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CONJUGACY CRITERIA FOR HALF–LINEAR DIFFERENTIAL EQUATIONS

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Abstract. Sufficient conditions on the function $c(t)$ ensuring that the half-linear second order differential equation

$$(|u'|^{p-2}u')' + c(t)|u(t)|^{p-2}u(t) = 0\,, \qquad p > 1$$

possesses a nontrivial solution having at least two zeros in a given interval are obtained. These conditions extend some recently proved conjugacy criteria for linear equations which correspond to the case $p = 2$.

1. Introduction

In this paper we investigate oscillatory behaviour of the solutions of half-linear second order differential equation

$$[\phi(u')]' + c(t)\phi(u) = 0 \tag{1.1}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is the scalar p-Laplacian defined by $\phi(s) := |s|^{p-2}s, p > 1$, and c is a continuous real valued function in an interval $I \subset \mathbb{R}$. If $p = 2$, then (1.1) reduces to the linear equation

$$u'' + c(t)u = 0\,. \tag{1.2}$$

The terminology *half-linear equation* for (1.1) is justified by the fact that if $u(t)$ is a solution of (1.1) and $\alpha \in \mathbb{R}$ then $\alpha u(t)$ also solves this equation. Here we look for conditions on the function c which guarantee that (1.1) has a solution having at least two zero points in a given interval. Conjugacy of linear equation (1.2) was investigated in severals papers. Tipler [6] proved that (1.2) is conjugate in $\mathbb{R}$ (i.e., there exists a nontrivial solution with at least zeros in $\mathbb{R}$) provided $\int_{-\infty}^{\infty} c(t)\,dt > 0$. This conjugacy criterion was extended by Müller-Pfeiffer [5] to the more general equation

$$(r(t)u')' + c(t)u = 0\,, \tag{1.3}$$

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where $r(t) > 0$, by showing that this equation is conjugate in an interval $(a,b) \subset \mathbb{R}$ if

$$\int_a r^{-1}(t)\,dt = \infty = \int^b r^{-1}(t)dt \quad \text{and} \quad \int_a^b c(t)\,dt > 0\,.$$

The result of Tipler is proved using the Riccati technique consisting in the fact that if u is a nonzero solutions of (1.2) then $v = \frac{u'}{u}$ solves the so-called Riccati equation

$$v' + v^2 + c(t) = 0 \tag{1.4}$$

and Müller-Pfeiffer's criterion is proved via the variational principle. This principle states that (1.2) is conjugate in (a,b) if and only if there exists a nontrivial function y which is piecevise of the class C^1, has compact support in (a,b), and

$$\int_a^b [r(t)(y'(t))^2 - c(t)y^2]\,dt \leq 0\,.$$

The above mentioned criteria were further generalized and extended in [1] using the combination of the transformation method and the Riccati technique.

Concerning a possible extension of these *linear* methods to half-linear equation, after some computations one can find that neither variational principle, nor transformation method extended directly to (1.1). On the other hand, the Riccati technique can be modified in a suitable way to apply to (1.1). Indeed, if u is a nonzero solution of (1.1) then $v(t) = \frac{\phi(u'(t))}{\phi(u(t))}$ solves the generalized Riccati equation

$$v' + c(t) + (p-1)|v|^q = 0\,, \tag{1.5}$$

where q is the conjugate number of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, see e.g. [4].

In this paper we use this idea to prove conjugacy criteria for (1.1) and to derive an estimate for distance of consecutive zeros of a solution of (1.1). If $p = 2$, our results reduce to those of [3] and [6].

2. Conjugacy criteria

In this section we prove conjugacy criteria for (1.1). The first one concerns conjugacy on a half-bounded interval.

Theorem 1. *Let $t_0 \in \mathbb{R}$, $c(t) \geq 0$ in $[t_0, \infty)$ and suppose that there exist t_1, t_2 such that $t_0 < t_1 < t_2$ and*

$$\frac{1}{(t_1 - t_0)^{p-1}} < \int_{t_1}^{t_2} c(t)\,dt\,. \tag{2.1}$$

Then the solution u of (1.1) given by the initial condition $u(t_0) = 0$, $u'(t_0) = 1$ has at least one zero in (t_0, ∞).

