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EXACT CONTROLLABILITY OF THE WAVE EQUATION WITH MIXED BOUNDARY CONDITION AND TIME-DEPENDENT COEFFICIENTS

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Abstract. In this paper we study the boundary exact controllability for the equation

$$\frac{\partial}{\partial t}\left(\alpha(t)\frac{\partial y}{\partial t}\right) - \sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\beta(t)a(x)\frac{\partial y}{\partial x_j}\right) = 0 \quad \text{in } \Omega\times(0,T)\,,$$

when the control action is of Dirichlet-Neumann form and Ω is a bounded domain in $\mathbf{R}^n$. The result is obtained by applying the HUM (Hilbert Uniqueness Method) due to J. L. Lions.

1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^n$ with C^2 boundary Γ, Γ_0 a nonempty open set of Γ and let Q be the finite cylinder $\Omega\times(0,T)$ with lateral boundary $\Sigma=\Gamma\times(0,T)$. We consider the following system with inhomogenous boundary conditions:

$$\text{(1.1)}\qquad \left|\begin{array}{l} (\alpha(t)y')' + A(t)y = 0 \quad \text{in } Q \\ \dfrac{\partial y}{\partial \nu_A} = v \quad \text{on } \Sigma_0 = \Gamma_0\times(0,T) \\ y = 0 \text{ on } \Sigma_1 = \Sigma\backslash\Sigma_0 \\ y(0) = y^0 \text{ and } y'(0) = y^1 \text{ in } \Omega\,, \end{array}\right.$$

where

$$A(t) = -\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\beta(t)a(x)\frac{\partial}{\partial x_j}\right)\,.$$

We shall consider the particular case $\overline{\Gamma}_0\cap(\Gamma\backslash\Gamma_0)=\emptyset$, that is, the case where $\Omega=\Omega_0\backslash\overline{\Omega}_1$ and Ω_0 and Ω_1 are nonempty open sets with C^2 boundaries Γ_0 and

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Γ_1, respectively, with $\overline{\Omega}_1 \subset \Omega_0$. Thus, Γ possesses two disjointed components Γ_0 and Γ_1.

The problem of the exact controllability for the system (1.1) is formulated as follows:

"Given $T > 0$ large enough, for each pair of initial data $\{y^0, y^1\}$ defined in a suitable space, find a control v such that the solution $y = y(x,t)$ of (1.1) satisfies the conditions

$$y(T) = y'(T) = 0\,."$$

In this paper we shall prove that system (1.1) is exactly controllable by making use of HUM (Hilbert Uniqueness Method) c.f. J. L. Lions [14]. For this, we employ the multiplier technique to obtain the inverse inequality.

When the coefficients depend on time, with suitable hypotheses on them, the inverse inequality still remains true but since standard arguments are not applicable, the regularity of the backward problem requires a new proof, which is the main task of this work.

We note that when $\alpha(t) = \beta(t) = a(x) = 1$, problem (1.1) was studied by J. L. Lions [14] using HUM and also by I.Lasiecka and R.Triggiani [13] using the ontoness approach. Many other authors have used HUM in the study of exact controllability of distributed systems with time-dependent or x-dependent coefficients. Among them, we mention J.Lagnese [11], C.Bardos, G.Lebeau and J.Rauch [2], V.Komornik [8], R. Fuentes [5], L. A. Medeiros [16], M. Milla Miranda [17], M. Milla Miranda and L. A. Medeiros [18], J. A. Soriano [19].

The goal of this work is to show that HUM can be applied to the case of *time-dependent coefficients* with mixed boundary condition. In fact we shall consider the wave equation for the following simple operators

$$A(t) = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\beta(t)a(x)\frac{\partial}{\partial x_j}\right).$$

However, with appropriated changes, we will obtain analogous results to the operator given by

$$A(t) = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(a(x,t)\frac{\partial}{\partial x_j}\right),$$

with $a(x,t) \geq \xi_0 > 0$ in $\overline{\Omega} \times (0,\infty)$. But, when we have matricial operators like

$$A(t) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j}(x,t)\frac{\partial}{\partial x_j}\right)$$

the usual arguments cannot be applied even if $i = j$ and $a_{i,j}(x,t) = a_i(x)$.

Our paper is divided into sixth sections. In section 2 we give notations and state the principal result. In section 3 we consider the homogeneous problem and in section 4 we establish the inverse inequality. In section 5 we study the backward problem and in the last section, section 6, we apply HUM.

2. Notations and Main Result

Let $x^0 \in \mathbf{R}^n$, $m(x) = x - x^0$ $(x \in \mathbf{R}^n)$ and $\nu(x)$ the unit exterior normal vector at $x \in \Gamma$, and

$$R(x^0) = \max\{\, \|m(x)\|;\ x \in \overline{\Omega}\,\}\,.$$

We define:

$$\Gamma(x^0) = \{\, x \in \Gamma;\ m(x) \cdot \nu(x) > 0\,\}\,,$$

$$\Gamma_*(x^0) = \{\, x \in \Gamma;\ m(x) \cdot \nu(x) \leq 0\} = \Gamma \backslash \Gamma(x^0)\,,$$

$$\Gamma_{i,*} = \Gamma_i \cap \Gamma_*(x^0); i = 0, 1\,.$$

In what follows we consider Ω_1 "star-shaped with respect to x_0", that is, there exist a point $x_0 \in \Omega_1$ such that $\Gamma_{1,*} = \Gamma_1$.

Remark 1. We are not considering that Ω_0 is star-shaped with respect to x^0 in order that $\Gamma_{0,*}$ is not necessarily equal to Γ_0. In fact we have $\Gamma(x^0) \cup \Gamma_{0,*} = \Gamma_0$.

We consider:

$$\Sigma(x^0) = \Gamma(x^0) \times (0, T)$$

and

$$\Sigma_*(x^0) = \Gamma_*(x^0) \times (0, T) = \Sigma \backslash \Sigma(x^0)\,.$$

Let us introduce some notations that will be used through this work.
We define:

$$V = \{u \in H^1(\Omega); u = 0 \text{ on } \Gamma_1\}$$

which is a Hilbert space of $H^1(\Omega)$.

By $(\cdot, \cdot)$ and $|\cdot|$ we denote the inner-product and the norm of $L^2(\Omega)$ respectively. The norm in V will be denoted by $\|\cdot\|$.

Let A be the operator defined by the triple $\{V, L^2(\Omega), a(u, v)\}$ where

$$a(u, v) = \sum_{j=1}^{n} \int_\Omega a(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j}\, dx \quad \forall\, u, v \in V$$

and

$$D(A) = \{u \in H^2(\Omega) \cap V;\ \frac{\partial u}{\partial \nu_A} = 0 \ \text{ on } \Gamma_0\}\,.$$

We recall that the Spectral Theorem for self-adjointed operators guarantees the existence of a complete orthonormal system (ω_ν) of $L^2(\Omega)$ given by the eigenfunctions of A. If (λ_ν) are the correponding eigenvalues of A, then $\lambda_\nu \to +\infty$ as $\nu \to +\infty$. Besides,

$$D(A) = \left\{ u \in L^2(\Omega);\ \sum_{\nu=1}^{+\infty} \lambda_\nu^2 |(u, \omega_\nu)|^2 < +\infty \right\}$$

and

$$Au = \sum_{\nu=1}^{+\infty} \lambda_\nu (u, \omega_\nu)\omega_\nu, \quad \forall\, u \in D(A)\,.$$

Considering in $D(A)$ the norm given by the graph, that is,

$$\|u\|_{D(A)} = (|u|^2 + |Au|^2)^{\frac{1}{2}}\,,$$

it turns out that (ω_ν) is a complete system in $D(A)$. In fact, it is known that (ω_ν) is also a complete system in V. Now, since A is positive, given $\delta > 0$ one has

$$D(A^\delta) = \left\{ u \in L^2(\Omega);\ \sum_{\nu=1}^{+\infty} \lambda_\nu^{2\delta} |(u, \omega_\nu)|^2 < +\infty \right\}$$

and

$$A^\delta u = \sum_{\nu=1}^{+\infty} \lambda_\nu^\delta (u, \omega_\nu)\omega_\nu, \quad \forall\, u \in D(A^\delta)\,.$$

In $D(A^\delta)$ we consider the natural topology given by the norm $\|u\|_{D(A^\delta)} = (|u|^2 + |A^\delta u|^2)^{\frac{1}{2}}$. From the Spectral Theory one notes that such operators are also self-adjoints, that is,

$$(A^\delta u, v) = (u, A^\delta v) \quad \forall\, u, v \in D(A^\delta)\,,$$

$D(A^{1/2}) = V$ and $D(A^0) = L^2(\Omega)$. We also observe that $A(t) = \beta(t)A$. Here, we are using the same symbol for both operators to simplify the notation.

We are going to consider the following hypotheses:

$(H1)$: $\alpha, \beta \in W^{1,\infty}_{loc}(0,\infty)$, $\alpha', \beta' \in L^1(0,\infty)$,

$$\alpha(t) \geq \alpha_0 > 0 \quad \text{and} \quad \beta(t) \geq \beta_0 > 0, \quad \forall\, t \geq 0\,,$$

and $a \in C^1(\overline{\Omega})$ with $a(x) \geq a_0 > 0$, $\forall\, x \in \overline{\Omega}$.

$(H2)$: If $n > 1$

$$\|\nabla a\|_{C^0(\overline{\Omega})} < a_0[R(x^0)]^{-1}\,.$$

$(H3)$: If $n = 1$

$$\exists\, 0 < \gamma < 1 \ \text{ such that } \ \|\nabla a\|_{C^0(\overline{\Omega})} < \gamma a_0 [R(x^0)]^{-1}\,.$$

Now we are in a position to state our main result. Consider the following system:

$$\text{(2.1)} \qquad \left| \begin{array}{l} (\alpha(t)y')' + A(t)y = 0 \quad \text{in } Q \\ \dfrac{\partial y}{\partial \nu_A} = \left| \begin{array}{l} v_0 \text{ on } \Sigma(x^0) \\ v_1 \text{ on } \Sigma_{0,*}(x^0) \end{array} \right. \\ y = 0 \text{ on } \Sigma_1 \\ y(0) = y^0 \ \text{ and } \ y'(0) = y^1 \ \text{ in } \ \Omega\,. \end{array} \right.$$

Theorem 2.1. *Suppose that assumptions* $(H1)$-$(H3)$ *are satisfied. Then there exists a time* $T_0 > 0$ *such that for* $T > T_0$ *and initial data* $\{y^0, y^1\} \in L^2(\Omega) \times V'$, *there exists a control*

$$v_0 \in [H^1(0,T; L^2(\Gamma(x^0))]' \text{ and } v_1 \in L^2(0,T; [H^1(\Gamma_{0,*}(x^0))]')$$

such that the ultra-weak [1] *solution* $y = y(x,t)$ *of (2.1) satisfies*

$$y(T) = y'(T) = 0\,.$$

The proof of this theorem will be developed in the following sections.

3. The Homogeneous Problem

We begin this section presenting a standard result for the solutions to the following homogeneous problem.

$$\left|\begin{array}{l} (\alpha(t)\theta')' + A(t)\theta = f \quad \text{in } Q \\ \dfrac{\partial \theta}{\partial \nu_A} = 0 \quad \text{on } \Sigma_0 \\ \theta = 0 \text{ on } \Sigma_1 \\ \theta(0) = \theta^0 \quad \text{and} \quad \theta'(0) = \theta^1 \quad \text{in } \Omega\,. \end{array}\right. \tag{3.1}$$

We have the following results.

