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ON THE ASYMPTOTICALLY PERIODIC SOLUTION OF SOME LINEAR DIFFERENCE EQUATIONS

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Abstract. For the linear difference equation

$$x_{n+1} - a_n x_n = \sum_{i=0}^{r} a_n^{(i)} x_{n+i}, \quad n \in N$$

sufficient conditions for the existence of an asymptotically periodic solutions are given.

In the paper by N, R, R_0 we denote the set of positive integers, real numbers, and nonnegative real numbers respectively.

For any function $y : N \to R$ the forward difference operator Δ is defined as follows:

$$\Delta y_n = y_{n+1} - y_n, \quad n \in N .$$

Using the method we have applied in [3] to get existence of constant approaching solutions of difference equations, we consider existence of asymptotically periodic solutions of the equation

$$(E) \qquad x_{n+1} - a_n x_n = \sum_{i=0}^{r} a_n^{(i)} x_{n+i}, \quad n \in N .$$

Definition. The sequence $v : N \to R$ is periodic (σ- periodic), if $v_{n+\sigma} = v_n$ for all $n \in N$. The sequence $v : N \to R$ is asymptotically periodic (asymptotically σ-periodic) if there exist two sequences $u, w : N \to R$ such that u is periodic (σ-periodic), $\lim_{n\to\infty} w_n = 0$, and $v_n = u_n + w_n$ for all $n \in N$.

A sequence $\{x_n\}_{n=1}^{\infty}$ is called generalized solution of (E) if it satisfies (E) for all n sufficiently large.

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Theorem 1. *Let $a^{(i)} : N \to R, \quad a : N \to R \setminus \{0\}$ be σ-periodic and such that $\prod_{j=1}^{\sigma} a_j = 1$, let furthermore*

$$\sum_{j=1}^{\infty} |a_j^{(i)}| < \infty, \quad i = 0, 1, \dots, r. \tag{1}$$

Then for arbitrary $C \in R, \ C \neq 0$ there exists asymptotically σ-periodic generalized solution x of (E) such that

$$x_n = C \prod_{j=1}^{n-1} a_j + o(1)\,. \tag{2}$$

If moreover

$$a_n^0 \neq -a_n \tag{3}$$

for each $n \in N$ then these solutions can be extended to the left up to $n = 1$.

Proof. Let us observe that if u is a solution of (E) such that $u_n = C \prod_{j=1}^{n-1} a_j + o(1)$ with $C > 0$, then the sequence $\{-u_n\}$ is also the solution of (E) and have the same asymptotic properties with $C < 0$. Therefore we restrict our considerations to the case $C > 0$.

Since $C > 0$ there exists a positive constant ε such that $C - \varepsilon > 0$. Let us denote

$$\begin{aligned} C_1 &= C + \varepsilon, \\ \alpha_n &= C_1 \sum_{i=0}^{r} \sum_{j=n}^{\infty} \left| b_j^{(i)} \right|, \quad n \in N\,, \end{aligned} \tag{4}$$

where

$$b_n^{(0)} = \frac{1}{a_n} a_n^{(0)},\ b_n^{(1)} = a_n^{(1)},\ b_n^{(i)} = \left(\prod_{k=n+1}^{n+i-1} a_k \right) a_n^{(i)} \tag{5}$$

for $i = 2, \dots, r$, $n \in N$.

Notice that by periodicity of $\{a_n\}$ and condition (1) the series

$$\sum_{j=1}^{\infty} b_j^{(i)}\,, \quad i = 0, 1, \dots, r$$

are absolutely convergent (also for $i = 0$ because of $a_n \neq 0$).

Therefore there exists $n_1 \in N$ such that $\alpha_n \leq \varepsilon$ for all $n \geq n_1$. So we can define

$$I = [C - \varepsilon, C + \varepsilon], \quad I_n = [C - \alpha_n, C + \alpha_n]$$

for $n \geq n_1$. It is evident that $I_{n+1} \subseteq I_n$ and

$$\operatorname{diam}\ I_n \to 0 \text{ as } n \to \infty\,. \tag{6}$$

Let l_∞ denotes the Banach space of bounded sequences $x = \{x_n\}_{n=1}^{\infty}$ with the norm $||x|| = \sup_{n\geq 1} |x_n|$.