Proof. First of all note that the solution u is by the initial condition determined uniquely and exists up to ∞, see [2]. Suppose, by contradiction, that $u(t) > 0$ on

(t_0,∞). Then we have also $u'(t)\ge 0$ on $[t_0,\infty)$. Indeed, if $u'(T)<0$ for some $T\in(t_0,\infty)$, then $\alpha:=\phi(u'(T))<0$ and for $t>T$

$$\int_T^t [\phi(u'(t))]'\,dt=\phi(u'(t))-\alpha=-\int_T^t c(t)u^{p-1}(t)\,dt\le 0\,,$$

hence $\phi(u'(t))\le\alpha<0$ and thus $u'(t)\le -|\alpha|^{\frac{1}{p-1}}$, which means

$$u(t)\le u(T)-|\alpha|^{\frac{1}{p-1}}(t-T)\to-\infty \quad \text{as} \quad t\to\infty\,,$$

a contradiction, consequently $u'(t)\ge 0$, $t\in[t_0,\infty)$.

This implies that u' is nonincreasing for $t=t_0$, since from (1.1)

$$0\ge[\phi(u')]'=((u'(t))^{p-1})'=(p-1)(u'(t))^{p-2}u''(t)$$

i.e. $u''(t)\le 0$. Using this fact and the mean value theorem, there exists $\xi\in(t_0,t_1)$ such that

$$\frac{u(t_1)-u(t_0)}{t_1-t_0}=\frac{u(t_1)}{t_1-t_0}=u'(\xi)\ge u'(t_1)\,,\quad \phi(u'(t_1))>0$$

hence $u(t_1)\ge u'(t_1)\ (t_1-t_0)$. Using this inequality and the fact that $\phi(u'(t))\ge 0$, $t\ge t_0$, we have

$$\phi(u'(t))\Big|_{t_1}^{t_2}=\phi(u'(t_2))-\phi(u'(t_1))=-\int_{t_1}^{t_2} c(t)u^{p-1}(t)\,dt\,,$$

hence

$$\begin{aligned}\phi(u'(t_1))=(u'(t_1))^{p-1}\ge\int_{t_1}^{t_2} c(t)u^{p-1}(t)\,dt\ge\\ \ge u^{p-1}(t_1)\int_{t_1}^{t_2} c(t)\,dt\ge(u'(t_1))^{p-1}(t_1-t_0)^{p-1}\int_{t_1}^{t_2} c(t)\,dt\end{aligned}$$

and thus

$$(u'(t_1))^{p-1}\left[1-(t_1-t_0)^{p-1}\int_{t_1}^{t_2} c(t)\,dt\right]\ge 0$$

which contradicts to (2.1), i.e. $u(t)$ has a zero in (t_0,∞). □

The next statement gives sufficient condition for conjugacy of (1.1) on the whole real line.

Theorem 2. *If*

$$\int_{-\infty}^{\infty} c(t)\,dt>0\,, \tag{2.2}$$

then there exists a nontrivial solution of (1.1) *having at least two zeros in* $\mathbb{R}$.

Proof. Condition (2.2) implies the existence of $t_0 \in \mathbb{R}$ such that

$$\int_{t_0}^{\infty} c(t)\,dt > 0\,, \quad \int_{-\infty}^{t_0} c(t)\,dt > 0\,, \tag{2.3}$$

see [6]. Let u be the solution of (1.1) given by the initial condition $u(t_0) = 1$, $u'(t_0) = 0$. We will show that u has at least one zero point both in $(-\infty, t_0)$ and (t_0, ∞). Suppose, by contradiction, that $u(t) > 0$ for $t > t_0$ (for $t < t_0$ we proceed in the same way) and set

$$v(t) = \frac{\phi(u'(t))}{\phi(u(t))}\,.$$

Then v satisfies generalized Riccati equation (1.5) and integrating this equation from t_0 to t we get

$$v(t) = -(p-1)\int_{t_0}^{t} |v(s)|^q\,ds - \int_{t_0}^{t} c(s)\,ds\,.$$

By (2.3) there exist $\xi > 0$ and $T > t_0$ such that $\int\limits_{t_0}^{t} c(s)\,ds > \xi$ whenever $t > T$, hence for $t > T$, we have

$$v(t) \le -(p-1)\int_{t_0}^{t} |v(s)|^q\,ds - \xi\,.$$

Denote $R(t) := -(p-1)\int\limits_{t_0}^{t} |v(s)|^q\,ds - \xi$. Then for $t > T$ $v(t) \le R(t) \le -\xi$ and hence

$$R'(t) = -(p-1)|v(t)|^q \le -(p-1)|R(t)|^q\,.$$

This implies

$$\frac{R'(t)}{(p-1)|R(t)|^q} \le -1$$

and integrating this inequality from T to t we obtain

$$\frac{1}{(p-1)(q-1)|R(t)|^{q-1}} \le -t + T + \frac{1}{(p-1)(q-1)|R(T)|^{q-1}}$$

