then the form:

$$\mathrm{d}\omega = \frac{1}{2}\omega_{ija,k}e^a\mathrm{d}x^k \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j + (\frac{1}{2}\omega_{ija} + \omega_{ja,i})\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}e^a\,.$$

Therefore the external derivative $d\omega$ equals 0 if and only if the following two conditions are satisfied:

$$\begin{aligned} &\omega_{ija}(x) = \omega_{ia,j}(x) - \omega_{ja,i}(x)\,,\\ &\omega_{ija,k}(x) + \omega_{jka,i}(x) + \omega_{kia,j}(x) = 0\,. \end{aligned} \tag{55}$$

The second condition is an immediate consequence of the first one.

On the other hand, the mapping $\widetilde{\omega}_c$ is given in the coordinate system, by the formula

$$p_i = -\omega_{ia}e^a\,.$$

Consequently, the pull-back of the canonical symplectic form $\omega_M = \mathrm{d}p_i \wedge \mathrm{d}x^i$ by ω_c is given by

$$\widetilde{\omega}_c^*\omega_M = \mathrm{d}(-\omega_i e^a) \wedge \mathrm{d}x^i = \omega_{ia}\mathrm{d}x^i \wedge \mathrm{d}e^a - \omega_{ia,j}e^a\mathrm{d}x^j \wedge \mathrm{d}x^i\,.$$

It follows that $\omega = \widetilde{\omega}_c^*\omega_M$ if and only if the condition

$$\omega_{ija}(x) = \omega_{ia,j}(x) - \omega_{ja,i}(x)\,, \tag{56}$$

which is equivalent to (55), is satisfied. □

If ω is nondegenerate, i.e., if $\widetilde{\omega}$ is an isomorphism of vector bundles, then also $\widetilde{\omega}_c$ is an isomorphism. In that case $\widetilde{\omega}_c$ is a symplectomorphism. Thus, we can consider the pair $(\mathbf{E}, \omega)$ as a *special symplectic manifold* ([6], [7]).

7.4. Vertical lifts and complete lifts. In this section, we present concepts of vertical and complete lifts of a vector field ([9], [1]) in the general framework of double vector bundles.

Let $\mathbf{K} = (\mathbf{K}_r, \mathbf{K}_l, \mathbf{E}, \mathbf{F})$ be a double vector bundle with the core $\mathbf{C}$. Let γ be a section of the core. Using the double vector bundle structure of $\mathbf{K}$ we assign to γ two sections $\mathcal{V}_r\gamma$ and $\mathcal{V}_l\gamma$ of τ_r and τ_l respectively.

From Proposition 3 we have $\ker\tau_r = F \times_M \mathbf{C}$ and $\ker\tau_l = E \times_M \mathbf{C}$. Sections $\mathcal{V}_r\gamma$ and $\mathcal{V}_l\gamma$ are defined by the following formulae:

$$\mathcal{V}_l\gamma\colon F \to K\colon f \mapsto (f, \gamma(\bar{\tau}_r(f))) \in \ker\tau_r \subset K$$

and

$$\mathcal{V}_r\gamma\colon E \to K\colon e \mapsto (e, \gamma(\bar{\tau}_l(e))) \in \ker\tau_l \subset K\,.$$

Sections $\mathcal{V}_r\gamma$ and $\mathcal{V}_l\gamma$ are called *vertical lifts of* γ with respect to the right and left projections.

Examples.

(1) Let $\mathbf{K} = \mathsf{T}\mathbf{E}$. The core of $\mathsf{T}\mathbf{E}$ is isomorphic to $\mathbf{E}$ We can therefore lift the section of τ to the section of τ_E and $\mathsf{T}\tau$. In local coordinate system $(x^i, e^a, \dot{x}^j, \dot{e}^b)$ vertical lifts are given by the fomulae

$$\mathcal{V}_r\gamma = \gamma^a \frac{\partial}{\partial e^a}\,,$$
$$\mathcal{V}_l\gamma = \dot{x}^j \frac{\partial}{\partial x^j} + \gamma^a \frac{\partial}{\partial e^a}\,.$$

(2) Let $\mathbf{K} = \mathsf{T}^*\mathbf{E}$. The core we identify with T^*M. The right vertical lift of a 1-form θ is the pull-back of θ by the projection τ.

Now, let $X: F \to K$ be a section of τ_l. We say that this section is *linear* if it projects to a mapping $\overline{X}: M \to E$ and the pair $\mathbf{X} = (X, \overline{X})$ is a vector bundle morphism

$$\mathbf{X}: \mathbf{F} \to \mathbf{K}_r\,.$$

In a similar way we define linear sections of τ_r.

