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CHARACTERIZATIONS OF INNER PRODUCT SPACES THROUGH AN ISOSCELES TRAPEZOID PROPERTY

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Abstract. Generalizing a property of isosceles trapezoids in the real plane into real normed spaces, a couple of characterizations of inner product spaces (i.p.s) are obtained.

1. Introduction

When working in non Euclidean geometries, most of the "natural" geometric properties may fail. In a general normed space some of these properties hold just when the space is an inner product space (i.p.s.). Some of these properties may characterize the space as an i.p.s. In the 3-dimensional case, this means that our world is as natural as we always have thought. The study of characterizations is a fruitful area and hundreds of them may be found in the literature [1, 2, 3, 4].

In this paper we consider a property, proved by F. Suzuki [5], of isosceles trapezoids in the real plane, we translate this property into a real normed space and then we study how the obtained condition can characterize the norm as a norm derivable from an inner product. We will translate this property in two different ways, obtaining two different characterizations of inner product spaces.

Suzuki's property states that, considering an isosceles trapezoid $ABCD$ in the real plane, $\mathbb{R}^2$, (see Fig. 1) for any point S in $\mathbb{R}^2$ we have:

$$\frac{SA^2 - SD^2}{AD} = \frac{SB^2 - SC^2}{BC} \tag{1}$$

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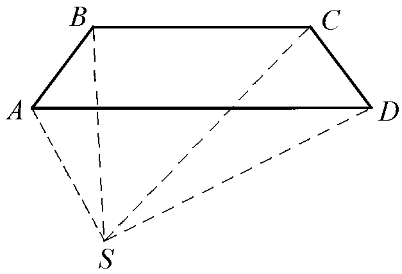

Fig. 1

In order to translate this property into a real normed space we will consider the two functions $\rho'_{\pm} : E \times E \to \mathbb{R}$ defined by

$$\rho'_{\pm}(x, y) = \lim_{t \to 0\pm} \frac{\|x + ty\|^2 - \|x\|^2}{2t}$$

which generalize the inner product in a real normed space. These two functions are half of the one-sided derivatives of the square of the norm at the point x in the direction of y. The limits exist by the convexity of the norm (see [4]).

The mappings $\rho'_{\pm}$ play a crucial role in characterizing inner product spaces (see [1, 2, 3, 4]). Indeed, when the norm derives from an inner product $(E, <, >)$, then $\rho'_{\pm}(x, y) = < x, y >$.

We quote here some elementary results concerning the functions $\rho'_{\pm}$:

(i) $\rho'_{\pm}(x, x) = \|x\|^2$ and $|\rho'_{\pm}(x, y)| \le \|x\|\|y\|$;
(ii) $\rho'_{+}(\alpha x, y) = \rho'_{+}(x, \alpha y) = \alpha\rho'_{+}(x, y)$, $\alpha \ge 0$;
(iii) $\rho'_{+}(\alpha x, y) = \rho'_{+}(x, \alpha y) = \alpha\rho'_{-}(x, y)$, $\alpha \le 0$;
(iv) $\rho'_{+}(x, \alpha x + y) = \rho'_{+}(x, y) + \alpha\|x\|^2$;
(v) $\rho'_{-}(x, y) \le \rho'_{+}(x, y)$.

If any of the following two conditions is verified then the norm in E derives from an inner product, i.e.: E is an i.p.s.

(I) $\rho'_{+}(u, v) \le \rho'_{+}(v, u)$ for all u, v unit vectors in E;
(II) $\rho'_{+}(x, y) = \rho'_{+}(y, x)$ for all x, y in E.

The above two conditions are the ones we will use in this paper to prove the characterization of inner product spaces. Some other chracterizations of i.p.s that use the functions $\rho'_{\pm}$ may be found in [4].

2. A first characterization of inner product spaces

We will first translate equality (1) into an inner product space considering vectors x, y, z and w as in figure 2.

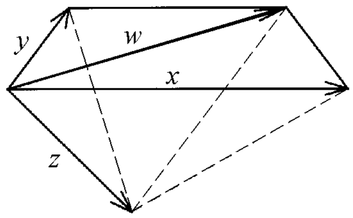

Fig. 2

Then equation (1) becomes:

$$\frac{||z||^2 - ||x-z||^2}{||x||} = \frac{||z-y||^2 - ||w-z||^2}{||w-y||} \tag{2}$$

Moreover, the isosceles condition for the trapezoid will be imposed as follows:

$$||w-x|| = ||y|| \text{ and } x = \lambda(w-y) \quad \text{with } \lambda \geq 0$$

which implies

$$\lambda = 1 \text{ or } \lambda = \frac{||x||^2}{||x||^2 - 2 < x, y >}$$

The case $\lambda = 1$ means that the two parallel sides of the trapezoid have the same length and then the trapezoid becomes a rectangle. This case was used to define in [3] an orthogonality relation in real normed spaces.