which leads to a contradiction if we let $t \to \infty$. $\square$

Theorem 3. *Suppose that $c(t) > 0$ on $[0,\infty)$. Then the solution of* (1.1) *given by the initial condition $u(0) = 1$, $u'(0) = 0$ has a zero point in the interval $I := [0, a + b^{-\frac{1}{p-1}}]$ provided that*

$$\int_0^a c(t)\,dt \geq b\,.$$

Proof. Again, we proceed by contradiction, i.e., suppose that $u(t) > 0$ in I. Then we have

$$\phi(u'(t)) - \phi(u'(0)) = -\int_0^t c(r)|u(r)|^{p-1}\,dr \leq 0\,, \quad u'(t) \leq 0,\ t \in I\,.$$

This inequality implies that (1.1) takes the form

$$-[|u'(t)|^{p-1}]' + c(t)u^{p-1}(t) = 0$$

and integrating this equation from $t = 0$ to $t = a$ we obtain

$$\begin{aligned}\left[|u'(t)|^{p-1}\right]_{t=0}^{t=a} &= |u'(a)|^{p-1} = \\ &= \int_0^a c(t)u^{p-1}(t)\,dt \geq (u(a))^{p-1}\int_0^a c(t)\,dt \geq (u(a))^{p-1}b\,.\end{aligned}$$

Hence $u'(a) \leq -u(a)b^{\frac{1}{p-1}}$. Since $u'(t)$ is decreasing, the graph of u lies below the line $y = u(a)\left[1 - b^{\frac{1}{p-1}}(t-a)\right]$ which crosses the t-axis at $t = a + b^{-\frac{1}{p-1}}$, consequently u must have also a zero point in this interval, a contradiction. $\square$

Theorem 4. *Suppose $c(t)$ is continuous and non-negative on the finite interval $I = [a, b)$. If* (1.1) *is disconjugate on this interval and for all solutions of* (1.1) *we have $\lim\limits_{t\to b^-} u(t) = 0$, then $\int\limits_a^b c(t)\,dt = \infty$.*

Proof. Suppose, by contradiction, that the statement does not hold. Then since $c(t) \geq 0$, the integral $\int\limits_a^t c(r)\,dr$ is monotonically increasing. This means that it must converge to some positive number as $t \to b^-$.

Let $t_0 \in [a, b)$. If we choose the solution u given by the initial condition $u(t_0) = 0$, $u'(t_0) > 0$, then $u(t) > 0$ for $t \in (t_0, b)$ and

$$0 \geq [\phi(u'(t))]' = (p-1)|u'(t)|^{p-2}u''(t)\,, \quad t \in [t_0, b)\,,$$

hence $u''(t) \leq 0$ for $t \in [t_0, b)$. This implies

$$u(t) \leq u'(t_0)(t - t_0) \leq u'(t_0)(b - t_0) \quad \text{for} \quad t \in [t_0, b)$$

and hence

$$\begin{aligned}\phi(u'(t)) = |u'(t)|^{p-1}\operatorname{sgn} u'(t) &= \phi(u'(t_0)) - \int_{t_0}^t c(r)u(r)^{p-1}\,dr \\ &\geq (u'(t_0))^{p-1}\left[1 - (b - t_0)^{p-1}\int_{t_0}^t c(r)\,dr\right]\,.\end{aligned}$$

Since $\lim_{t\to b^-} u(t) = 0$, $u'(t)$ and hence also $\phi(u'(t))$ must vanish for some $t \in [t_0, b)$. However, by choosing t_0 to be sufficiently close to b we can prevent this if the integral converges. Thus $\lim_{t\to b^-} \int_a^t c(r)\,dr$ must diverge. $\square$

Theorem 5. *Let $c(t)$ be continuous and $c(t) \geq 0$ on the finite interval $I = [a, b)$ and suppose*

$$\lim_{t\to b^-} \int_a^t \left(\int_a^s c(r)\,dr \right)^{\frac{1}{p-1}} ds = +\infty\,.$$

Then either (1.1) is oscillatory on $[a, b)$ or else all solutions $u(t)$ satisfy $\lim_{t\to b^-} u(t) = 0$ or both.