Now $\mathbf{T} \subset l_\infty$ be such a set that $x = \{x_n\}_{n=1}^{\infty} \in \mathbf{T}$ if

$$\begin{cases} x_n = C & \text{for} \quad n = 1, 2, \dots, n_1 - 1 \\ x_n \in I_n & \text{for} \quad n \geq n_1 \,. \end{cases}$$

It is easy to check that $\mathbf{T}$ is a closed, convex and compact subset of l_∞. By (6), for arbitrary $\varepsilon_1 > 0$ we can set up a finite ε_1-net for the set $\mathbf{T}$. Hence by Hausdorff's theorem $\mathbf{T}$ is really compact.

Define now an operator $\mathcal{A}$. Let $y = \{y_n\}_{n=1}^{\infty} \in \mathbf{T}$ and $\mathcal{A}y = \eta = \{\eta_n\}_{n=1}^{\infty}$ if

$$\eta_n = \begin{cases} C & \text{for} \quad n = 1, 2, \dots, n_1 - 1 \\ C - \sum\limits_{i=0}^{r} \sum\limits_{j=n}^{\infty} b_j^{(i)} y_{j+i} & \text{for} \quad n \geq n_1 \,. \end{cases}$$

By absolute convergence of series $\sum\limits_{j=1}^{\infty} b_j^{(i)}$ and boundedness of the sequence y the operator $\mathcal{A}$ is well defined on the set $\mathbf{T}$. Furthermore

$$|\eta_n - C| \leq \sum_{i=0}^{r} \sum_{j=n}^{\infty} |b_j^{(i)}||y_{j+i}| \leq C_1 \sum_{i=0}^{r} \sum_{j=n}^{\infty} |b_j^{(i)}|$$

because $|y_n| \leq C + \alpha_n \leq C_1$ for all $n \in N$, so $|\eta_n - C| \leq \alpha_n$ that is $\eta_n \in I_n$.

Therefore $\mathcal{A}$ maps the set $\mathbf{T}$ into $\mathbf{T}$. We now prove that $\mathcal{A}$ is continuous on $\mathbf{T}$.

Take any $\varepsilon_1 > 0$. Let $x = \{x_n\}_{n=1}^{\infty}$ and $y = \{y_n\}_{n=1}^{\infty}$ be any two elements of the set $\mathbf{T}$ such that $||x - y|| < \delta_1$ where $\delta_1 = \frac{C_1 \varepsilon_1}{\alpha_{n_1}}$. Then the absolute convergence of series

$$\sum_{i=0}^{r} \sum_{j=n_1}^{\infty} b_j^{(i)} x_{j+i}, \qquad \sum_{i=0}^{r} \sum_{j=n_1}^{\infty} b_j^{(i)} y_{j+i}$$

yields

$$\begin{aligned} ||\mathcal{A}x - \mathcal{A}y|| &= \sup_{n\in N} |(\mathcal{A}x)_n - (\mathcal{A}y)_n| \\ &\leq \sup_{n\geq n_1} |[C - \sum_{i=0}^{r} \sum_{j=n}^{\infty} b_j^{(i)} x_{j+i}] - [C - \sum_{i=0}^{r} \sum_{j=n}^{\infty} b_j^{(i)} y_{j+i}]| \\ &\leq \sup_{n\geq n_1} \sum_{i=0}^{r} \sum_{j=n}^{\infty} |b_j^{(i)}||x_{j+i} - y_{j+i}| \leq ||x - y|| \sup_{n\geq n_1} \sum_{i=0}^{r} \sum_{j=n}^{\infty} |b_j^{(i)}| \\ &\leq \delta_1 \frac{\alpha_{n_1}}{C_1} < \frac{C_1 \varepsilon_1}{\alpha_{n_1}} \frac{\alpha_{n_1}}{C_1} = \varepsilon_1 \,, \end{aligned}$$

from there we can deduce that operator $\mathcal{A}$ is continuous on the set $\mathbf{T}$. Hence by Schauder fixed point theorem there exists a solution of the operator equation $x = \mathcal{A}x$ in $\mathbf{T}$. Let $z = \{z_n\}_{n=1}^{\infty}$ be this fixed point of $\mathcal{A}$. Then

$$z = \{C, \dots, C, z_{n_1}, \dots, z_n, \dots\}$$

while from the other hand

$$\mathcal{A}z = \left\{ C, \dots, C, C - \sum_{i=0}^{r} \sum_{j=n_1}^{\infty} b_j^{(i)} z_{j+i}, \dots, C - \sum_{i=0}^{r} \sum_{j=n}^{\infty} b_j^{(i)} z_{j+i}, \dots \right\} .$$