Theorem 3.1. *Suppose that assumption* $(H1)$ *holds. Then given* $k \in \{0,1,2\}$ *and*

$$\{\theta^0, \theta^1, f\} \in D(A^{\frac{k+1}{2}}) \times D(A^{\frac{k}{2}}) \times L^1(0,T; D(A^{\frac{k}{2}})\,,$$

the problem (3.1) *possesses a unique solution* $\theta : Q \to \mathbf{R}$ *in the class*

$$\theta \in C^0([0,T]; D(A^{\frac{k+1}{2}})) \cap C^1([0,T]; D(A^{\frac{k}{2}}))\,.$$

Moreover, the linear map

$$\begin{array}{rcl} D(A^{\frac{k+1}{2}} \times D(A^{\frac{k}{2}}) \times L^1(0,T; D(A^{\frac{k}{2}}) & \to & C^0([0,T]; D(A^{\frac{k+1}{2}})) \times C^1([0,T]; D(A^{\frac{k}{2}})) \\ \{\theta^0, \theta^1, f\} & \mapsto & \{\theta, \theta'\} \end{array}$$

is continuous.

Theorem 3.1 can be proved in a standard way by applying the Faedo-Galerkin Method and using the spectral considerations given in section 2.

[1] The solution of (2.1) is defined by the transposition method, see J. L. Lions and E. Magenes [14].

Next we consider the homogeneous problem

$$
(3.2) \qquad \left| \begin{array}{l} (\alpha(t)\theta')' + A(t)\theta = f' \quad \text{in } Q \\ \dfrac{\partial \theta}{\partial \nu_A} = 0 \quad \text{on } \Sigma_0 \\ \theta = 0 \text{ on } \Sigma_1 \\ \theta(0) = \theta'(0) = 0 \text{ in } \Omega\,, \end{array} \right.
$$

that will be used in the study of the regularity of the solution of (2.1).

Theorem 3.2. *Given $f \in \mathcal{D}(0,T,D(A))$, the unique solution of problem (3.2) satisfies for all $t \in [0,T]$*

$$
|\alpha^{\frac{1}{2}} A^{\frac{1}{2}} \theta'(t) - \alpha^{\frac{-1}{2}} A^{\frac{1}{2}} f(t)|_{L^2(\Omega)} + |A\theta(t)|_{L^2(\Omega)} \leq C\|f\|_{L^1(0,T;D(A))}
$$

and

$$
|\alpha^{\frac{1}{2}} \theta'(t) - \alpha^{\frac{-1}{2}} f(t)|_{L^2(\Omega)} + |A^{\frac{1}{2}} \theta(t)|_{L^2(\Omega)} \leq C\|f\|_{L^1(0,T;V)}\,.
$$

Proof. Since $\theta^0 = \theta^1 = 0$ and $f' \in \mathcal{D}(0,T,D(A))$, from Theorem 3.1 the above problem has a unique solution θ such that

$$
(3.3) \qquad \theta \in C^0([0,T]; D(A^{\frac{3}{2}})) \cap C^1([0,T]; D(A))\,.
$$

Besides, such a solution satisfies the identity

$$
\begin{aligned}
\frac{1}{2}\left\{\alpha(t)|A^{\frac{1}{2}}\theta'(t)|^2 + \beta(t)|A\theta(t)|^2\right\} =& \frac{1}{2}\int_0^t \beta'(s)|A\theta(s)|^2 ds - \frac{1}{2}\int_0^t \alpha'(s)|A\theta'(s)|^2 ds \\
(3.4) \qquad & + \int_0^t (A^{\frac{1}{2}} f'(s), A^{\frac{1}{2}}\theta'(s))\, ds\,.
\end{aligned}
$$

From (3.3) we get $A\theta \in C^0([0,T]; D(A^{\frac{1}{2}}))$ and therefore

$$
(\alpha\theta')' \in L^\infty(0,T; D(A^{\frac{1}{2}}))\,.
$$

This togheter with assumption $(H1)$ implies that

$$
\begin{aligned}
\frac{d}{ds}\left(\alpha^{-1}(s)A^{\frac{1}{2}}f(s), \alpha(s)A^{\frac{1}{2}}\theta'(s)\right) =& -\left(\frac{\alpha'(s)}{\alpha^2(s)}A^{\frac{1}{2}}f(s), \alpha(s)A^{\frac{1}{2}}\theta'(s)\right) \\
& + \left(\alpha^{-1}(s)A^{\frac{1}{2}}f'(s), \alpha(s)A^{\frac{1}{2}}\theta'(s)\right) \\
& + \left(\alpha^{-1}(s)A^{\frac{1}{2}}f(s), A^{\frac{1}{2}}[(\alpha(s)\theta'(s))']\right)\,.
\end{aligned}
$$

Integrating this equality and noting that $f(0)=0$ we have

$$\begin{aligned}\int_0^t (A^{\frac{1}{2}}f'(s), A^{\frac{1}{2}}\theta'(s))ds = &\left(\alpha^{-1}(t)A^{\frac{1}{2}}f(t), \alpha(t)A^{\frac{1}{2}}\theta'(t)\right)\\ &+\int_0^t \left(\alpha'(s)\alpha^{-1}(s)A^{\frac{1}{2}}f(s), A^{\frac{1}{2}}\theta'(s)\right) ds\\ &-\int_0^t \left(\alpha^{-1}(s)A^{\frac{1}{2}}f(s), A^{\frac{1}{2}}[(\alpha(s)\theta'(s))']\right) ds\,.\end{aligned}$$

Replacing $(\alpha\theta')'$ by $f'-\beta A\theta$ in the last integral we obtain

$$\begin{aligned}\int_0^t (A^{\frac{1}{2}}f'(s), A^{\frac{1}{2}}\theta'(s))ds = &\left(A^{\frac{1}{2}}f(t), A^{\frac{1}{2}}\theta'(t)\right)\\ &+\int_0^t \left(\alpha'(s)\alpha^{-1}(s)A^{\frac{1}{2}}f(s), A^{\frac{1}{2}}\theta'(s)\right) ds\\ &-\int_0^t \left(\alpha^{-1}(s)A^{\frac{1}{2}}f(s), A^{\frac{1}{2}}f'(s)\right) ds\\ &+\int_0^t \left(\alpha^{-1}(s)A^{\frac{1}{2}}f(s), \beta(s)A^{\frac{1}{2}}[A\theta(s)]\right) ds\,.\end{aligned}\tag{3.5}$$

Now integrating by parts and noting that $f(0)=0$,

$$\begin{aligned}\int_0^t \left(\alpha^{-1}(s)A^{\frac{1}{2}}f(s), A^{\frac{1}{2}}f'(s)\right) ds = &\frac{1}{2}\left(\alpha^{-1}(t)A^{\frac{1}{2}}f(t), A^{\frac{1}{2}}f(t)\right)\\ &+\frac{1}{2}\int_0^t \left(\alpha'(s)\alpha^{-2}(s)A^{\frac{1}{2}}f(s), A^{\frac{1}{2}}f(s)\right) ds\,.\end{aligned}\tag{3.6}$$

Replacing (3.6) into (3.5) we have,

$$\begin{aligned}\int_0^t (A^{\frac{1}{2}}f'(s), A^{\frac{1}{2}}\theta'(s))ds = &\left(A^{\frac{1}{2}}f(t), A^{\frac{1}{2}}\theta'(t)\right)\\ &+\int_0^t \left(\alpha'(s)\alpha^{-1}(s)A^{\frac{1}{2}}f(s), A^{\frac{1}{2}}\theta'(s)\right) ds\\ &-\frac{1}{2}\left(\alpha^{-1}(t)A^{\frac{1}{2}}f(t), A^{\frac{1}{2}}f(t)\right)\\ &-\frac{1}{2}\int_0^t \left(\alpha'(s)\alpha^{-2}(s)A^{\frac{1}{2}}f(s), A^{\frac{1}{2}}f(s)\right) ds\\ &+\int_0^t \left(\alpha^{-1}(s)A^{\frac{1}{2}}f(s), \beta(s)A^{\frac{1}{2}}[A\theta(s)]\right) ds\,.\end{aligned}\tag{3.7}$$

From (3.4) and (3.7) it follows that

$$
\begin{aligned}
\frac{1}{2}|\alpha^{\frac{1}{2}}(t)A^{\frac{1}{2}}\theta'(t) - \alpha^{\frac{-1}{2}}(t)A^{\frac{1}{2}}f(t)|^2 + \frac{1}{2}\beta(t)|A\theta(t)|^2 = {} & \frac{1}{2}\int_0^t \beta'(s)|A\theta(s)|^2 ds \\
& - \frac{1}{2}\int_0^t \alpha'(s)|A^{\frac{1}{2}}\theta'(s)|^2 ds \\
& + \int_0^t \Big(\alpha'(s)\alpha^{-1}(s)A^{\frac{1}{2}}f(s), A^{\frac{1}{2}}\theta'(s)\Big)\, ds \\
& - \frac{1}{2}\int_0^t \Big(\alpha'(s)\alpha^{-2}(s)A^{\frac{1}{2}}f(s), A^{\frac{1}{2}}f(s)\Big)\, ds \\
& + \int_0^t \Big(\alpha^{-1}(s)A^{\frac{1}{2}}f(s), \beta(s)A^{\frac{1}{2}}[A\theta(s)]\Big)\, ds\,.
\end{aligned}
$$

Defining $\alpha^{\frac{1}{2}}\theta' - \alpha^{\frac{-1}{2}}f = \varphi$ and replacing θ' by $\alpha^{\frac{-1}{2}}\varphi + \alpha^{-1}f$ in the above expression we have

$$
\begin{aligned}
\frac{1}{2}|A^{\frac{1}{2}}\varphi(t)|^2 + \frac{1}{2}\beta(t)|A\theta(t)|^2 = {} & \frac{1}{2}\int_0^t \beta'(s)|A\theta(s)|^2 ds \\
& - \frac{1}{2}\int_0^t \alpha'(s)\alpha^{-1}|A^{\frac{1}{2}}\varphi(s)|^2 ds \\
& + \int_0^t \alpha^{-1}(s)\beta(s)\left(Af(s), A\theta(s)\right) ds\,.
\end{aligned}
\tag{3.8}
$$

From the hypotheses $(H1)$, $(H2)$ and (3.8) there exists a constant $C > 0$ independent of f and θ such that

$$
\begin{aligned}
& \frac{1}{2}|A^{\frac{1}{2}}\varphi(t)|^2 + \frac{1}{2}|A\theta(t)|^2 \\
& \leq C\left\{\int_0^t |A\theta(s)|^2 ds + \frac{1}{2}\int_0^t |A^{\frac{1}{2}}\varphi(s)|^2 ds + \int_0^t |Af(s)|[|A\theta(s)| + |A^{\frac{1}{2}}\varphi(s)|]ds\right\}.
\end{aligned}
$$

Applying the Gronwall's inequality it follows that

$$
|A^{\frac{1}{2}}\varphi(t)| + |A\theta(t)| \leq C\|f\|_{L^1(0,T;D(A))} \quad \forall\, t \in [0,T]\,.
$$

In a similar way we also infer that

$$
|\varphi(t)| + |A\theta(t)| \leq C\|f\|_{L^1(0,T;V)} \quad \forall\, t \in [0,T]\,.
$$

Using the definition of φ by its definition we get the desired inequalities. $\square$

4. The Inverse Inequality

In this section we construct an special T_0 time depending on n, $R(x^0)$, on the functions $\alpha(t)$, $\beta(t)$, $a(t)$, and also on the geometry of Ω.