Proof. From hypothesis we have

$$\lim_{s\to b^-} \int_a^s c(t)\,dt = +\infty\,.$$

Suppose, by contradiction, that there exists a solution $u(t)$ such that $u(t) > 0$ in $[m, b)$ for some m, $a \leq m < b$, and $\lim_{t\to b^-} u(t) \geq d > 0$.

Let $M = \min[\inf_{m\leq t<b} u(t), d] > 0$. If $u' \geq 0$ in $[m, b)$, from (1.1) we obtain:

$$\begin{aligned}
\left([u'(t)]^{p-1}\right)' + c(t)u^{p-1}(t) &= 0\,, \quad t \in [m, b)\,,\\
u'(s)^{p-1} - u'(m)^{p-1} &= -\int_m^s c(t)u^{p-1}dt\,, \quad m \leq s < b\,,\\
u'(s)^{p-1} &= -\int_m^s c(t)u^{p-1}(t)\,dt + u'(m)^{p-1}
\end{aligned}$$

and the above equality will become negative as $s \to b^-$. This implies that $u'(s_0) < 0$ for some s_0 in $[m, b)$ and from (1.1) we obtain:

$$\begin{aligned}
(|u'(t)|^{p-1})' - c(t)u^{p-1}(t) &= 0\,, \quad s_0 \leq t < s_0 + \varepsilon,\ \varepsilon > 0\,,\\
|u'(s)|^{p-1} - |u'(s_0)|^{p-1} &- \int_{s_0}^s c(t)u^{p-1}(t)\,dt\,, \quad s_0 \leq s < b\,,\\
|u'(s)|^{p-1} &\geq M^{p-1} \int_{s_0}^s c(t)\,dt\,.
\end{aligned}$$

Hence

$$|u'(s)| \geq M \left[\int_{s_0}^s c(r)\,dr \right]^{\frac{1}{p-1}}$$

and thus

$$u(t) \leq u(s_0) - \int_{s_0}^t M \left(\int_{s_0}^s c(r)\,dr \right)^{\frac{1}{p-1}} ds\,.$$

This inequality together with hypothesis implies that $u(t)$ has a zero in $[s_0, b)$, contrary to the assumption. $\square$

Remarks.

(i) Consider a more general half-linear equation

$$[r(t)\phi(u')]' + c(t)\phi(u) = 0\,, \tag{2.4}$$

where r is a positive function. By a direct computation one can verify that the transformation of the independent variable

$$s = \int^t [r(s)]^{-\frac{1}{p-1}}\, ds \tag{2.5}$$

transforms (2.4) into the equation

$$\frac{d}{ds}\left[\phi\left(\frac{d}{ds}u\right)\right] + [r(t(s))]^{\frac{1}{p-1}} c(t(s))\phi(u) = 0\,,$$

where $t = t(s)$ is the inverse function of $s = s(t)$ given by (2.5). Consequently, using this transformation we have the following statement.

Theorem 6. *Suppose that $r(t) > 0$ for $t \in (a,b) \subset \mathbb{R}$ and*

$$\int_a [r(s)]^{-\frac{1}{p-1}}\, ds = \infty = \int^b [r(s)]^{-\frac{1}{p-1}}\, ds\,. \qquad \square$$

If $\int_a^b c(t)\, dt > 0$ then (2.4) possesses a nontrivial solution with at least two zeros in (a,b).

(ii) A closer examination of the proof of Theorem 2 reveals the fact that this statement remains valid if we replace (2.2) by a weaker requirement

$$\liminf_{t_1\to\infty, t_2\to\infty} \int_{t_1}^{t_2} c(t)\, dt > 0\,.$$

(iii) Observe that conjugacy criterion from Theorem 2 is really a focal point criterion. Indeed, the proof of this theorem establishes that there is a right focal point of t_0 in (t_0,∞) and similarly may be proved that $\int_{-\infty}^{t_0} c(t)\, dt > 0$ implies the existence of a left focal point in $(-\infty, t_0)$.

Recall that a point $t_2 > t_1$ is said to be the (right) focal point t_1 if there exists a solution u of (1.1) such that $u'(t_1) = 0$, $u(t_2) = 0$. If an interval $[t_1, b)$ contains no focal point of t_1, then (1.1) is said to be disfocal in this interval.

3. Distance between consecutive zeros

In this section we extend the result of B. J. Harris and Q. Kong [3].