Therefore

$$z_n = C - \sum_{i=0}^{r} \sum_{j=n}^{\infty} b_j^{(i)} z_{j+i}, \quad \text{for} \quad n \geq n_1 . \tag{7}$$

Notice that by (6) and $z \in \mathbf{T}$, we have $z_n \to C$ because $z_n \in I_n$. In other words

$$z = C + o(1) . \tag{8}$$

Let us take

$$v_n = \left(\prod_{k=1}^{n-1} a_k \right) z_n , \quad \text{i.e.} \quad z_n = \left(\prod_{k=1}^{n-1} \frac{1}{a_k} \right) v_n . \tag{9}$$

Substituting (9) into (7) we get

$$v_n \prod_{k=1}^{n-1} \frac{1}{a_k} = C - \sum_{i=0}^{r} \sum_{j=n}^{\infty} b_j^{(i)} v_{j+i} \prod_{k=1}^{j+i-1} \frac{1}{a_k} .$$

From there

$$\Delta \left(v_n \prod_{k=1}^{n-1} \frac{1}{a_k} \right) = \sum_{i=0}^{r} b_n^{(i)} v_{n+i} \prod_{k=1}^{n+i-1} \frac{1}{a_k} .$$

Hence, by (5)

$$\begin{aligned} v_{n+1} - a_n v_n &= \left(\prod_{k=1}^{n} a_k \right) \sum_{i=0}^{r} b_n^{(i)} v_{n+i} \prod_{k=1}^{n+i-1} \frac{1}{a_k} \\ &= \left(\prod_{k=1}^{n} a_k \right) \left\{ \frac{1}{a_n} a_n^{(0)} v_n \prod_{k=1}^{n-1} \frac{1}{a_k} + a_n^{(1)} v_{n+1} \prod_{k=1}^{n} \frac{1}{a_k} + \right. \\ &\quad \left. + \dots + \left(\prod_{k=n+1}^{n+r-1} a_k \right) a_n^{(r)} v_{n+r} \prod_{k=1}^{n+r-1} \frac{1}{a_k} \right\} \\ &= \sum_{i=0}^{r} a_n^{(i)} v_{n+i} . \end{aligned}$$

That is the sequence $\{v_n\}_{n=n_1}^{\infty}$ fulfils (E) for $n \geq n_1$ (is generalized solution of (E)). By (8) and (9) we have

$$\left(\prod_{k=1}^{n-1} \frac{1}{a_k} \right) v_n = C + o(1) ,$$

from there

$$v_n = C \prod_{k=1}^{n-1} a_k + \left(\prod_{k=1}^{n-1} a_k \right) o(1) . \tag{10}$$

However σ-periodicity of the sequence $\{a_n\}$, and condition $\prod_{j=1}^{\sigma} a_j = 1$ yields σ- periodicity of the sequence $\left\{\prod_{j=1}^{n} a_j\right\}_{n=1}^{\infty}$. This in turns yields boundedness of $\left(\prod_{k=1}^{n-1} a_k\right)$, and consequently

$$\left(\prod_{k=1}^{n-1} a_k\right) o(1) = o(1)\,.$$

Therefore from (10) we get (2).

If furthermore the condition (3) is fulfilled then we can transform (E) to the form

$$x_n = -\frac{1}{a_n + a_n^{(0)}}\left\{-x_{n+1} + \sum_{i=1}^{r} a_n^{(i)} x_{n+i}\right\}. \tag{11}$$

Substituting in (11) $n = n_1 - 1$, $x_n = v_n$ for $n \geq n_1$ we obtain x_{n_1-1}. Proceeding this way, we find step by step $x_{n_1-2}, \ldots, x_1$. Consequently, we obtain sequence $\{x_n\}_{n=1}^{\infty}$ which fulfills (E) for all $n \in N$, and because $x_n = v_n$ for $n \geq n_1$ this (ordinary) solution of (E) has the property (2).