Taking into account the regularity of Γ, we can define on Γ a unit exterior normal vector field $\nu(x)$ of class C^1. In the same way we can define a family of $(n-1)$ tangents vector field $\{\tau^1(x), \cdots, \tau^{n-1}(x)\}$ of class C^1 such that the family $\{\nu(x), \tau^1(x), \cdots, \tau^{n-1}(x)\}$ defines a orthonormal basis for $\mathbf{R}^n$, for all $x \in \Gamma$. If $\varphi : \overline{\Omega} \to \mathbf{R}$ is a regular function, we have

$$\frac{\partial \varphi}{\partial x_j} = \nu_j \frac{\partial \varphi}{\partial \nu} + \sum_{k=1}^{n-1} \tau_j^k \frac{\partial \varphi}{\partial \tau^k} \quad \text{on } \Gamma, \quad j = 1, \cdots, n\,, \tag{4.1}$$

where

$$\frac{\partial \varphi}{\partial \nu} = \nabla \varphi \cdot \nu \quad \text{and} \quad \frac{\partial \varphi}{\partial \tau^k} = \nabla \varphi \cdot \tau^k\,.$$

Defining

$$\sigma_j \varphi = \sum_{k=1}^{n-1} \tau_j^k \frac{\partial \varphi}{\partial \tau^k} \tag{4.2}$$

we obtain from (4.1) and (4.2)

$$\frac{\partial \varphi}{\partial x_j} = \nu_j \frac{\partial \varphi}{\partial \nu} + \sigma_j \varphi \quad \text{on } \Gamma, \quad j = 1, \cdots, n\,. \tag{4.3}$$

We observe that when $\frac{\partial \varphi}{\partial \nu_A} = 0$ on Γ_0 then $\frac{\partial \varphi}{\partial \nu} = 0$ since

$$\frac{\partial \varphi}{\partial \nu_A} = a(x) \frac{\partial \varphi}{\partial \nu} \quad \text{on } \Gamma_0, \text{ and } a(x) \geq a_0 > 0$$

Then, defining $\nabla_\sigma \varphi = (\sigma_1 \varphi, \cdots, \sigma_n \varphi)$, we obtain from (4.3)

$$\nabla_\sigma \varphi = \nabla \varphi \ \text{ on } \ \Gamma_0\,, \tag{4.4}$$

and consequently,

$$|\nabla \varphi|^2 = |\nabla_\sigma \varphi|^2 = \sum_{j=1}^{n} |\sigma_j \varphi|^2 \ \text{ on } \ \Gamma_0\,. \tag{4.5}$$

Remark 2. In this point we observe that when A is a matricial operator, that is, when it is given by

$$A(t) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x,t) \frac{\partial}{\partial x_j} \right)$$

then we have

$$\frac{\partial y}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial y}{\partial x_j} \nu_i$$

and therefore if $\frac{\partial y}{\partial \nu_A} = 0$ we don't have necessarely that $\frac{\partial y}{\partial \nu} = 0$ and consequently we can not use the identity

$$|\nabla y|^2 = |\nabla_\sigma y|^2 \text{ on } \Sigma_0$$

even if $i = j$ and $a_{ij}(x,t) = a_j(x)$. As this identity is fundamental to prove the inverse inequality, this case requires another treatment which will not be considered in this work.

If $\varphi \in H^2(\Omega)$ we can define in a natural way a continuous linear operator

$$\sigma_j^1 : H^2(\Omega) \to H^{\frac{1}{2}}(\Gamma) \tag{4.6}$$

such that

$$\sigma_j^1 \varphi = (\sigma_j \varphi)\,|_{\Gamma_0} \text{ on } \Gamma\,, \ \ \forall\, \varphi \in C^2(\overline{\Omega})\,. \tag{4.7}$$

In addition, we can also consider a continuous linear operator

$$\sigma_j^2 : H^1(\Gamma_0) \to L^2(\Gamma_0) \tag{4.8}$$

where Γ_0 is a nonempty open subset of Γ (sometimes the hole Γ) such that

$$\sigma_j^2 \varphi|_{\Gamma_0} = (\sigma_j \varphi)\,|_{\Gamma_0} \text{ on } \Gamma_0, \ \ \forall\, \varphi \in C^2(\overline{\Omega})\,. \tag{4.9}$$

Thus, from (4.7) and (4.9) and by density arguments it results that

$$(\sigma_j^1 u)\,|_{\Gamma_0} = \sigma_j^2(u|_{\Gamma_0}) \text{ on } \Gamma_0, \ \ \forall\, u \in H^2(\Omega)\,. \tag{4.10}$$

Considering the above equality we are able to define the tangential gradient

$$\nabla_\sigma u = ((\sigma_1^1 u)|_{\Gamma_0}, \cdots, (\sigma_n^1 u)|_{\Gamma_0}) = (\sigma_1^1 u|_{\Gamma_0}, \cdots, \sigma_n^1 u|_{\Gamma_0}), \quad \forall\, u \in H^2(\Omega)\,.$$

Dropping the index "2" in (4.8) to simplify the notation, we define the adjoint operator

$$\sigma_j^* : L^2(\Gamma_0) \to (H^1(\Gamma_0))'$$

$$\langle \sigma_j^* \psi, \varphi \rangle = (\psi, \sigma_j \varphi)_{L^2(\Gamma_0)} \quad \forall\, \varphi \in H^1(\Gamma_0)\,, \tag{4.11}$$

and from (4.8) and (4.11) we define the continuous linear operator

$$-\Delta_{\Gamma_0} : H^1(\Gamma_0) \to (H^1(\Gamma_0))'$$

$$\varphi \longmapsto -\Delta_{\Gamma_0}\varphi = \sum_{j=1}^{n} (\sigma_j^* \circ \sigma_j)\varphi\,.$$

Hence for all $\varphi, \psi \in H^1(\Gamma_0)$,

$$\langle -\Delta_{\Gamma_0}\varphi, \psi \rangle = \int_{\Gamma_0} \nabla_\sigma \varphi \nabla_\sigma \psi \, d\Gamma\,. \tag{4.12}$$

In particular,

$$\langle -\Delta_{\Gamma_0}\varphi, \varphi \rangle = \int_{\Gamma_0} |\nabla_\sigma \varphi|^2 \, d\Gamma\,. \tag{4.13}$$

Theorem 4.1. *Let θ be the weak solution of the problem* (3.1), *that is,* $\{\theta^0, \theta^1\} \in V \times L^2(\Omega)$. *Then if* $f = 0$,

$$e^{-C_0} E(0) \le E(t) \le e^{C_0} E(0) \quad \forall\, t \ge 0\,,$$

where

$$C_0 = \max\{\alpha_0^{-1}, \beta_0^{-1}\} \int_0^{+\infty} (|\alpha'(t)| + |\beta'(t)|)dt$$

and

$$E(t) = \frac{1}{2}\left[\int_\Omega \alpha(t)|\theta'(x,t)|^2 dx + \int_\Omega \beta(t)a(x)|\nabla\theta(x,t)|^2 dx\right]. \tag{4.14}$$

Proof. We suppose first that $\{\theta^0, \theta^1\} \in D(A) \times V$. Then, in view of Theorem 3.1, there exists a unique solution θ in the class

$$\theta \in C^0([0,T]; D(A)) \cap C^1([0,T]; V)\,.$$

Multiplying $(3.1)_1$ by $\theta'(t)$ we obtain

$$\alpha'(t)|\theta'(t)|^2 + \alpha(t)\frac{1}{2}\frac{d}{dt}|\theta'(t)|^2 + \beta(t)\frac{1}{2}\frac{d}{dt}|a^{\frac{1}{2}}(x)\nabla\theta(t)|^2 = 0\,.$$

Integrating this relation from 0 to t and then integrating by parts we get

$$\frac{1}{2}\left[\alpha'(t)|\theta'(t)|^2 + \beta(t)|a^{\frac{1}{2}}(x)\nabla\theta(t)|^2\right] = \frac{1}{2}\left[\alpha(0)|\theta^1|^2 + \beta(0)|a^{\frac{1}{2}}(x)\nabla\theta^0|^2\right]$$
$$-\frac{1}{2}\int_0^t \alpha'(s)|\theta'(s)|^2 ds + \frac{1}{2}\int_0^t \beta'(s)|a^{\frac{1}{2}}(x)\nabla\theta(s)|^2 ds\,.$$

Taking (4.14) into account we can rewrite the above expression as follows.

$$0 \le E(t) = E(0) - \frac{1}{2}\int_0^t \alpha'(s)|\theta'(s)|^2 ds + \frac{1}{2}\int_0^t \beta'(s)|a^{\frac{1}{2}}(x)\nabla\theta(s)|^2 ds\,,$$

On the other hand, differentiating $E(t)$ we have

$$E'(t) = -\frac{1}{2}\alpha'(t)|\theta'(t)|^2 + \frac{1}{2}\beta'(t)|a^{\frac{1}{2}}(x)\nabla\theta(t)|^2\,,$$

and therefore

$$|E'(t)| \le \max\{\alpha_0^{-1}, \beta_0^{-1}\}[\,|\alpha'(t)| + |\beta'(t)|\,]\,[\,\alpha(t)|\theta'(t)|^2 + \beta(t)|a^{\frac{1}{2}}(x)\nabla\theta(t)|^2\,]\,,$$

So

$$|E'(t)| \le G(t)E(t)\,,$$

where

$$G(t) = \max\{\alpha_0^{-1}, \beta_0^{-1}\}[\,|\alpha'(t)| + |\beta'(t)|\,]\,.$$

The above inequality gives,

$$-G(t)E(t) \le E'(t) \le G(t)E(t)\,. \tag{4.15}$$

Now, considering

$$C_0 = \int_0^{+\infty} G(s)ds$$

it follows from (4.15) that

$$e^{-C_0} E(0) \leq E(t) \leq e^{C_0} E(0) \quad \forall\, t \geq 0\,.$$

Finally, assuming

$$\{\theta^0, \theta^1\} \in V \times L^2(\Omega)$$

we obtain the desired result using density arguments. □

Theorem 4.2. *Let $q = (q_k)_{1 \leq k \leq n}$ be a vector field such that $q \in [C^1(\overline{\Omega})]^n$. Then each weak solution ϕ of problem* (3.1) *satisfies:*

$$\begin{aligned}
&\frac{1}{2}\int_{\Sigma_0} q_k \nu_k [\alpha(t)|\phi'(t)|^2 - \beta(t)a(x)|\nabla_\sigma \phi|^2]\, d\Sigma + \frac{1}{2}\int_{\Sigma_1} q_k \nu_k \beta(t) a(x) |\frac{\partial \phi}{\partial \nu}|^2 \\
&\quad = \left(\alpha(t)\phi'(t), q_k \frac{\partial \phi(t)}{\partial x_k}\right) \Big|_0^T + \frac{1}{2}\int_Q \alpha(t) \frac{\partial q_k}{\partial x_k} |\phi'|^2 dx dt \\
&\quad - \frac{1}{2}\int_Q \beta(t) a(x) \frac{\partial q_k}{\partial x_k} |\nabla \phi|^2 dx dt + \int_Q \beta(t) a(x) \frac{\partial \phi}{\partial x_i} \frac{\partial q_k}{\partial x_j} \frac{\partial \phi}{\partial x_k} dx dt \\
&\quad - \frac{1}{2}\int_Q \beta(t) \frac{\partial a(x)}{\partial x_k} q_k |\nabla \phi|^2 dx dt - \int_Q f q_k \frac{\partial \phi}{\partial x_k} dx dt\,.
\end{aligned}$$