Theorem 7. *If u is a solution of* (1.1) *satisfying $u'(d) = 0$, $u(b) = 0$ with $u(t) > 0$ and $u'(t) \le 0$ for $t \in (d, b)$, then*

$$\sup_{d \le t \le b} \int_d^t c(s)\,ds > 0\,.$$

Proof. Suppose the contrary. Let $Q(t) := \int_d^t c(s)\,ds \le 0$, $t \in [d, b]$ and define the Riccati variable

$$r(t) := -\frac{|u'(t)|^{p-2}u'(t)}{|u(t)|^{p-2}u(t)}\,, \tag{3.1}$$

we thus have:

$$r'(t) = c(t) + (p-1)|r(t)|^q\,, \qquad t \in [d, b) \tag{3.2}$$

$$\begin{gathered} r(d) = 0\,, \quad \lim_{t \to b^-} r(t) = \infty\,, \quad r(t) = (p-1)\int_d^t |r(s)|^q\,ds + Q(t) \\ t \in [d, b), \quad r(t) \ge 0\,. \end{gathered} \tag{3.3}$$

Since $Q(t) \le 0$ for $t \in [d, b]$ and $r(t) \ge 0$ for $t \in [d, b]$, we have $r(t) \le (p-1)\int_d^t (r(s))^q\,ds$, and so $r(t) = 0$, $t \in [d, b)$ as a simple consequence of the general theory of integral inequalities (we recall that $q > 1$), contrary to $\lim_{t \to b^-} r(t) = \infty$. The proof is now complete. $\square$

Theorem 7a. *If u is a solution of* (1.1) *satisfying $u(a) = 0$, $u'(b) = 0$ with $u(t) > 0$ and $u'(t) \ge 0$ for $t \in (a, b)$, then*

$$\sup_{a \le t \le b} \int_t^b c(s)\,ds > 0\,.$$

The proof is omitted.

Theorem 8. *Let $d < b$ and let u be a non-trivial solution of (1.1) satisfying $u'(d) = 0$, $u(b) = 0$, and suppose that $u(t) \ne 0$ for $t \in [d, b)$. Then we have*

$$(b-d)\,(q-1)\,(p-1)\left[\sup_{d \le t \le b}\left|\int_d^t c(s)\,ds\right|\right]^{q-1} > 1\,. \tag{3.4}$$

Moreover, if there are no extreme values of u in (d, b), then

$$(b-d)\,(q-1)\,(p-1)\left[\sup_{d \le t \le b}\int_d^t c(s)\,ds\right]^{q-1} > 1\,. \tag{3.5}$$

Proof. We assume, without loss of generality, that $u(t) > 0$ for $t \in [d, b)$. Let r be defined by

$$r(t) := -\frac{|u'(t)|^{p-2}u'(t)}{|u(t)|^{p-2}u(t)}, \qquad t \in [d, b)$$

and let

$$w(t) := (p-1)\int_d^t |r(s)|^q \, ds\,, \quad t \in [d, b) \tag{3.6}$$

with $r(t)$ satisfying

$$r'(t) - c(t) - (p-1)|r(t)|^q = 0\,, \quad t \in [d, b)\,,$$

or equivalently ,

$$r(t) = (p-1)\int_d^t |r(\alpha)|^q \, d\alpha + \int_d^t c(\alpha)\, d\alpha\,. \tag{3.7}$$

Thus, $r(d) = 0$, $w(d) = 0$, $\lim_{t\to b^-} r(t) = \infty$, $\lim_{t\to b^-} w(t) = \infty$ and

$$r(t) = w(t) + \int_d^t c(s)\, ds\,. \tag{3.8}$$

We set $Q^* := \sup_{d\le t\le b}\left|\int_d^t c(s)\, ds\right|$ and observe that $|r(t)| \le Q^* + w(t)$ and

$$w'(t) = (p-1)|r(t)|^q \le (p-1)(Q^* + w(t))^q$$

hence

$$\frac{w'(t)}{(p-1)(Q^* + w(t))^q} \le 1\,,$$

thus

$$\lim_{s\to b^-} \frac{1}{-(q-1)(p-1)[Q^* + w(t)]^{q-1}}\Bigg|_{t=d}^{s} \le (s-d)$$

and

$$\frac{1}{(q-1)(p-1)[Q^*]^{q-1}} \le b - d\,.$$

We remark that equality cannot hold, for otherwise

$$|Q(t)| := \left|\int_d^t c(s)\, ds\right| = Q^*\,, \quad t \in [d, b)$$

which contradicts the fact that Q is continuous and $Q(d) = 0$.