Notice that in fact Theorem 1 gives some sufficient conditions for the linear equation

$$c_n^{(r)} x_{n+r} + \ldots + c_n^{(1)} x_{n+1} + c_n^{(0)} x_n = 0 \tag{$E1$}$$

to possess asymptotically periodic solutions, because (E1) can be transformed into the form (E). Therefore if $c_n^{(0)}$ differs from some σ-periodic sequence $\{a_n\}$ (possessing properties defined in the Theorem 1) up to absolute summable sequence $\left\{a_n^{(0)}\right\}$, $c_n^{(1)}$ differs from 1 up to absolute summable sequence $\left\{a_n^{(1)}\right\}$, and

$$\sum_{j=1}^{\infty} |c_j^{(i)}| < \infty \quad \text{for} \quad i = 2, \ldots, r$$

then (E1) possesses asymptotically σ-periodic solutions.

In [4] we have given condition (with b almost σ-periodic) for the equation

$$x_{n+1} - x_n = \sum_{i=0}^{\infty} a_n^i x_{n+i} + b_n$$

possesses such type of solutions. For general viewpoint on this problem for first order linear equations see e.g. [1].

Following the referee suggestion we can generalize Theorem 1. Using similar method we can get suitable result for the equation:

$$(E2)\qquad x_{n+1} - a_n x_n = \sum_{i=0}^{r} a_n^{(i)} f_i(x_{n+i}), \quad n \in N\,.$$

Theorem 2. *Let $a^{(i)} : N \to R$, $a : N \to R\backslash\{0\}$ be σ-periodic and such that $\prod_{j=1}^{\sigma} a_j = 1$, and*

$$\sum_{j=1}^{\infty} |a_j^{(i)}| < \infty, \quad i = 0, 1, \ldots, r\,.$$

Let furthermore $f_i : R \to R$, $i = 0, 1, \ldots, r$ be odd and satisfy Lipschitz conditions i.e.

$$|f_i(u) - f_i(v)| \le L_i |u - v|$$

for $u, v \in R$ and some positive constant L_i. Then for arbitrary $C \in R$, $C \neq 0$ there exists asymptotically σ- periodic generalized solution x of (E2) such that

$$x_n = C \prod_{j=1}^{n-1} a_j + o(1)\,.$$

Proof. Proof of Theorem 2 follows similar way as the proof of Theorem 1. The main difference is in definition of the operator $\mathcal{A}$.

Let $y = \{y_n\}_{n=1}^{\infty} \in \mathbf{T}$ then we define $\eta = \{\eta_n\}_{n=1}^{\infty} = \mathcal{A}y$ if

$$\eta_n = \begin{cases} C & \text{for} \quad n = 1, 2, \ldots, n_1 - 1 \\ C - \sum\limits_{i=0}^{r} \sum\limits_{j=n}^{\infty} b_j^{(i)} f_i\big((\prod\limits_{k=1}^{j+i-1} a_k) y_{j+i}\big) & \text{for} \quad n \ge n_1\,. \end{cases}$$

Furthermore we should take

$$\alpha_n = C_1 \bar{L} \bar{a} \sum_{i=0}^{r} \sum_{j=n}^{\infty} |b_j^{(i)}|, \quad n \in N\,,$$

where

$$b_n^{(i)} = \left(\prod_{k=1}^{n} \frac{1}{a_k} \right) a_n^{(i)}, \quad \text{for} \quad i = 0, 1, \ldots, r, \quad n \in N\,,$$

$$\bar{a} = \max_{1 \le k \le \sigma} \left\{ \max_{1 \le j \le \sigma} |\prod_{i=1}^{j} a_{k+i}| \right\}, \quad \bar{L} = \max_{0 \le i \le r} L_i\,.$$

New definition of $\mathcal{A}$ is the consequence of $\{v_n\}_{n=n_1}^{\infty}$, defined by (9), have to be the solution (generalized) of (E2). Now new α_n allows us to get $\mathcal{A}$ maps $\mathbf{T}$ into $\mathbf{T}$, while the Lipschitz conditions yield continuity of $\mathcal{A}$.

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