The other possibility for λ is now introduced in equation (2) obtaining:

$$\frac{||z||^2 - ||x-z||^2}{||x||} = \frac{||z-y||^2 - \left\| \frac{||x||^2 - 2<x,y>}{||x||^2} x + y - z \right\|^2}{||x|| \frac{||x||^2 - 2<x,y>}{||x||^2}} \tag{3}$$

Now we rewrite this equation in a real normed space, using the map ρ'_+ as the generalization of the inner product. Thus, equation (3) becomes in a real normed space:

$$||x||^2 (||z||^2 - ||x-z||^2) \left(||x||^2 - 2\rho'_+(x,y)\right) =$$

$$= ||x||^4 ||z-y||^2 - \left\| (||x||^2 - 2\rho'_+(x,y))x + ||x||^2 (y-z) \right\|^2 \tag{4}$$

Theorem 1. *Let $(E, ||\cdot||)$ be a real normed space. Then E is an i.p.s. if, and only if, for all vectors x, y, z in E equation (4) holds.*

Proof. To prove this theorem we proceed in four steps. First, in equation (4) we substitute x by tx with $t > 0$ and z by $-y$. By a straight forward computation and making t decrease down to 0 we obtain:

$$||x||^2 \rho'_+(y,x) \rho'_+(x,y) = ||x||^4 ||y||^2 - \left\| ||x||^2 y - \rho'_+(x,y) x \right\|^2$$

Taking u, v unit vectors in E, i.e. $u, v \in S_E$, this equality becomes:

$$\rho'_+(v,u) \rho'_+(u,v) = 1 - ||v - \rho'_+(u,v) u||^2$$

in particular $\frac{u+v}{||u+v||} \in S_E$ and changing v by $\frac{u+v}{||u+v||}$ we obtain:

$$1-\rho'_+(u,v)\rho'_+(v,u) = ||u+v||^2 - (1+\rho'_+(u,v))\rho'_+(u+v,u) \tag{5}$$

Secondly, we will use equation (4) again, where we substitute z for y:

$$\left(||x||^2 - 2\rho'_+(x,y)\right)\left(||x||^2 + ||y||^2 - ||x-y||^2 - 2\rho'_+(x,y)\right) = 0$$

From this equation we will obtain two different results, by the substitution: $x \to u+v$ and $y \to u$, where $u, v \in S_E$ we have:

$$\rho'_+(u+v,u) = \frac{||u+v||^2}{2} \tag{6}$$

and by the substitution: $x \to tx$ with $t > 0$ and doing the limit when t tends to 0, yields:

$$\rho'_+(x,y)\cdot\left(\rho'_+(x,y) - \rho'_-(y,x)\right) = 0 \tag{7}$$

In the third step, we prove that if a couple u, v of unit vectors verifies $\rho'_+(u,v) = 0$ then $\rho'_-(v,u) = 0$ is verified as well. Assuming $\rho'_+(u,v) = 0$, equation (5) becomes $\rho'_+(u+v,u) = ||u+v||^2 - 1$, then by (6) we have that $\rho'_+(u+v,u) = 1$, and therefore from (7) it is $\rho'_+(u+v,u) = \rho'_-(u,u+v) = 1 + \rho'_-(u,v) = 1$, so $\rho'_-(u,v) = 0$ and by (7), we obtain $\rho'_+(v,u) = 0$. From these results following exactly as we have done from the begining of this third part of the proof we arrive to $\rho'_-(v,u) = 0$.

Finally we use the previous result and the property (v) we quoted for ρ'_+:

$$\rho'_+(u,v) = \rho'_-(v,u) \leq \rho'_+(v,u)$$

Since this inequality holds for all $u, v \in S_E$ then E must be an i.p.s. by condition (I). □

3. A second characterization of inner product spaces

We will now translate the property of the isosceles trapezoid into a real normed space in a different way. To do this we consider the property (1) in an i.p.s. Let $h(a,b)$ be the height vector of a triangle (see Fig. 3a).

Using the height of the triangle formed by the two vectors $-y, x-y$ (see Fig. 3b), equation (1) can be written as:

$$\frac{||z||^2 - ||x-z||^2}{||x||} = \frac{||z-y||^2 - \left\|y - z + (||x|| - 2||y + h(-y,x-y)||)\frac{x}{||x||}\right\|^2}{||x|| - 2||y+h(-y,x-y)||} \tag{8}$$

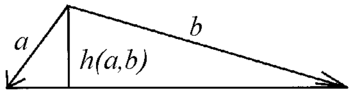

Fig. 3a

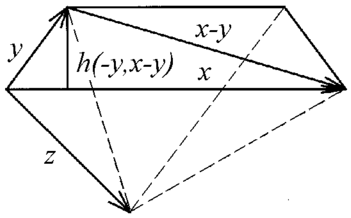

Fig. 3b

In this case, the obstacle to translate this equation into a real normed space is the height, $h(-y, x-y)$. We will use one of the several generalizations that Alsina, Guijarro and Tomás (see [1, 2]) have defined for heights in a triangle in real normed spaces:

$$h(x,y) = y + \frac{||y||^2 - \rho'_+(x,y)}{||x-y||^2}(x-y)$$

Let f be the following expression:

$$f(x,y) = 1 - 2\left|\frac{||x||^2 + ||y||^2 - ||x-y||^2 - \rho'_-(y,x)}{||x||^2}\right| \tag{9}$$