Proof. First we prove the identity for the strong [2] solutions of (3.1) and then the result follows by density arguments. So, let us suppose that

$$\phi \in C^0([0,T]; D(A)) \cap C^1([0,T]; V)\,.$$

By multiplying equation $(3.1)_1$ by $q_k \frac{\partial \phi}{\partial x_k}$ and integrating over Q,

$$\begin{aligned}
(4.16) \quad &\int_Q ((\alpha(t)\phi')' q_k \frac{\partial \phi}{\partial x_k} dx dt \\
&\quad - \int_Q \beta(t) \frac{\partial}{\partial x_i}(a(x) \frac{\partial \phi}{\partial x_i}) q_k \frac{\partial \phi}{\partial x_k} dx dt = \int_Q f q_k \frac{\partial \phi}{\partial x_k} dx dt\,.
\end{aligned}$$

Integrating by parts the left hand side of equality (4.16) we get

$$\begin{aligned}
(4.17) \quad &\int_Q ((\alpha(t)\phi')' q_k \frac{\partial \phi}{\partial x_k} dx dt \\
&\quad = \left(\alpha(t)\phi'(t), q_k \frac{\partial \phi(t)}{\partial x_k}\right) \Big|_0^T - \int_Q \alpha(t) q_k \phi' \frac{\partial \phi'}{\partial x_k} dx dt\,.
\end{aligned}$$

[2] It means that the initial data $\{y^0, y^1\} \in D(A) \times V$

On the other hand, since

$$\int_Q \alpha(t) q_k \phi' \frac{\partial \phi'}{\partial x_k} dx dt = \frac{1}{2}\int_Q \alpha(t) q_k \frac{\partial}{\partial x_k}(\phi')^2 dx dt \,,$$

we have from (4.17) that

$$\begin{aligned} \int_Q ((\alpha(t)\phi')' q_k \frac{\partial \phi}{\partial x_k} dx dt & \\ &= \left(\alpha(t)\phi'(t), q_k \frac{\partial \phi(t)}{\partial x_k}\right)\Big|_0^T - \frac{1}{2}\int_Q \alpha(t) q_k \frac{\partial}{\partial x_k}(\phi')^2 dx dt \,. \end{aligned} \tag{4.18}$$

We also have

$$\begin{aligned} \frac{1}{2}\int_Q \alpha(t) q_k \frac{\partial}{\partial x_k}(\phi')^2 dx dt & \\ &= -\frac{1}{2}\int_Q \alpha(t) \frac{\partial q_k}{\partial x_k}|\phi'|^2 dx dt + \frac{1}{2}\int_{\Sigma_0} \alpha(t) q_k |\phi'|^2 \nu_k d\Sigma \,. \end{aligned} \tag{4.19}$$

Thus, combining (4.19) and (4.18) we obtain

$$\begin{aligned} \int_Q ((\alpha(t)\phi')' q_k \frac{\partial \phi}{\partial x_k} dx dt = \left(\alpha(t)\phi'(t), q_k \frac{\partial \phi(t)}{\partial x_k}\right)\Big|_0^T & \\ + \frac{1}{2}\int_Q \alpha(t) \frac{\partial q_k}{\partial x_k}|\phi'|^2 dx dt - \frac{1}{2}\int_{\Sigma_0} \alpha(t) q_k |\phi'|^2 \nu_k d\Sigma \,. \end{aligned} \tag{4.20}$$

Now, estimating the right hand side of (4.16), we have from the Green identity

$$\begin{aligned} -\int_Q \beta(t) \frac{\partial}{\partial x_i}(a(x)\frac{\partial \phi}{\partial x_i}) q_k \frac{\partial \phi}{\partial x_k} dx dt &= \int_Q \beta(t) a(x) \frac{\partial \phi}{\partial x_i}\frac{\partial q_k}{\partial x_i}\frac{\partial \phi}{\partial x_k} dx dt \\ &- \frac{1}{2}\int_Q \beta(t)\frac{\partial a}{\partial x_k} q_k |\nabla \phi|^2 dx dt - \frac{1}{2}\int_Q \beta(t) a(x) \frac{\partial q_k}{\partial x_k}|\nabla\phi|^2 dx dt \\ &+ \frac{1}{2}\int_{\Sigma_0} \beta(t) a(x) q_k \nu_k |\nabla\phi|^2 dx dt + \frac{1}{2}\int_{\Sigma_1} \beta(t) a(x) q_k \nu_k |\nabla \phi|^2 d\Sigma \\ &- \int_{\Sigma_1} \beta(t) a(x) q_k \frac{\partial \phi}{\partial x_k}\frac{\partial \phi}{\partial \nu} d\Sigma \,. \end{aligned} \tag{4.21}$$

Combining (4.16), (4.20), (4.21) and (4.5) and observing that $\frac{\partial \phi}{\partial x_k} = \nu_k \frac{\partial \phi}{\partial \nu}$ on Σ_1 we obtain the desired identity. □

The above mentioned T_0 time is defined by

$$T_0 = T(x^0, \alpha, \beta, a) =$$
$$2\max\{\alpha_0^{-1}, \beta_0^{-1} a_0^{-1}\} \mathrm{e}^{C_0} R(x^0) \|\alpha\|_{L^\infty(0,T)} (1 - \|\nabla a\|_{C^0(\overline{\Omega})} a_0^{-1} R(x^0))^{-1} \quad \text{if } n > 1 \,,$$
$$T_0 = T(x^0, \alpha, \beta, a) =$$
$$2\max\{\alpha_0^{-1}, \beta_0^{-1} a_0^{-1}\} \mathrm{e}^{C_0} R(x^0) \|\alpha\|_{L^\infty(0,T)} (\gamma - \|\frac{\partial a}{\partial x}\|_{C^0(\overline{\Omega})} a_0^{-1} R(x^0))^{-1} \quad \text{if } n = 1 \,.$$

and uniquely depends on $n, R(x^0), \alpha(t), \beta(t), a(x)$ and the geometry of Ω.
Remark 3. We note that if $\alpha(t) = \beta(t) = a(x) = 1$, then $T_0 = 2R(x^0)$. This optimal time was determined in J.L.Lions [14] and V.Komornik [7] for the wave equation $u'' - \Delta u = 0$.

Theorem 4.3. *Suppose that hypotheses $(H1)$, $(H2)$ and $(H3)$ hold and that $T > T_0$ is given. Then for each weak solution ϕ of (3.1) with $f = 0$ there exists $C > 0$ such that*
(i) *If $n > 1$ then*

$$\begin{aligned}&\|\phi^0\|_V^2 + |\phi^1|_{L^2(\Omega)}^2\\ &\leq C\left\{\int_{\Sigma_0} m\cdot\nu[\,\alpha(t)|\phi'|^2 - \beta(t)a(x)|\nabla_\sigma(\phi)|^2]d\Sigma + \int_{\Gamma_0} m\cdot\nu[\,|\phi(0)|^2 + |\phi(T)|^2 d\Gamma\right\}.\end{aligned}$$

(ii) *If $n = 1$ then*

$$\|\phi^0\|_V^2 + |\phi^1|_{L^2(\Omega)}^2 \leq C\left\{\int_{\Sigma_0} m\,\alpha(t)|\phi'|^2 d\Sigma + \int_{\Gamma_0} m[\,|\phi(0)|^2 + |\phi(T)|^2 d\Gamma\right\}.$$

Proof. By using the identity given in the Theorem 4.2 with $q(x) = m(x) = x - x^0$, we get after some calculations

$$\begin{aligned}&\frac{1}{2}\int_{\Sigma_0} m\cdot\nu[\,\alpha(t)|\phi'|^2 - \beta(t)a(x)|\nabla_\sigma\phi|^2]d\Sigma + \frac{1}{2}\int_{\Sigma_1} m\cdot\nu\beta(t)a(x)|\frac{\partial\phi}{\partial\nu}|^2 d\Sigma\\ &\qquad = (\alpha(t)\phi'(t), m\cdot\nabla\phi(t))\,|_0^T + \frac{n}{2}\int_Q \alpha(t)|\phi'|^2 dxdt\\ &\qquad\quad - \frac{n}{2}\int_Q \beta(t)a(x)|\nabla\phi|^2 dxdt + \int_Q \beta(t)a(x)|\nabla\phi|^2 dxdt\\ &\qquad\quad - \frac{1}{2}\int_Q \beta(t)\nabla a\cdot m|\nabla\phi|^2 dxdt\,.\end{aligned} \tag{4.22}$$

On the other hand,

$$\begin{aligned}&\frac{n}{2}\int_Q [\,\alpha(t)|\phi'|^2 - \beta(t)a(x)|\nabla\phi|^2\,]\,dxdt\\ &\qquad = \frac{n-1}{2}\int_Q [\,\alpha(t)|\phi'|^2 - \beta(t)a(x)|\nabla\phi|^2\,]\,dxdt\\ &\qquad\quad + \int_0^T E(t)dt - \int_Q \beta(t)a(x)|\nabla\phi|^2 dxdt\,.\end{aligned} \tag{4.23}$$

Multiplying equation $(3.1)_1$ by ϕ and integrating over Q we have

$$(\alpha(t)\phi'(t), \phi(t))|_0^T = \int_0^T [\,\alpha(t)|\phi'|^2 - \beta(t)|a^{\frac{1}{2}}(x)\nabla\phi|^2\,]dt\,. \tag{4.24}$$

Replacing (4.24) in (4.23) it follows that

$$\frac{n}{2}\int_Q [\,\alpha(t)|\phi'|^2 - \beta(t)a(x)|\nabla\phi|^2\,]\,dxdt = (\alpha(t)\phi'(t), \frac{n-1}{2}\phi(t))|_0^T$$
$$+\int_0^T E(t)dt - \int_Q \beta(t)a(x)|\nabla\phi|^2 dxdt\,. \tag{4.25}$$

Now, substituting (4.25) in (4.22) we obtain

$$\frac{1}{2}\int_{\Sigma_0} m\cdot\nu[\,\alpha(t)|\phi'|^2 - \frac{1}{2}\beta(t)a(x)|\nabla_\sigma\phi|^2]d\Sigma + \frac{1}{2}\int_{\Sigma_1} m\cdot\nu\beta(t)a(x)|\frac{\partial\phi}{\partial\nu}|^2 d\Sigma$$
$$= \left(\alpha(t)\phi'(t), m\cdot\nabla\phi(t) + \frac{n-1}{2}\phi(t)\right)|_0^T$$
$$+\int_0^T E(t)dt - \frac{1}{2}\int_Q \beta(t)\nabla a\cdot m|\nabla\phi|^2 dxdt\,. \tag{4.26}$$

Since $R(x^0) = \max\{\,\|m(x)\|;\, x\in\overline{\Omega}\,\}$, then from the hypothesis $(H1)$ we have

$$\frac{1}{2}\int_Q \beta(t)\nabla a\cdot m|\nabla\phi|^2 dxdt \le \|\nabla a\|_{C^0(\overline{\Omega})} R(x^0) a_0^{-1}\int_0^T E(t)dt\,. \tag{4.27}$$