Proposition 23. *There exist a unique linear section Y of the right vector bundle structure of $\mathbf{K}^{*_r}$ such that $\langle X, Y\rangle = 0$*

Proof. For each point $m \in M$ the image of $\bar{\tau}_r^{-1}(m)$ under X is a vector subspace of $\tau_r^{-1}(\overline{X}(m))$. We denote by $X^\circ(m)$ the anihilator of this subspace in $\pi_l^{-1}(\overline{X}(m)) \subset K^{*_r}$. If n_E, n_F, n_C are the dimensions of fibers of E, F, C respectively then the dimension of $X^\circ(m)$ is equal n_C.

We show that $X^\circ(m)$ projects to the whole fiber of $\mathbf{C}^*$ (which is of dimension n_C). Since the projection π_r is linear with respect to the left vector bundle structure, it is enough to show that π_r, restricted to the anihilator $X^\circ(m)$, is an injection.

Let a be an element of $\ker \pi_r \cap X^\circ(m)$. We represent a by a pair $(e, \varphi) \in E \times_M F^*$, where $e = \overline{X}(m)$. Since (e, φ) is an element of the anihilator $X^\circ(m)$ then $\langle X(f), (\xi(m), \varphi)\rangle$ should be equal to zero for all f. The following calculation shows that, in this case, φ must be zero:

$$0 = \langle X(f), (\xi(m), \varphi)\rangle = \langle \tau_l(X(f)), \varphi\rangle = \langle f, \varphi\rangle\,.$$

It follows that $X^\circ(m)$ is the image of a section of π_r over $\pi_c^{-1}(m)$. Collecting the anihilators of $X(\bar{\tau}_r^{-1}(m))$ point by point in M we get a section Y of π_r. The uniqueness of Y is obvious. □

Let $(x^i, e^a, f^A, k^\alpha)$ be a local coordinate system on K and let $(x^i, e^a, \varphi_A, \kappa_\alpha)$ be the adopted local coordinate system on K^{*_r}. A section of τ_l is linear if it is of the form

$$e^a = \xi^a(x^i)$$
$$k^\alpha = X^\alpha_B(x^i) f^B\,.$$

A linear section Y of π_r which have the same projection onto section of $\bar{\tau}_r$ can be written as:

$$
\begin{aligned}
e^a &= \xi^a(x^i)\,, \\
\varphi_A &= Y^\beta_A(x^i)\varphi_\beta\,.
\end{aligned}
$$

The condition $\langle X, Y\rangle = 0$ gives

$$
Y^\alpha_A(x^i) = -X^\alpha_A(x^i)\,.
$$

Of special interest is the case of $\mathbf{K} = \mathsf{J}(\mathsf{T}\mathbf{E})$. The left projection is the canonical projection $\tau_E\colon \mathsf{T}E \to E$ and a section X of this projection is a vector field on E. The right dual to $\mathsf{J}(\mathsf{T}\mathbf{E})$ we identify with $\mathsf{T}\mathbf{E}^*$ (Example 3 of Section 4) and a section of the right projection is a vector field on E^*. Linear sections of τ_E, τ_{E^*} are vector fields of degree 0. Proposition 23 establishes the one-to-one correspondence between vector fields of degree 0 on E and vector fields of degree 0 on E^*. In particular, for $\mathbf{E} = \mathsf{T}M$, the complete tangent lift $\mathrm{d}_\mathsf{T}X$ of a vector field on M is a vector fields of degree 0 on $\mathsf{T}M$ (see [9], [1]). One can easily recognize the corresponding vector field Y on the cotangent bundle T^*M as the complete cotangent lift of X. In local coordinates, for $X = X^i\dfrac{\partial}{\partial x^i}$, we have

$$
\mathrm{d}_\mathsf{T}X = X^i\frac{\partial}{\partial x^i} + X^i_{,j}\dot{x}^j\frac{\partial}{\partial \dot{x}^i}
$$

and

$$
Y = X^i\frac{\partial}{\partial x^i} - X^i_{,j}p_i\frac{\partial}{\partial p_j}\,.
$$

7.5. Linear connections and the dual connections. A connection on a vector fibration $\mathbf{E}$ is given by the horizontal distribution and can be represented by a section δ of the fibration $\mathsf{j}^1(\mathbf{E}) \to \mathsf{j}^0(\mathbf{E}) = \mathbf{E}$. The connection is linear if δ is a morphism of vector bundles

$$
\delta\colon \mathbf{E} \to j^1(\mathbf{E})\,,
$$

where $j^1(\mathbf{E})$ is a vector fibration over M.