If d is the largest extreme point of u in $[d, b)$, then $u'(t) \le 0$ and thus $r(t) \ge 0$ for $t \in [d, b)$. We set $Q_* := \sup_{d\le t\le b} \int_d^t c(s)\, ds$. By Theorem 7, $Q_* > 0$; and from (3.8)

$$0 \le r(t) \le Q_* + w(t)\,.$$

The proof of the second part of the theorem now follows in a way similar to that of the first one. $\square$

Theorem 8a. *Let u denote a non-trivial solution of* (1.1) *satisfying $u(a) = 0$, $u'(c) = 0$, and $u(t) \neq 0$ for $t \in (a, c]$. Then*

$$(c-a)\,(p-1)\,(q-1)\sup_{a\le t\le c}\left|\int_t^c c(s)\,ds\right|^{q-1} > 1\,.$$

Moreover, if there are no extreme values of u in (a, c), then

$$(c-a)\,(p-1)\,(q-1)\left(\sup_{a\le t\le c}\int_t^c c(s)\,ds\right)^{q-1} > 1\,. \qquad \square$$

The proof of this result is similar to the proof of Theorem 8 and is omitted.

Theorem 9. *Let a and b denote two consecutive zeros of a non-trivial solution u of* (1.1) *and $q \geq 2$. Then there exist two disjoint subintervals of $[a, b]$, I_1 and I_2, satisfying both*

$$(b-a)\,(p-1)\,(q-1)\left(\int_{I_1\cup I_2} c(s)\,ds\right)^{q-1} > 4\,, \tag{3.9}$$

$$\int_{[a,b]\setminus(I_1\cup I_2)} c(s)\,ds \le 0\,. \tag{3.10}$$

Proof. Let c and d denote the least and greatest extreme points of u on $[a, b]$, respectively. If there is only one zero of u' in (a, b), then c and d coincide. Then $u'(d) = 0$, $u(b) = 0$, and $u(t) \neq 0$ for $t \in [d, b)$. By Theorem 8 the inequality (3.5) holds. There thus exists $b_1 \in (d, b]$ such that

$$(p-1)\,(q-1)\left(\int_d^{b_1} c(s)\,ds\right)^{q-1} > \frac{1}{b-d} \quad \text{and} \quad \int_d^{b_1} c(s)\,ds \ge \int_d^b c(s)\,ds\,.$$

Similarly, by Theorem 8a we can choose $a_1 \in [a, c)$ such that

$$(p-1)\,(q-1)\left(\int_{a_1}^{c} c(s)\,ds\right)^{q-1} > \frac{1}{c-a} \quad \text{and} \quad \int_{a_1}^{c} c(s)\,ds \ge \int_a^c c(s)\,ds\,.$$

Let $I_1 := [d, b_1]$, $I_2 := [a_1, c]$, and $q \geq 2$. We have

$$\begin{aligned}
&(p-1)\,(q-1)\,(b-a)\left(\int_{I_1\cup I_2} c(s)\,ds\right)^{q-1}\\
&= (p-1)\,(q-1)\,(b-a)\left[\int_{I_1} c(s)\,ds + \int_{I_2} c(s)\,ds\right]^{q-1}\\
&\ge (p-1)\,(q-1)\,(b-a)\left[\left(\int_{I_1} c(s)\,ds\right)^{q-1} + \left(\int_{I_2} c(s)\,ds\right)^{q-1}\right]
\end{aligned}$$

$$
\begin{aligned}
> (p-1)(q-1)(b-a)\left[\frac{1}{(c-a)(p-1)(q-1)} + \frac{1}{(b-d)(p-1)(q-1)}\right] \\
\geq \frac{b-a}{b-d} + \frac{b-a}{c-a} \\
\geq [(b-d)+(c-a)]\left[\frac{1}{b-d} + \frac{1}{c-a}\right] \\
\geq 2 + \frac{c-a}{b-d} + \frac{b-d}{c-a} \geq 4
\end{aligned}
$$

and (3.9) is verified. It is also easy to see that $\int_{b_1}^{b} c(s)\,ds \leq 0$ and $\int_{a}^{a_1} c(s)\,ds \leq 0$.

To verify (3.10) it is sufficient to show that $\int_c^d c(s)\,ds \leq 0$. Let $r(t)$ be defined as in Theorem 8. Since $u'(c) = u'(d) = 0$, we have $r(c) = r(d) = 0$ and $0 = r(d) - r(c) = \int_c^d c(s)\,ds + (p-1)\int_c^d |r(s)|^q ds$. This means that $\int_c^d c(s)\,ds \leq 0$ and hence that (3.10) holds. □

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