Hence, from (4.26), (4.27) and noting that $m\cdot\nu\le 0$ on Σ_1 we have

$$\left(\alpha(t)\phi'(t), m\cdot\nabla\phi(t) + \frac{n-1}{2}\phi(t)\right)|_0^T + (1 - \|\nabla a\|_{C^0(\overline{\Omega})} a_0^{-1} R(x^0))\int_0^T E(t)\,dt$$
$$\le \frac{1}{2}\int_{\Sigma_0} m\cdot\nu[\,\alpha(t)|\phi'|^2 - \beta(t)a(x)|\nabla_\sigma\phi|^2]d\Sigma\,,$$

and from hypothesis $(H2)$ and Theorem 4.1 we obtain

$$(\alpha(t)\phi'(t), m\cdot\nabla\phi(t) + \frac{n-1}{2}\phi(t)\Big)|_0^T + (1 - \|\nabla a\|_{C^0(\overline{\Omega})} a_0^{-1} R(x^0))\mathrm{e}^{-C_0}E(0)$$
$$\le \frac{1}{2}\int_{\Sigma_0} m\cdot\nu[\,\alpha(t)|\phi'|^2 - \beta(t)a(x)|\nabla_\sigma\phi|^2]d\Sigma\,. \tag{4.28}$$

Next, we estimate

$$z(t) = (\alpha(t)\phi'(t), m\cdot\nabla\phi(t) + \frac{n-1}{2}\phi),\ \ \forall\, t\in[0,T]\,.$$

From the hypothesis $(H1)$ and Theorem 4.1, we have,

$$|z(t)| \le \|\alpha\|_{L^\infty(0,T)}\Bigg\{\max(\alpha_0^{-1}, \beta_0^{-1}a_0^{-1}\}\mathrm{e}^{2C_0}R(x^0)$$
$$- \frac{n^2-1}{8R(x^0)}|\phi(t)|^2 + \frac{n-1}{4R(x^0)}\int_{\Gamma_0} m\cdot\nu|\phi(t)|^2 d\Gamma\Bigg\}\,,$$

and from (4.29) we obtain

$$\left|\left(\alpha(t)\phi'(t),\, m\cdot\nabla\phi(t)+\frac{n-1}{2}\phi\right)\Bigg|_0^T\right| \tag{4.30}$$
$$\leq \|\alpha\|_{L^\infty(0,T)}\left\{2\max(\alpha_0^{-1},\beta_0^{-1}a_0^{-1}\}\mathrm{e}^{C_0}R(x^0)-\frac{n^2-1}{8R(x^0)}(|\phi(0)|^2+|\phi(T)|^2)\right.$$
$$\left.+\frac{n-1}{4R(x^0)}\int_{\Gamma_0} m\cdot\nu(|\phi(0)|^2+|\phi(T)|^2)d\Gamma\right\}.$$

From the above inequality we have

$$\begin{aligned}
&\Big[(1-\|\nabla a\|_{C^0(\overline{\Omega})}R(x^0)a_0^{-1})\mathrm{e}^{-C_0}T-2\max\{\alpha_0^{-1},\beta_0^{-1}a_0^{-1}\}\mathrm{e}^{C_0}R(x^0)\|\alpha\|_{L^\infty(0,T)}\Big]E(0)\\
&\qquad+\frac{n^2-1}{8R(x^0)}\|\alpha\|_{L^\infty(0,T)}[|\phi(0)|^2+|\phi(T)|^2]\\
&\quad\leq\left(\alpha(t)\phi'(t),\, m\cdot\nabla\phi(t)+\frac{n-1}{2}\phi\right)\Bigg|_0^T\\
&\qquad+(1-\|\nabla a\|_{C^0(\overline{\Omega})}R(x^0)a_0^{-1})\mathrm{e}^{-C_0}E(0)T\\
&\qquad+\frac{n-1}{4R(x^0)}\|\alpha\|_{L^\infty(0,T)}\int_{\Gamma_0} m\cdot\nu(|\phi(0)|^2+|\phi(T)|^2)d\Gamma\,,
\end{aligned}$$

which, together with (4.28) implies that

$$\begin{aligned}
&\Big[(1-\|\nabla a\|_{C^0(\overline{\Omega})}R(x^0)a_0^{-1})\mathrm{e}^{-C_0}T-2\max\{\alpha_0^{-1},\beta_0^{-1}a_0^{-1}\}\mathrm{e}^{C_0}R(x^0)\|\alpha\|_{L^\infty(0,T)}\Big]E(0)\\
&\qquad+\frac{n^2-1}{8R(x^0)}\|\alpha\|_{L^\infty(0,T)}[|\phi(0)|^2+|\phi(T)|^2]\\
&\quad\leq\frac{1}{2}\int_{\Sigma_0} m\cdot\nu[\,\alpha(t)|\phi'|^2-\beta(t)a(x)|\nabla_\sigma\phi|^2]d\Sigma\\
&\qquad+\frac{n-1}{4R(x^0)}\|\alpha\|_{L^\infty(0,T)}\int_{\Gamma_0} m\cdot\nu(|\phi(0)|^2+|\phi(T)|^2)d\Gamma\,,
\end{aligned}$$

where we deduce (i).

To prove (ii), we consider the identity

$$\begin{aligned}
\frac{1}{2}\int_Q[\,\alpha(t)|\phi'|^2-\beta(t)a(x)|\nabla\phi|^2\,]dxdt&=\frac{\gamma}{2}\int_Q[\,\alpha(t)|\phi'|^2+\beta(t)a(x)|\nabla\phi|^2]dxdt\\
&\quad+\frac{1-\gamma}{2}\int_Q[\,\alpha(t)|\phi'|^2-\beta(t)a(x)|\nabla\phi|^2\,]dxdt\\
&\quad+(1-\gamma)\int_Q\beta(t)a(x)|\nabla\phi|^2dxdt\,.
\end{aligned}\tag{4.31}$$

Then, it follows from (4.22) and (4.31) that

$$
\begin{aligned}
(\alpha(t)\phi'(t), m\cdot\nabla\phi(t))|_0^T &+ \frac{\gamma}{2}\int_Q [\,\alpha(t)|\phi'|^2 + \beta(t)a(x)|\nabla\phi|^2\,]dxdt \\
&+ \frac{1-\gamma}{2}\int_Q [\,\alpha(t)|\phi'|^2 - \beta(t)a(x)|\nabla\phi|^2\,]dxdt \\
&+ (1-\gamma)\int_Q \beta(t)a(x)|\nabla\phi|^2 dxdt \\
&- \frac{1}{2}\int_Q \beta(t)\nabla a\cdot m|\nabla\phi|^2\, dxdt = \frac{1}{2}\int_{\Sigma_0} \alpha(t)m|\phi'|^2 d\Sigma\,.
\end{aligned}
$$

From $(H3)$ we have that $0<\gamma<1$ and therefore

$$
\begin{aligned}
(\alpha(t)\phi'(t), m\cdot\nabla\phi(t))|_0^T &+ \frac{\gamma}{2}\int_Q [\,\alpha(t)|\phi'|^2 + \beta(t)a(x)|\nabla\phi|^2\,]dxdt \\
&+ \frac{1-\gamma}{2}\int_Q [\,\alpha(t)|\phi'|^2 - \beta(t)a(x)|\nabla\phi|^2\,]dxdt \\
&- \frac{1}{2}\int_Q \beta(t)\nabla a\cdot m|\nabla\phi|^2\, dxdt \le \frac{1}{2}\int_{\Sigma_0} \alpha(t)m|\phi'|^2 d\Sigma\,.
\end{aligned}
\tag{4.32}
$$

Then by making use of the same arguments of (4.27) and (4.28), from (4.32) we obtain

$$
\begin{aligned}
(\alpha(t)\phi'(t), m\cdot\nabla\phi(t) + \frac{1-\gamma}{2}\phi(t))|_0^T &+ (\gamma - \|\nabla a\|_{C^0(\overline{\Omega})}R(x^0)a_0^{-1})\mathrm{e}^{-C_0}TE(0) \\
&\le \frac{1}{2}\int_{\Sigma_0} \alpha(t)m|\phi'|^2 d\Sigma\,.
\end{aligned}
$$

Defining

$$
z(t) = (\alpha(t)\phi'(t), m\cdot\nabla\phi(t) + \frac{1-\gamma}{2}\phi(t))|_0^T\,,
$$

in view of hypothesis $(H2)$ and using similar arguments to the case $n>1$ we obtain (ii). □

Theorem 4.4 (Inverse Inequality)**.** *Suppose that hypotheses $(H1)$-$(H3)$ hold and let $T>T_0$. Then for each strong solution ϕ of (3.1) with $f=0$ there exists $C>0$ such that*
(i) *If $n>1$*

$$
\|\phi^0\|_V^2 + |\phi^1|_{L^2(\Omega)}^2 \le C\left\{\int_{\Sigma(x^0)}[|\phi|^2+|\phi'|^2]d\Sigma + \int_{\Sigma_{0,*}(x^0)}|\nabla_\sigma\phi|^2 d\Sigma\right\}
$$

(ii) *If $n=1$*

$$
\|\phi^0\|_V^2 + |\phi^1|_{L^2(\Omega)}^2 \le C\int_{\Sigma(x^0)}[|\phi|^2+|\phi'|^2]d\Sigma\,.
$$

Proof. We prove the case (i). Dropping the terms that contribute to negative parts in the Theorem 4.3, one has

$$(4.33)\quad \|\phi^0\|_V^2 + |\phi^1|_{L^2(\Omega)}^2 \leq C_1 \Big\{ \int_{\Sigma(x^0)} [|\phi|^2 + |\phi'|^2] d\Sigma + \int_{\Sigma_{0,*}(x^0)} |\nabla_\sigma \phi|^2 d\Sigma + \int_{\Gamma(x^0)} [|\phi(0)|^2 + |\phi'(T)|^2] d\Gamma \Big\} .$$

On the other hand, there exists a constant $C_2 > 0$ such that

$$(4.34)\quad \int_{\Gamma(x^0)} [|\phi(0)|^2 + |\phi'(T)|^2] d\Sigma \leq C_2 \int_{\Sigma(x^0)} [|\phi|^2 + |\phi'|^2] d\Sigma .$$

Indeed, since ϕ is a regular solution, then

$$\phi \in C^0([0,T]; D(A)) \cap C^1([0,T]; V)$$

and therefore

$$(4.35)\quad \phi|_\Sigma \in C^0([0,T]; H^{\frac{3}{2}}(\Gamma)) \quad \text{and} \quad \phi'|_\Sigma \in C^0([0,T]; H^{\frac{1}{2}}(\Gamma)) .$$

Defining

$$h(t) = |\phi(t)|_{L^2(\Gamma(x^0))}^2 \quad \forall\, t \in [0,T] ,$$

we have

$$h'(t) = 2(\phi(t), \phi'(t))_{L^2(\Gamma(x^0))} \quad \forall\, t \in [0,T] ,$$

and from (4.35) it follows that $h, h' \in L^2(0,T)$ and hence $h \in C^0[0,T]$. Let $t_0 \in [0,T]$ a minimizer of h. Thus,

$$h(t) - h(t_0) = \int_{t_0}^t h'(s) ds$$

and consequently

$$(4.36)\quad h(t) \leq h(t_0) + \int_0^T |\phi(s)|_{L^2(\Gamma(x^0))}^2 ds + \int_0^T |\phi'(s)|_{L^2(\Gamma(x^0))}^2 ds .$$