A connection on $\mathbf{E}$ defines a splitting of the tangent bundle $\mathsf{T}E$ into the vertical and horizontal parts. Since the bundle $\mathsf{V}E$ of vertical vectors can be identified with the product $\mathbf{E}\times_M E$, we can look at the splitting map as an isomorphism of vector bundles

$$
D\colon \mathsf{T}E \to (\mathsf{T}M \oplus_M \mathbf{E}) \times_M E \tag{57}
$$

over the identity of E.

Proposition 24. *A mapping $D\colon \mathsf{T}E \to (\mathsf{T}M \oplus_M \mathbf{E}) \times_M E$ is the splitting related to a linear connection if and only if D defines a double vector bundle morphism*

$$\mathbf{D}\colon \mathsf{T}\mathbf{E} \to \mathbf{K}(\mathsf{T}M, \mathbf{E}, \mathbf{E})\,, \tag{58}$$

such that the corresponding mappings

$$\begin{aligned} &D_r\colon E \to E\,,\\ &D_l\colon \mathsf{T}M \to \mathsf{T}M\,,\\ &D_c\colon E \to E \end{aligned} \tag{59}$$

are identities.

Let D be the the splitting of a linear connection on $\mathbf{E}$. The transposed left dual to D defines an isomorphism

$$\mathbf{D}^*\colon \mathsf{T}\mathbf{E}^* \to \mathbf{K}(\mathsf{T}M, \mathbf{E}^*, \mathbf{E}^*) \tag{60}$$

and, because of (59) and (45), D_r^*, D_l^*, D_c^* are identities. Thus $\mathbf{D}^*$ is the splitting of a linear connection on $\mathbf{E}^*$. We call it *the dual connection.*

Let $g\colon \mathbf{E} \to \mathbf{E}^*$ be a metric on $\mathbf{E}$ (g is a self-adjoint isomorphism of vector bundles). The splitting D is the splitting of a metric connection if the following diagram is commutative

$$\begin{array}{ccc} \mathsf{T}E & \xrightarrow{\ \mathbf{D}\ } & \mathbf{K}(\mathsf{T}M, \mathbf{E}, \mathbf{E}) \\ {\scriptstyle \mathsf{T}g}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{\mathsf{T}M} \times g \times g} \\ \mathsf{T}E^* & \xrightarrow{\ \mathbf{D}^*\ } & \mathbf{K}(\mathsf{T}M, \mathbf{E}^*, \mathbf{E}^*) \end{array} \tag{61}$$

7.6. Symmetric connections. In this section $\mathbf{E} = \mathsf{T}M$. We have then the canonical isomorphism

$$\kappa_M\colon \mathsf{TT}M \to \mathsf{J}(\mathsf{TT}M)\,.$$

We introduce also an isomorphism

$$\kappa\colon \mathbf{K}(\mathsf{T}M, \mathsf{T}M, \mathsf{T}M) \to \mathsf{J}(\mathbf{K}(\mathsf{T}M, \mathsf{T}M, \mathsf{T}M))$$

by

$$\kappa(v, w, u) = (u, w, v)\,.$$

Proposition 25. *A connection D is symmetric (torsion-free) if and only if*

$$\kappa \circ \mathbf{D} = \mathsf{J}(\mathbf{D}) \circ \kappa_M \tag{62}$$

i. e., if the following diagram is commutative

$$
\begin{tikzcd}
\mathsf{TT}M \arrow[r, "\mathbf{D}"] \arrow[d, "\kappa_M"'] & \mathbf{K}(\mathsf{T}M, \mathsf{T}M, \mathsf{T}M) \arrow[d, "\kappa"] \\
\mathsf{J}(\mathsf{TT}M) \arrow[r, "\mathsf{J}(\mathbf{D})"] & \mathsf{J}(\mathbf{K}(\mathsf{T}M, \mathsf{T}M, \mathsf{T}M))
\end{tikzcd}
\tag{63}
$$