Equation (8) in the real normed space becomes:

$$(||z||^2 - ||x-z||^2)\cdot f(x,y) = ||y-z||^2 - ||y-z+f(x,y)x||^2 \tag{10}$$

In this case, the characterization is as follows:

Theorem 2. *Let $(E, ||\cdot||)$ be a real normed space such that equation (10) holds for all vectors x, y in E which satisfy $f(x,y) \neq 0$ and $\rho'_+(x,y) \geq 0$ then E is an i.p.s.*

Note: The negation of the first of these two conditions, $f(x,y) = 0$, corresponds to the case that the short parallel side of the trapezoid vanishes. Observe that equation (10) vanishes under this condition, too. The second condition, $\rho'_+(x,y) \geq 0$, is related to the angle of x and y.

Proof. Consider $x, y \in E$ such that $f(x,y) \neq 0$ and $\rho'_+(x,y) \geq 0$. Rewriting equation (10) for $z = x+y$ we obtain:

$$(||x+y||^2 - ||y||^2)\cdot f(x,y) = f(x,y)\cdot(2 - f(x,y))\cdot||x||^2 \tag{11}$$

Dividing by $f(x,y)$ and using the expression (9) of f, the above equation becomes:

$$||x+y||^2 - ||x||^2 - ||y||^2 = 2\left|||x||^2 + ||y||^2 - ||x-y||^2 - \rho'_+(y,x)\right| \tag{12}$$

Observe that $\lim_{t\to 0^+} f(x,ty) = 1$, so we can state that for any $t > 0$ small enough the couple x, ty satisfy equation (12) too. Therefore, we can change y by ty under the same conditions, and equation (12) can be rewritten, after being divided by $2t$, as:

$$\frac{\|x+ty\|^2-\|x\|^2}{2t}-t\frac{\|y\|^2}{2}=2\left|\frac{\|x-ty\|^2-\|x\|^2}{-2t}+\frac{t\|y\|^2}{2}-\frac{\rho'_-(y,x)}{2}\right|$$

making t decrease to 0, then we have

$$\rho'_+(x,y)=2\left|\rho'_-(x,y)-\frac{1}{2}\rho'_-(y,x)\right|$$

Similarly if we rewrite equation (10) for $z=y$ and consider the same remarks as before, substituting y by ty, with $t>0$, dividing by $2t$ and calculating the limit when $t\to 0^+$ we obtain:

$$\rho'_-(x,y)=2\left|\rho'_-(x,y)-\frac{1}{2}\rho'_-(y,x)\right|$$

We arrive at the conclusion that for all x,y in E such that $f(x,y)\neq 0$ and $\rho'_+(x,y)\geq 0$.

$$\rho'_+(x,y)=\rho'_-(x,y)\in\left\{\rho'_-(y,x),\frac{1}{3}\rho'_-(y,x)\right\}\tag{13}$$

Consider x,y in E such that $\rho'_+(x,y)\geq 0$. Using the same reasoning as we did above, we can state that for any $t>0$ small enough $f(x,ty)\neq 0$, so equation (13) becomes:

$$\rho'_+(x,ty)=\rho'_-(x,ty)\in\left\{\rho'_-(ty,x),\frac{1}{3}\rho'_-(ty,x)\right\}\tag{14}$$

Dividing by t we obtain that for all x,y in E such that $\rho'_+(x,y)\geq 0$ equation (13) is satisfied as well.

Using the above result we will prove that for all x,y in E such that $\rho'_+(x,y)\geq 0$, ρ'_+ is a symmetrical map. Let x,y be in E such that $\rho'_+(x,y)\geq 0$. By (13) we have that $\rho'_-(y,x)\geq 0$ too, and because of $\rho'_-\leq\rho'_+$ we deduce that $\rho'_+(y,x)\geq 0$, so, using (13) again for the couple $y,x\in E$ we have:

$$\rho'_+(y,x)=\rho'_-(y,x)\in\left\{\rho'_-(x,y),\frac{1}{3}\rho'_-(x,y)\right\}\tag{15}$$

And from (13) and (15) we can state that for all x,y in E such that $\rho'_+(x,y)\geq 0$ then $\rho'_+(x,y)=\rho'_+(y,x)$.

Finally, for all x,y in E such that $\rho'_+(x,y)\leq 0$ we have that $\rho'_+(-x,y)\geq 0$ and using (13) and (15) once more we obtain the symmetry of ρ'_+ for all x,y in E, and then, by condition II, E must be an i.p.s. $\square$

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