But, since t_0 is a minimizer, we have that

$$\int_0^T h(t) dt \geq h(t_0) T ,$$

and then

$$(4.37)\quad h(t_0) \leq \frac{1}{T_0} \int_0^T h(t) dt .$$

Thus, from (4.36) and (4.37) we obtain

$$h(t) \leq C' \left\{ \int_0^T |\phi(s)|_{L^2(\Gamma(x^0))}^2 ds + \int_0^T |\phi'(s)|_{L^2(\Gamma(x^0))}^2 ds \right\} \quad \forall\, t \in [0,T] ,$$

which proves (4.34). Combining (4.33) and (4.34) one finishes the proof. □

5. The Backward Problem

Let $T > T_0$ as in previous section and consider the following homogeneous problem

$$(5.1)\qquad \left|\begin{array}{l} (\alpha(t)\phi')' + A(t)\phi = 0 \ \text{ in } \ Q \\ \frac{\partial \phi}{\partial \nu_A} = 0 \ \text{ on } \ \Sigma_0 \\ \theta = 0 \ \text{ on } \ \Sigma_1 \\ \phi(0) = \phi^0 \quad \phi'(0) = \phi^1 \quad \text{on } \ \Omega\,. \end{array}\right.$$

According to the inverse inequality (Theorem 4.4), the expression

$$\|\{\phi^0,\phi^1\}\|_* = \left\{\int_{\Sigma(x^0)} [|\phi|^2 + |\phi'|^2] d\Sigma + \int_{\Sigma_{0,*}(x^0)} |\nabla_\sigma \phi|^2 d\Sigma\right\}^{\frac{1}{2}}$$

defines a norm in $D(A)\times V$. We define the Hilbert space

$$(5.3)\qquad F = \overline{D(A)\times V}^{\|\cdot\|_*}$$

equipped with the topology

$$(5.4)\qquad \|\{\phi^0,\phi^1\}\|_F = \lim_{\nu\to\infty} \|\{\phi^0_\nu,\phi^1_\nu\}\|_*$$

where $(\{\phi^0_\nu,\phi^1_\nu\})_{\nu\in\mathbf{N}}$ is any Cauchy sequence in $(D(A)\times V, \|\cdot\|_*)$ defined by the equivalence relation

$$\{\phi^0_\nu,\phi^1_\nu\} \sim \{\psi^0_\nu,\psi^1_\nu\} \Leftrightarrow \lim_{\nu\to\infty} \|\{\phi^0_\nu-\psi^0_\nu, \phi^1_\nu, \psi^1_\nu\}\|_* = 0\,.$$

For every $\forall\, \{\phi^0,\phi^1\}\in D(A)\times V$ we have:

$$\|\{\phi^0,\phi^1\}\|_* \le C_1 \|\{\phi^0,\phi^1\}\|_{D(A)\times V}$$

and

$$\|\{\phi^0,\phi^1\}\|_{V\times L^2(\Omega)} \le C_2 \|\{\phi^0,\phi^1\}\|_*\,.$$

Now, since $D(A)\times V$ is dense in F, we have

$$(5.5)\qquad D(A)\times V \hookrightarrow F \hookrightarrow V\times L^2(\Omega)\,,$$

where the inclusions are continuous and denses.

It should be noted that by the construction of F,

$$\{\phi^0,\phi^1\}\in F \Leftrightarrow \int_{\Sigma(x^0)} [|\phi|^2 + |\phi'|^2] d\Sigma + \int_{\Sigma_{0,*}(x^0)} |\nabla_\sigma\phi|^2 d\Sigma < \infty\,,$$

that means if $\{\phi^0,\phi^1\}\in F$ then

$$(5.6)\qquad \phi|_{\Sigma(x^0)},\ \phi'|_{\Sigma(x^0)} \in L^2(\Sigma(x^0)) \quad \text{and} \quad \nabla_\sigma\phi|_{\Sigma_{0,*}(x^0)} \in (L^2(\Sigma_{0,*}(x^0)))^n\,,$$

as well,

$$(5.7)\qquad \phi|_{\Sigma_{0,*}(x^0)} \in L^2(0,T;H^1(\Gamma_{0,*}(x^0)))\,.$$

The proof of the above regularities are given in the appendix.

We then consider the backward problem

$$(5.8)\qquad \left|\begin{array}{l} (\alpha(t)\psi')' + A(t)\psi = 0 \ \text{ in } \ Q \\ \\ \frac{\partial\phi}{\partial\nu_A} = \left|\begin{array}{l} \beta^{-1}[-\phi + \frac{\partial}{\partial t}(\phi')] \ \text{ on } \Sigma(x^0) \\ \beta^{-1}\Delta_{\Gamma_*(x^0)}\phi \ \text{ on } \Sigma_{0,*}(x^0) \end{array}\right. \\ \\ \psi = 0 \ \text{ on } \ \Sigma_1 \\ \phi(0) = \phi^0 \quad \phi'(0) = \phi^1 \quad \text{on } \ \Omega\,. \\ \psi(T) = \psi'(T) = 0 \ \text{ in } \ \Omega\,, \end{array}\right.$$

where ϕ is the unique solution of problem (5.1) with initial data $\{\phi^0, \phi^1\} \in F$.

We observe that the operator $\frac{\partial}{\partial t}$ is well defined on $\Sigma(x^0)$ taking into account (5.6) and considering the following meanning: $\forall\, w \in H^1(0,T; L^2(\Gamma(x^0)))$,

$$(5.9)\qquad \left\langle \frac{\partial}{\partial t}(\phi'), w \right\rangle_{[H^1(0,T;L^2(\Gamma(x^0)))]', H^1(0,T;L^2(\Gamma(x^0)))} = -\int_0^T \int_{\Gamma(x^0)} \phi' w' \, d\Gamma dt\,.$$

It is important to note that this operator is not taken in the distributional sense.

On the other hand, from (5.7) we obtain

$$(5.10)\qquad \Delta_{\Gamma_{0,*}(x^0)}\phi \in L^2(0,T; [H^1(\Sigma_{0,*}(x^0))]')\,.$$

The solution ψ of (5.8) is defined by the transposition method, that will be precised later. Let $\{\phi^0, \phi^1\} \in F$ and $f \in L^1(0,T,H^1(\Omega))$, and let $\theta : Q \to \mathbf{R}$ the unique solution of

$$(5.11)\qquad \left|\begin{array}{l} (\alpha(t)\theta')' + A(t)\theta = f \ \text{ in } \ Q \\ \frac{\partial\theta}{\partial\nu_A} = 0 \ \text{ on } \ \Sigma_0 \\ \theta = 0; \ \text{ on } \ \Sigma_1 \\ \theta(0) = \theta^0 \quad \theta'(0) = \theta^1 \quad \text{on } \ \Omega\,. \end{array}\right.$$

Multiplying (5.11) by ψ and integrating by parts, we obtain formally

$$(5.12)\qquad \begin{aligned} \int_Q f\psi\, dx dt = &-\int_\Omega \alpha(0)\theta'(0)\psi(0) dx \\ &+ \int_\Omega \alpha(0)\theta(0)\psi'(0) dx + \int_\Sigma \beta(t)\frac{\partial\psi}{\partial\nu_A}\theta d\Sigma_0\,. \end{aligned}$$

Replacing $\frac{\partial\psi}{\partial\nu_A}$ by its value in (5.8) we get from (4.13) and (5.9)

$$\int_{\Sigma_0} \beta(t)\frac{\partial\psi}{\partial\nu_A}\theta d\Sigma = -\int_{\Sigma(x^0)} (\phi\theta + \phi'\theta') d\Sigma - \int_{\Sigma_{0,*}(x^0)} \nabla_\sigma\phi \cdot \nabla_\sigma\theta d\Sigma\,.$$

Observing this expression we define the functional

$$(5.13)\qquad L(\theta^0, \theta^1, f) = -\int_{\Sigma(x^0)} (\phi\theta + \phi'\theta') d\Sigma - \int_{\Sigma_{0,*}(x^0)} \nabla_\sigma\phi \cdot \nabla_\sigma\theta d\Sigma\,.$$

Thus, from (5.12) and (5.13) we obtain formally that

$$\int_Q f\psi dxdt + \int_\Omega \alpha(0)\theta'(0)\psi(0)dx - \int_\Omega \alpha(0)\theta(0)\psi'(0)dx = L(\theta^0,\theta^1,f)\,. \tag{5.14}$$

Considering Theorem 3.1 and the construction of F, we have that the functional given by (5.13) is continuous, that is,

$$L \in F' \times [L^1(0,T;V)]'.$$

Indeed, first of all we note that the solution θ of (5.11) verifies $\theta = \theta_1 + \theta_2$, where θ_1 and θ_2 are, respectively, the solutions of the following problems:

$$\left|\begin{array}{l} (\alpha(t)\theta_1') + A(t)\theta_1 = 0 \text{ in } Q \\ \frac{\partial\theta_1}{\partial\nu_A} = 0 \text{ on } \Sigma_0 \\ \theta_1 = 0 \text{ on } \Sigma_1 \\ \theta_1(0) = \theta^0; \theta_1'(0) = \theta^1 \text{ in } \Omega\,. \end{array}\right.$$

and

$$\left|\begin{array}{l} (\alpha(t)\theta_2') + A(t)\theta_2 = f \text{ in } Q \\ \frac{\partial\theta_2}{\partial\nu_A} = 0 \text{ on } \Sigma_0 \\ \theta_2 = 0 \text{ on } \Sigma_1 \\ \theta_2(0) = \theta_2'(0) = 0 \text{ in } \Omega\,. \end{array}\right.$$

Besides, from (5.13) we can write for all $\{\phi^0,\phi^1\} \in D(A)\times V$ and $i = 1,2$:

$$L(\theta^0,\theta^1,f) = \sum_{i=1}^{2}\left(\int_{\Sigma(x^0)} (\phi\theta_i + \phi'\theta_i)\, d\Sigma + \int_{\Sigma_{0,*}(x^0)} \nabla_\sigma\phi\nabla_\sigma\theta_i d\Sigma\right) \tag{5.16}$$

and therefore from (5.2) and (5.16) we obtain:

$$\begin{aligned} &|L(\theta^0,\theta^1,f)| \\ &\leq C_1\|\{\phi^0,\phi^1\}\|_F \sum_{i=1}^{2}\left\{\int_{\Sigma(x^0)} \left(|\phi\theta_i|^2 + |\phi'\theta_i|^2\right) d\Sigma + \int_{\Sigma_{0,*}(x^0)} |\nabla_\sigma\phi\nabla_\sigma\theta_i|^2 d\Sigma\right\}^{1/2}. \end{aligned} \tag{5.17}$$

From (5.17) and Theorem 3.1 we have:

$$|L(\theta^0,\theta^1,f)| \leq C_2\left\{\|\{\phi^0,\phi^1\}\|_F^2 + \|f\|^2_{L^1(0,T;V)}\right\}^{1/2}. \tag{5.18}$$

By density arguments we conclude that inequality (5.18) is vality for all $\{\phi^0,\phi^1\}$, $\{\theta^0,\theta^1\} \in F$ which proves (5.15).