Proof. Let $(x^i, \dot{x}^j, x'^k, \dot{x}'^l)$ be an adopted coordinate system on $\mathsf{TT}M$ and let $(y^i, \dot{y}_l^j, \dot{y}_c^k, \dot{y}_r^l)$ be a coordinate system on $\mathbf{K}(\mathsf{T}M, \mathsf{T}M, \mathsf{T}M)$. We have then

$$
\begin{aligned}
y^i(\kappa \circ D) &= x^i\,, \\
\dot{y}_l^j(\kappa \circ D) &= \dot{x}^j\,, \\
\dot{y}_r^j(\kappa \circ D) &= x'^j\,, \\
\dot{y}_c^l(\kappa \circ D) &= \dot{x}'^l + \Gamma^l_{ij}\dot{x}^i x'^j\,.
\end{aligned}
\tag{64}
$$

On the other hand,

$$
\begin{aligned}
y^i(\mathsf{J}(D) \circ \kappa_M) &= x^i\,, \\
\dot{y}_l^j(\mathsf{J}(D) \circ \kappa_M) &= \dot{x}^j\,, \\
\dot{y}_r^j(\mathsf{J}(D) \circ \kappa_M) &= x'^j\,, \\
\dot{y}_c^l(\mathsf{J}(D) \circ \kappa_M) &= \dot{x}'^l + \Gamma^l_{ij}\dot{x}^j x'^i\,.
\end{aligned}
\tag{65}
$$

The diagram (63) is commutative if and only if $\Gamma^l_{ij} = \Gamma^l_{ji}$, i. e., if the connection is symmetric. □

In order to obtain conditions for a connection to be symmetric, in terms of the dual connection, let us consider first a more general commutative diagram of isomorphisms of double vector bundles.

$$
\begin{tikzcd}
\mathbf{K} \arrow[r, "\mathbf{\Phi}"] \arrow[d, "\kappa_K"'] & \mathbf{L} \arrow[d, "\kappa_L"'] \\
\mathsf{J}(\mathbf{K}) \arrow[r, "\mathsf{J}(\mathbf{\Phi})"] & \mathsf{J}(\mathbf{L})
\end{tikzcd}
\tag{66}
$$

The left dual to this diagram is the commutative diagram (see (21))

$$
\begin{tikzcd}
\mathbf{K}^{*_l} & \mathbf{L}^{*_l} \arrow[l, "\mathbf{\Phi}^{*_l}"'] \\
\mathsf{J}(\mathbf{K}^{*_r}) \arrow[u, "\kappa_K^{*_l}"] & \mathsf{J}(\mathbf{L}^{*_r}) \arrow[l, "\mathsf{J}(\mathbf{\Phi}^{*_r})"'] \arrow[u, "\kappa_L^{*_l}"]
\end{tikzcd}
\tag{67}
$$

Using the canonical isomorphisms of $\mathbf{K}, \mathbf{L}$ and $\mathbf{K}^{*_r *_l}, \mathbf{L}^{*_r *_l}$, we obtain

$$\mathbf{K}^{*_r} = \mathcal{L}^{\pm}_{K^{*_r}}(\mathbf{K}^{*_r *_l *_l *_l}) = \mathcal{L}^{\pm}_{K^{*_r}}(\mathbf{K}^{*_l *_l}) \tag{68}$$

and

$$\mathbf{L}^{*_r} = \mathcal{L}^{\pm}_{L^{*_r}}(\mathbf{L}^{*_r *_l *_l *_l}) = \mathcal{L}^{\pm}_{L^{*_r}}(\mathbf{L}^{*_l *_l}) \,. \tag{69}$$

With these identifications, we can replace the diagram (67) by an equivalent one:

$$\begin{tikzcd} \mathbf{K}^{*_l} & \mathbf{L}^{*_l} \arrow[l, "\mathbf{\Phi}^{*_l}"'] \\ \mathsf{J}(\mathbf{K}^{*_l *_l}) \arrow[u, "\beta_K^{-1}"] & \mathsf{J}(\mathbf{L}^{*_l *_l}) \arrow[l, "\mathsf{J}(\mathbf{\Phi}^{*_l *_l})^{-1}"'] \arrow[u, "\beta_L^{-1}"'] \end{tikzcd} \tag{70}$$

or

$$\begin{tikzcd} \mathbf{K}^{*_l} \arrow[r, "(\mathbf{\Phi}^{*_l})^{-1}"] \arrow[d, "\beta_K"'] & \mathbf{L}^{*_l} \arrow[d, "\beta_L"'] \\ \mathsf{J}(\mathbf{K}^{*_l *_l}) & \mathsf{J}(\mathbf{L}^{*_l *_l}) \arrow[l, "\mathsf{J}(\mathbf{\Phi}^{*_l *_l})^{-1}"'] \end{tikzcd} \tag{71}$$

where

$$\beta_K^{-1} = \kappa_K^{*_l} \circ \mathsf{J}(\mathcal{L}^{\pm}_{K^{*_r}}),\ \beta_L^{-1} = \kappa_L^{*_l} \circ \mathsf{J}(\mathcal{L}^{\pm}_{L^{*_r}}) \,.$$

In the case of $\mathbf{K} = \mathsf{TT}M$ and $\mathbf{L} = \mathbf{K}(\mathsf{T}M, \mathsf{T}M, \mathsf{T}M)$, we have, as in (41),

$$\begin{aligned} \mathbf{K}^{*_l} &= \mathsf{J}(\mathsf{TT}^*M), \quad \mathbf{L}^{*_l} = \mathsf{J}(\mathbf{K}(\mathsf{T}M, \mathsf{T}^*M, \mathsf{T}^*M))\,, \\ \mathbf{K}^{*_r} &= \mathsf{J}(\mathsf{T}^*\mathsf{T}M), \quad \mathbf{L}^{*_r} = \mathbf{K}(\mathsf{T}M, \mathsf{T}^*M, \mathsf{T}^*M)\,, \\ \mathbf{K}^{*_l *_l} &= \mathsf{T}^*\mathsf{T}^*M, \quad \mathbf{L}^{*_l *_l} = \mathbf{K}(\mathsf{T}M, \mathsf{T}^*M, \mathsf{T}^*M)\,, \end{aligned} \tag{72}$$

and the canonical evaluation between $\mathbf{L}^{*_l}$ and $\mathbf{L}^{*_l *_l}$ is given by the formula

$$\langle (v, f, g), (w, h, g) \rangle = \langle v, h \rangle + \langle w, f \rangle. \tag{73}$$

Moreover, for $\kappa_K = \kappa_M$, $\kappa_L = \kappa$, $\Phi = D$, we have

$$\begin{aligned} &(\kappa_K)^{*_l} = \mathsf{J}(\alpha_M^{-1})\,, \quad (\kappa_L)^{*_l} = \mathrm{id}\,, \\ &\mathbf{\Phi}^{*_l} = (\mathsf{J}(\mathbf{D}^*))^{-1}, \quad (\mathsf{J}(\mathbf{\Phi}^{*_l *_l}))^{-1} = \mathsf{J}((\mathsf{J}(\mathbf{D}^*))^{*_l}) = (\mathbf{D}^*)^{*_r}\,. \end{aligned} \tag{74}$$

The commutative diagram (71) is then equivalent to

$$\begin{tikzcd} \mathsf{J}(\mathsf{TT}^*M) \arrow[r, "\mathsf{J}(\mathbf{D}^*)"] \arrow[d, "\beta_K"'] & \mathsf{J}(\mathbf{K}(\mathsf{T}M, \mathsf{T}^*M, \mathsf{T}^*M)) \arrow[d, "\beta_L"'] \\ \mathsf{J}(\mathsf{T}^*\mathsf{T}^*M) & \mathsf{J}(\mathbf{K}(\mathsf{T}M, \mathsf{T}^*M, \mathsf{T}^*M)) \arrow[l, "(\mathbf{D}^*)^{*_r}"'] \end{tikzcd} \tag{75}$$

and

$$
\begin{tikzcd}
\mathsf{T}\mathsf{T}^*M \arrow[r, "\mathbf{D}^*"] \arrow[d, "\mathsf{J}(\beta_K)"'] & \mathbf{K}(\mathsf{T}M, \mathsf{T}^*M, \mathsf{T}^*M) \arrow[d, "\mathsf{J}(\beta_L)"'] \\
\mathsf{T}^*\mathsf{T}^*M & \mathbf{K}(\mathsf{T}M, \mathsf{T}^*M, \mathsf{T}^*M) \arrow[l, "\mathsf{J}((\mathbf{D}^*)^{*_r})"']
\end{tikzcd}
\tag{76}
$$