It follows that there exists a unique triple $\{\rho^0,\rho^1,\psi\}$ such that

$$\{\alpha(0)\rho^1, -\alpha(0)\rho^0\} \in F \quad \text{and} \quad \psi \in L^\infty(0,T;V')\,,$$

$$\int_0^T \langle \psi(t), f(t) \rangle_{V',V} + \langle \{-\alpha(0)\rho^1, \alpha(0)\rho^0\}, \{\theta^0, \theta^1\} \rangle_{F',F} \tag{5.19}$$
$$= - \left[\int_{\Sigma(x^0)} (\phi\theta + \phi'\theta') d\Sigma + \int_{\Sigma_{0,*}(x^0)} \nabla_\sigma(\phi) \cdot \nabla_\sigma(\theta) d\Sigma \right].$$

Definition. The unique function ψ that satisfies (5.19) in named *solution by transposition* of the problem (5.8).

Now we state our main result of this Section, which is a consequence of Theorem 3.2

Theorem 5.1 *The unique solution by transposition ψ of problem (5.8) has the following regularity.*

$$\psi \in L^\infty(0, T; V') \cap W^{1,\infty}(0, T; [D(A)]'),$$
$$\{\psi'(0), \psi(0)\} \in F'.$$

In addition, the linear map

$$\{\phi^0, \phi^1\} \in F \mapsto \{\alpha(0)\psi'(0), -\alpha(0)\psi(0)\} \in F'$$

is continuous.

Proof. For $f \in \mathcal{D}(0, T; D(A))$ we have

$$L(0, 0, f') = - \int_{\Sigma(x^0)} (\phi\theta + \phi'\theta') d\Sigma - \int_{\Sigma_{0,*}(x^0)} \nabla_\sigma(\phi) \cdot \nabla_\sigma(\theta) d\Sigma,$$

where

$$\left| \begin{array}{l} (\alpha(t)\theta')' + A(t)\theta = f' \text{ in } Q \\ \dfrac{\partial\theta}{\partial\nu_A} = 0 \text{ on } \Sigma_0 \\ \theta = 0 \text{ on } \Sigma_1 \\ \theta(0) = \theta'(0) = 0 \text{ in } \Omega \end{array} \right. \tag{5.20}$$

and

$$\left| \begin{array}{l} (\alpha(t)\theta')' + A(t)\theta = 0 \text{ in } Q \\ \dfrac{\partial\theta}{\partial\nu_A} = 0 \text{ on } \Sigma_0 \\ \theta = 0 \text{ on } \Sigma_1 \\ \theta(0) = \phi^0 \quad \phi'(0) = \phi^1 \text{ in } \Omega. \end{array} \right. \tag{5.21}$$

By definition of F, from Theorem 3.1 and taking into account the definition of F (cf. Lions [14]) it follows that

$$|L(0, 0, f')| \leq C[\, \|\theta\|_{L^1(0,T;D(A))} + \|\theta'\|_{L^1(0,T;V)}\,]. \tag{5.22}$$

Indeed, it is sufficient to prove (5.22) when the initial data $\{\phi^0, \phi^1\} \in D(A) \times V$ because by density arguments we conclude the same when $\{\phi^0, \phi^1\} \in F$.

We have by Schwarz inequality and Theorem 3.1:

$$|L(0,0,f')| \leq$$

$$\leq C_1 \int_0^T \left[\int_{\Gamma(x^0)} |\phi|^2 d\Gamma + \int_{\Gamma(x^0)} |\phi'|^2 d\Gamma + \int_{\Gamma_{0,*}(x^0)} |\nabla_\sigma \phi|^2 d\Gamma \right]^{1/2}$$

$$\left[\int_{\Gamma(x^0)} |\theta|^2 d\Gamma + \int_{\Gamma(x^0)} |\theta'|^2 d\Gamma + \int_{\Gamma_{0,*}(x^0)} |\nabla_\sigma \theta|^2 d\Gamma \right]^{1/2} \leq$$

$$\leq ||\{\phi^0, \phi^1\}||_{D(A)\times V} [\, ||\theta||_{L^1(0,T;D(A))} + ||\theta'||_{L^1(0,T;V)} \,]$$

which concludes (5.22).

On the other hand, from Theorem 3.2 we get

$$||\theta||_{L^1(0,T;D(A))} + ||\theta'||_{L^1(0,T;V)} \leq C||f||_{L^1(0,T;D(A))} \, . \tag{5.23}$$

which is the crucial point for control problems involving time-dependent coefficients.

In fact, before we prove (5.23) we observe that in the right side of equation (5.20) we have f' while in the right side of (5.23) we have f. Besides, we note that when the coefficients do not depend on time, (see for example the most simple case for the wave equation) it is not difficult to obtain the above inequality using Theorem 3.1 and the following standard argument:

If ω is a solution to problem

$$\left| \begin{array}{l} \omega'' - \Delta\omega = f \;\text{ in }\; Q \\ \dfrac{\partial \omega}{\partial \nu} = 0 \;\text{ on }\; \Sigma_0 \\ \omega = 0 \text{ on } \Sigma_1 \\ \omega(0) = \omega'(0) = 0 \;\text{ in }\; \Omega \end{array} \right.$$

with $f \in D(0,T,D(A))$, then $\theta = \omega'$ is the solution of

$$\left| \begin{array}{l} \theta'' - \Delta\theta = f' \;\text{ in }\; Q \\ \dfrac{\partial \theta}{\partial \nu} = 0 \;\text{ on }\; \Sigma_0 \\ \theta = 0 \text{ on } \Sigma_1 \\ \theta(0) = 0 \quad \theta'(0) = 0 \;\text{ in }\; \Omega \, . \end{array} \right.$$

But in our case, where we have time-dependent coefficients, this argument fails completely and we need to solve it in other way. From Theorem 3.2 we obtain:

$$||\theta||_{L^1(0,T;L^2(\Omega))} \leq k_1 ||f||_{L^1(0,T;D(A))}$$

$$||A\theta||_{L^1(0,T;L^2(\Omega))} \leq k_2 ||f||_{L^1(0,T;D(A))}$$

which implies

$$||\theta||_{L^1(0,T;D(A))} \leq k_3 ||f||_{L^1(0,T;D(A))} \tag{5.24}$$

In addition

$$\|\theta'\|_{L^1(0,T;L^2(\Omega))} \le k_4 \|f\|_{L^1(0,T;D(A))}$$

$$\|A^{1/2}\theta'\|_{L^1(0,T;L^2(\Omega))} \le k_5 \|f\|_{L^1(0,T;D(A))}$$

and therefore

$$\|\theta'\|_{L^1(0,T;V)} \le k_6 \|f\|_{L^1(0,T;D(A))} . \tag{5.25}$$

From (5.24) and (5.25) we get (5.23). Combining (5.22) and (5.23) we obtain

$$|L(0,0,f')| \le C\|f\|_{L^1(0,T;D(A))} \quad \forall\, f \in \mathcal{D}(0,T;D(A)) .$$

which is sufficient to prove the desired regularity, that is,

$$\psi' \in L^\infty(0,T;[D(A)]') . \tag{5.26}$$

In fact, let us define

$$S(f) = -L(0,0,f') \quad \forall\, f \in \mathcal{D}(0,T;D(A)) .$$

Since $\mathcal{D}(0,T;D(A))$ is dense in $L^1(0,T;D(A))$, we can consider the unique linear continuous extension $\overline{S}$ of S, that is defined by

$$\overline{S}(f) = S(f) = -L(0,0,f) \quad \forall\, f \in \mathcal{D}(0,T;D(A)) , \tag{5.27}$$

and, consequently, it follows that

$$\overline{S} \in (L^1(0,T;D(A)))' = L^\infty(0,T;[D(A)]') . \tag{5.28}$$

Now, given $f = \varphi\theta$ with $\varphi \in D(A)$ and $\theta \in \mathcal{D}(0,T)$, according to (5.13), (5.19), (5.27) and considering the fact that $\theta^0 = \theta^1 = 0$ we obtain,

$$\langle \overline{S}, \varphi\theta\rangle = \langle S, \varphi\theta\rangle = -L(0,0,f')$$

$$= -\int_0^T \langle \psi(t), f'(t)\rangle dt = -\int_0^T \langle \psi(t), \varphi\rangle \theta'(t)\,dt .$$

So, by (5.28) it follows that

$$\int_0^T \langle \overline{S}(t), \varphi\rangle \theta(t)\,dt = -\int_0^T \langle \psi(t), \varphi\rangle \theta'(t)\,dt ,$$

which implies that

$$\left\langle \int_0^T \overline{S}(t)\theta(t)\,dt, \varphi \right\rangle = \left\langle -\int_0^T \psi(t)\theta'(t)\,dt, \varphi \right\rangle \quad \forall\, \varphi \in D(A) .$$

Therefore $\overline{S} = \psi'$ in $\mathcal{D}'(0,T;[D(A)]')$, and (5.26) is then proved.

One observes that if in (5.19), we consider $f = \varphi(\alpha\eta')' + \beta A(\varphi\eta); \theta = \varphi\eta$ with $\varphi \in D(A^{3/2}), \eta \in D(0,T)$ and $\phi^0 = \phi^1 = 0$, we have

$$(\alpha\psi')' + A(t)\psi = 0 \ \text{ in } \ L^\infty(0,T;[D(A^{\frac{3}{2}})]') ,$$

Since $\alpha(t) \geq \alpha_0 > 0$, it follows that

$$\psi'' \in L^\infty(0,T;[D(A^{\frac{3}{2}})]') \,. \tag{5.29}$$

Then, from (5.19), (5.26) and (5.29) we obtain

$$\psi \in C_s(0,T,V) \cap C^0([0,T],[D(A)]')$$

and

$$\psi' \in C_s(0,T,[D(A)]') \cap C^0([0,T],[D(A^{\frac{3}{2}})]') \,,$$

(see for example J.L.Lions and E.Magenes [15], V.1, Lemma 8.1) which makes $\psi(0)$ and $\psi'(0)$ meaningful.

Using the regularity of ψ, considering $f = \varphi(\alpha\eta')' + \beta A(\varphi\eta)$ and $\theta = \varphi\eta$ where $\varphi \in D(A^{3/2}), \eta \in C^2(0,T)^3$, we obtain from (5.19) with $\phi^0 = \phi^1 = 0$,

$$\psi(0) = \rho^0 \quad \text{and} \quad \psi'(0) = \rho^1 \,.$$

Finally, by considering $f = 0$ in (5.19) we conclude that

$$||\{\alpha(0)\psi'(0), -\alpha(0)\psi(0)\}||_{F'} \leq C||\{\phi^0,\phi^1\}||_F \quad \forall\, \{\phi^0,\phi^1\} \in F \,.$$

This ends the proof. □

6. HUM and Exact Controllability

Let us define the linear operator $\Lambda : F \to F'$ by

$$\Lambda\{\phi^0,\phi^1\} = \{\alpha(0)\psi'(0), -\alpha(0)\psi(0)\} \,, \tag{6.1}$$

that is continuous in view of Theorem 5.1.