The isomorphism $\mathcal{L}^{\pm}_{L^{*r}}$ is given by the formula (39)

$$\mathcal{L}^{\pm}_{L^{*r}}: (v, f, g) \mapsto (-v, f, g)$$

and $\mathcal{L}^{\pm}_{K^{*r}}: \mathsf{T}^*\mathsf{T}^*M \to \mathsf{J}(\mathsf{T}^*\mathsf{T}M)$ is a symplectomorphism such that it projects to the identity on T^*M and the core isomorphism is also the identity on T^*M. The isomorphism $\alpha_M: \mathsf{T}\mathsf{T}^*M \to \mathsf{T}^*\mathsf{T}M$ is a symplectomorphism between the tangent canonical symplectic structure on $\mathsf{T}\mathsf{T}^*M$ and the canonical symplectic structure on $\mathsf{T}^*\mathsf{T}M$ which projects to the identity on T^*M and the core isomorphism is also the identity on T^*M ([1], [5]) . It follows that

$$\beta = \mathsf{J}(\beta_L): (v, f, g) \mapsto (-v, f, g)$$

and that β_K is a symplectomorphism such that it projects to the identity on T^*M and the core isomorphism is also the identity on T^*M. We conclude that β_K is the canonical symplectic structure on T^*M. We get the diagram

$$
\begin{tikzcd}
\mathsf{T}\mathsf{T}^*M \arrow[r, "\mathbf{D}^*"] \arrow[d, "\beta_M"'] & \mathbf{K}(\mathsf{T}M, \mathsf{T}^*M, \mathsf{T}^*M) \arrow[d, "\beta"'] \\
\mathsf{T}^*\mathsf{T}^*M & \mathbf{K}(\mathsf{T}M, \mathsf{T}^*M, \mathsf{T}^*M) \arrow[l, "\mathsf{J}((\mathbf{D}^*)^{*_r})"']
\end{tikzcd}
\tag{77}
$$

$(\mathbf{D}^*)^{-1}$ is completely determined by its values on

$$\mathbf{K}(\mathsf{T}M, \mathsf{T}^*M, \mathsf{T}^*M) \supset V = \{(v, f, g): f = 0\}$$

and, consequently, by $W = (\mathbf{D}^*)^{-1}(V)$ Of course, $(\mathbf{D}^*)^{-1}(V)$ is the horizontal distribution of $\mathbf{D}^*$, and $\beta(V) = V$. Thus, the diagram (76) is commutative if and only if

$$\mathsf{J}((\mathbf{D}^*)^{*_r}) \circ \beta \circ \mathbf{D}^*(W) = \beta_M(W)\,. \tag{78}$$

Let X, Y be vector spaces, $T: X \to Y$ a linear mapping and $Z \subset X$ a vector subspace. We have the equality

$$T^*((F(Z))^\circ) = Z^\circ\,.$$

It follows that, since $V^\circ = V$,

$$\mathsf{J}((\mathbf{D}^*)^{*_r})(V) = W^\circ \tag{79}$$

and, consequently, the equality (77) is equivalent to

$$W^\circ = \beta(W)\,.$$

We have proved the following theorem.

Proposition 26. *The diagram (78) is commutative (the connection is symmetric) if and only if the horizontal distribution of the dual connection* $\mathbf{D}^*$ *is lagrangian.*

References

[1] Grabowski, J., Urbański, P., *Tangent lifts of Poisson and related structures*, J. Phys. A. **28** (1995), 6743–6777.

[2] Mackenzie, K. C. H., *Double Lie algebroids and second-order geometry I*, Adv. Math. **94** (1992), 180-239.

[3] Pradines, J., *Géométrie differentielle au-dessus d'un grupoïde*, C. R. Acad. Sci. Paris, série A **266** (1968), 1194-1196.

[4] Pradines, J., *Représentation des jets non holonomes par des morphismes vectoriels doubles soudés*, C. R. Acad. Sci. Paris, série A **278** (1974), 1523-1526.

[5] Tulczyjew, W. M., *Geometric Formulations of Physical Theories*, Bibliopolis, Napoli, 1989.

[6] Tulczyjew, W. M., *Les sous-varieétés lagrangiennes et la dynamique lagrangienne*, C. R. Acad. Sci. Paris, série A **283** (1976), 675-678.

[7] Tulczyjew, W. M., *Les sous-varieétés lagrangiennes et la dynamique hamiltonienne*, C. R. Acad. Sci. Paris, série A **283** (1976), 15-18.

[8] Urbański, P., *Double vector bundles in classical mechanics*, Rend. Sem. Mat. Torino (1997).

[9] Yano, K., Ishihara, S., *Tangent and Cotangent Bundles*, Marcel Dekker, Inc., New York, 1973.

Division of Mathematical Methods in Physics, University of Warsaw
Hoża 74, 00-682 Warsaw, POLAND
E-mail: `urbanski@fuw.edu.pl`