Considering $f = 0, \theta^0 = \phi^0$ and $\theta^1 = \phi^1$ in (5.13) and (5.14), we have

$$\begin{aligned}\langle \Lambda\{\phi^0,\phi^1\},\{\phi^0,\phi^1\}\rangle_{F',F} &= \langle\{\alpha(0)\psi'(0), -\alpha(0)\psi(0)\},\{\phi^0,\phi^1\}\rangle \\ &= \int_{\Sigma(x^0)} (|\phi|^2 + |\phi'|^2)d\Sigma + \int_{\Sigma_{0,*}(x^0)} |\nabla_\sigma(\phi)|^2 d\Sigma\end{aligned}$$

that is,

$$\langle \Lambda\{\phi^0,\phi^1\},\{\phi^0,\phi^1\}\rangle_{F',F} = ||\{\phi^0,\phi^1\}||_F^2 \,.$$

This implies immediatly that Λ is injective and self-adjoint. Then Λ is a isomorphism from F to F'. Therefore, given $\{y^1,-y^0\} \in F'$ then $\{\alpha(0)y^1, -\alpha(0)y^0\} \in F'$ and consequently there exists a unique $\{\phi^0,\phi^1\} \in F$ such that

$$\Lambda\{\phi^0,\phi^1\} = \{\alpha(0)y^1, -\alpha(0)y^0\} \,. \tag{6.2}$$

From (6.1), and (6.2) we have

$$\psi'(0) = y^1 \quad \text{and} \quad \psi(0) = y^0 \,. \tag{6.3}$$

[3] First we get, for instance, $\eta(t) = (T-t)^2 t$ and secondly we can consider $\eta(t) = (T-t)^2$.

Now we are going to finish the proof of Theorem 2.1. Since $\{y^1, -y^0\} \in L^2(\Omega) \times V'$, then taking into account the chain

$$D(A) \times V \hookrightarrow F \hookrightarrow V \times L^2(\Omega) \hookrightarrow L^2(\Omega) \times L^2(\Omega)$$
$$\hookrightarrow V' \times L^2(\Omega) \hookrightarrow F' \hookrightarrow D(A)' \times V',$$

we obtain $\{y^1, -y^0\} \in F'$ and therefore in this case we conclude (6.3).

Defining in (2.1) the controls

$$v_0 = \beta^{-1}\left[-\phi + \frac{\partial}{\partial t}(\phi')\right] \quad \text{on } \Sigma(x^0)$$

and

$$v_1 = \beta^{-1}\Delta_{\Gamma_{0,*}(x^0)}\phi \quad \text{on } \Sigma_{0,*}(x^0),$$

from (6.3), the uniqueness of the problems (2.1) and

$$\left|\begin{array}{l} (\alpha(t)\psi')' + A(t)\psi = 0 \text{ in } Q \\ \\ \frac{\partial \psi}{\partial \nu_A} = \left|\begin{array}{l} v_0 \text{ on } \Sigma(x^0) \\ v_1 \text{ on } \Sigma_{0,*}(x^0) \end{array}\right. \\ \\ \psi = 0 \text{ on } \Sigma_1 \\ \psi(0) = y^0 \quad \psi'(0) = y^1 \text{ in } \Omega \\ \psi(T) = \psi'(T) = 0 \text{ in } \Omega, \end{array}\right.$$

we finally conclude that

$$y(T) = y'(T) = 0.$$

Thus Theorem 2.1 is proved. □

7. Appendix

Since $D(A) \times V$ is dense in F, there exists $\{\phi_\nu^0, \phi_\nu^1\} \in D(A) \times V$ such that

$$\lim_{\nu\to\infty} \{\phi_\nu^0, \phi_\nu^1\} = \{\phi^0, \phi^1\} \text{ in } F \tag{7.1}$$

and therefore, considering the inverse inequality,

$$\lim_{\nu\to\infty} \{\phi_\nu^0, \phi_\nu^1\} = \{\phi^0, \phi^1\} \text{ in } V \times L^2(\Omega). \tag{7.2}$$

According to Theorem 3.1, for each $\nu \in \mathbf{N}$ there exists $\phi_\nu \in C^0[0,T]; D(A)) \cap C^1([0,T]; V)$ which is the solution of (3.1) with initial data $\{\phi_\nu^0, \phi_\nu^1\} \in D(A) \times V$ and $f \in L^1(0,T;V)$. Thus, from the linearity of (3.1) we have,

$$||\phi_\nu - \phi_\mu||_{C^0([0,T];V)} + ||\phi_\nu' - \phi_\mu'||_{C^0([0,T];L^2(\Omega))} \leq$$
$$\leq C\left[||\phi_\nu^0 - \phi_\mu^0|| + |\phi_\nu^0 - \phi_\mu^0|\right]$$

which implies that the unique solution $\phi : Q \to \mathbf{R}$ of (3.1) satisfies

$$\lim_{\nu\to\infty} \phi_\nu = \phi \text{ in } C^0([0,T];V) \text{ and } \lim_{\nu\to\infty} \phi_\nu' = \phi' \text{ in } C^0([0,T];L^2(\Omega)). \tag{7.3}$$

On the other hand, from (5.2) we obtain,

$$||\{\phi^0_\nu - \phi^0_\mu, \phi^1_\nu - \phi^1_\mu\}||^2_* = \tag{7.4}$$

$$= \left\{ \int_{\Sigma(x^0)} \left[|\phi_\nu - \phi_\mu|^2 + |\phi'_\nu - \phi'_\mu|^2 \right] d\Sigma + \int_{\Sigma_{0,*}(x^0)} |\nabla_\sigma \phi_\nu - \nabla_\sigma \phi_\mu|^2 d\Sigma \right\} .$$

From the convergence in (7.1) we conclude that the right hand side of (7.4) converges to zero when μ and ν goes to infinity. So, (ϕ_ν) , (ϕ'_ν) and $(\nabla_\sigma \phi_\nu)$ are, respectively, sequences of Cauchy in $L^2(\Sigma(x^0))$, $L^2(\Sigma(x^0))$ and $L^2(\Sigma_{0,*}(x^0))$, which proves (5.6).

To prove (5.7) we need the following result.

Lemma. $\forall R > 0, \exists C > 0$ *such that*

$$||\{\phi^0, \phi^1\}|| \leq C||\{\phi^0, \phi^1\}||_*; \forall \{\phi^0, \phi^1\} \in D(A) \times V$$

satisfying $|| \ \{\phi^0, \phi^1\}||_* \geq R$.

Proof. Consider $\{\phi^0, \phi^1\} \in D(A) \times V$ such that $\left|\left|\{\phi^0, \phi^1\}\right|\right|_* \geq R$. So $\{\phi^0, \phi^1\}$ is different from $\{0, 0\}$ and, consequently, it is sufficient to prove that: $\forall R > 0, \exists C > 0$ such that

$$\frac{1}{C} \leq ||\{\phi^0, \phi^1\}||_* \forall \{\phi^0, \phi^1\} \in D(A) \times V \tag{7.5}$$

with $||\{\phi^0, \phi^1\}||_{D(A)\times V} = 1$ and $||\{\phi^0, \phi^1\}||_* \geq R$.

Let us suppose it does not happen, that is, there exists $R_0 > 0$ such that $\forall C > 0 \ \ \exists \{\phi^0_C, \phi^1_C\} \in D(A) \times V$ with $||\{\phi^0_C, \phi^1_C\}||_{D(A)\times V} = 1, ||\{\phi^0_C, \phi^1_C\}||_* \geq R_0$ and $||\{\phi^0_C, \phi^1_C\}||_* < \frac{1}{C}$.

In the particular case when $C = \frac{1}{R_0}$ it follows that $R_0 \leq ||\{\phi^0_{R_0}, \phi^1_{R_0}\}||_* < R_0$ which is a contradiction. So, (7.5) is proved and consequently the lemma.

Let us consider initially $\{\phi^0, \phi^1\} \in D(A) \times V$ and suppose ϕ is the strong solution of (5.1). Then, $\phi \in C^0([0,T]; D(A)) \cap C^1([0,T]; V)$ and therefore,

$$\phi|_\Sigma \in C^0([0,T]; H^{3/2}(\Gamma)) \subset C^0([0,T]; H^1(\Gamma)) .$$

Thus, from Theorem 3.1 we obtain:

$$||\phi|_{\Sigma_{0,*}(x^0)}||_{L^2(0,T;H^1(\Gamma_{0,*}(x^0)))} \leq k||\{\phi^0, \phi^1\}||_{D(A)\times V} . \tag{7.6}$$

Consider, now, $\{\phi^0, \phi^1\} \in F$ and ϕ the weak solution of (5.1). If $\{\phi^0, \phi^1\} = \{0, 0\}$ then $\phi = 0$ and the regularity in (5.7) follows imediately. Let us consider $\{\phi^0, \phi^1\}$ different from $\{0, 0\}$. Since $D(A)\times V$ is dense in F there exists $\{\phi^0_\nu, \phi^1_\nu\} \subset D(A) \times V$ such that

$$\lim_{\nu\to\infty} \{\phi^0_\nu, \phi^1_\nu\} = \{\phi^0, \phi^1\} \text{ in F} . \tag{7.7}$$

Defining $R_0 = \frac{1}{2}||\phi^0, \phi^1\}||_F$, there exist $\{\phi^0_\mu, \phi^1_\mu\}$ subsequence of $\{\phi^0_\nu, \phi^1_\nu\}$ such that $||\{\phi^0_\mu, \phi^1_\mu\} - \{\phi^0, \phi^1\}||_F < R_0; \forall \mu \in \mathbf{N}$. Therefore,

$$||\{\phi^0_\mu, \phi^1_\mu\}||_F = ||\{\phi^0_\mu, \phi^1_\mu\}||_* \geq R_0 . \tag{7.8}$$

Thus, from (7.8) and the above Lema $\exists C = C\left(||\{\phi^0, \phi^1\}||_F\right) > 0$ such that

$$||\{\phi^0_\mu, \phi^1_\mu\}||_{D(A)\times V} \leq C||\{\phi^0_\mu, \phi^1_\mu\}||_*; \forall \mu \in \mathbf{N}\,. \tag{7.9}$$

Let $\{\phi_\mu\}$ be the sequence of strong solutions of (5.1) with initial data $\{\phi^0_\mu, \phi^1_\mu\}$. Then, from (7.6) and (7.9) there exists $C_1 = C_1\left(||\{\phi^0, \phi^1\}||_F\right) > 0$ such that

$$||\phi_\mu|_{\Sigma_{0,*}(x^0)}||_{L^2(0,T;H^1(\Gamma_{0,*}(x^0)))} \leq C_1||\{\phi^0_\mu, \phi^1_\mu\}||_*\,. \tag{7.10}$$

But, from (7.7) we obtain,

$$||\{\phi^0_\mu, \phi^1_\mu\}||_F = ||\{\phi^0_\mu, \phi^1_\mu\}||_* \leq L; \forall \mu \in \mathbf{N}\,. \tag{7.11}$$

So, from (7.10) and (7.11) we conclude that

$$||\phi_\mu|_{\Sigma_{0,*}(x^0)}||_{L^2(0,T;H^1(\Gamma_{0,*}(x^0)))} \leq M; \forall \mu \in \mathbf{N}\,.$$

Then, there exists a subsequence that we will denote by the same notation $\{\phi_\mu\}$ such that,

$$\phi_\mu|_{\Sigma_{0,*}(x^0)} \rightharpoonup \chi \text{ in } L^2(0,T;H^1(\Gamma_{0,*}(x^0))) \text{ when } \mu \text{ goes to infinity.} \tag{7.12}$$

On the other hand, from (7.3) we have,

$$\lim_{\mu\to\infty} \phi_\mu|_{\Sigma_{0,*}(x^0)} = \phi|_{\Sigma_{0,*}(x^0)} \text{ in } L^2(0,T;H^{1/2}(\Gamma_{0,*}(x^0))) \tag{7.13}$$

and from (7.12) and (7.13) results $\phi = \chi$ which proves (5.7).

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