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OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF PERTURBED QUASILINEAR SECOND ORDER DIFFERENCE EQUATIONS

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Abstract. This paper deals with oscillatory and asymptotic behaviour of solutions of second order quasilinear difference equation of the form

$$\Delta(a_{n-1}|\Delta y_{n-1}|^{\alpha-1}\Delta y_{n-1}) + F(n, y_n) = G(n, y_n, \Delta y_n), \quad n \in N(n_0) \tag{E}$$

where $\alpha > 0$. Some sufficient conditions for all solutions of (E) to be oscillatory are obtained. Asymptotic behaviour of nonoscillatory solutions of (E) are also considered.

1. Introduction

Recently, there has been a lot of interest in the study of oscillation and nonoscillation of second order difference equations see, for example [1–4, 6, 7–13] and the references contained therein. Following this trend in this paper, we shall consider the perturbed quasilinear difference equation

$$\Delta(a_{n-1}|\Delta y_{n-1}|^{\alpha-1}\Delta y_{n-1}) + F(n, y_n) = G(n, y_n, \Delta y_n), \tag{1}$$

where $n \in N(n_0) = \{n_0, n_0 + 1, \dots\}$, ($n_0$ is a fixed nonnegative integer), Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, $\{a_n\}$ is a positive real sequence and $\alpha > 0$. Moreover, F and G are real valued functions with $y_n : N(n_0) \to R$, $F : N(n_0) \times R \to R$ and $G : N(n_0) \times R^2 \to R$.

By a solution of (1) we mean a real sequence $\{y_n\}$ satisfying (1) for $n \in N(n_0)$. A nontrivial solution $\{y_n\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

The purpose of this paper is to establish some new results on the oscillatory and asymptotic behaviour of solutions of (E). The results obtained here differ greatly from those in [6, 7, 8, 11, 12, 13] and other known literature.

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Throughout this paper we shall assume that there exist real sequences $\{q_n\}$, $\{p_n\}$ and a function $f : R \to R$ such that

(c_1) $uf(u) > 0$ for all $u \neq 0$;

(c_2) $f(u) - f(v) = g(u,v)(u-v)$ for $u, v \neq 0$,

where g is a nonnegative function; and

(c_3) $\dfrac{F(n,u)}{f(u)} \geq q_n$, $\dfrac{G(n,u,v)}{f(u)} \leq p_n$ for $u, v \neq 0$.

2. Asymptotic behaviour of nonoscillatory solutions

In this section, we assume that

$$\sum_{j=n_0}^{\infty} (q_j - p_j) = \infty \,. \tag{2}$$

Theorem 1. *Let condition* (c_1) – (c_3) *and* (2) *hold, then any nonoscillatory solution of* (1) *must belong to one of the following two types:*

$$\begin{aligned} &M_c : y_n \to c \neq 0 \quad \textit{for} \quad n \to \infty \,, \\ &M_0 : y_n \to 0 \qquad \textit{for} \quad n \to \infty \,. \end{aligned}$$

Proof. Let $\{y_n\}$ be a nonoscillatory solution of (1), then $\{y_n\}$ is eventually positive or negative. Thus from (1), we have

$$\begin{aligned} \Delta\left(\frac{a_{n-1}|\Delta y_{n-1}|^{\alpha-1}\Delta y_{n-1}}{f(y_{n-1})}\right) &= \frac{\Delta(a_{n-1}|\Delta y_{n-1}|^{\alpha-1}\Delta y_{n-1})}{f(y_n)} \\ &\quad - \frac{a_{n-1}|\Delta y_{n-1}|^{\alpha-1}(\Delta y_{n-1})^2 g(y_n, y_{n-1})}{f(y_{n-1})\, f(y_n)} \\ &\leq -[q_n - p_n] - \frac{a_{n-1}|\Delta y_{n-1}|^{\alpha-1}(\Delta y_{n-1})^2 g(y_n, y_{n-1})}{f(y_{n-1})\, f(y_n)} \\ &< -(q_n - p_n) \,. \end{aligned} \tag{3}$$

Summing (3) from $n_0 + 1$ to n, we get

$$\frac{a_n|\Delta y_n|^{\alpha-1}\Delta y_n}{f(y_n)} \leq \frac{a_{n_0}|\Delta y_{n_0}|^{\alpha-1}\Delta y_{n_0}}{f(y_{n_0})} - \sum_{s=n_0+1}^{n} (q_s - p_s) \,. \tag{4}$$

If $\{y_n\}$ is eventually positive, then there exists $n_1 \in N(n_0)$ such that $y_n > 0$ for $n \in N(n_1)$. Then from (4) and (1), we get

$$\Delta y_n < 0 \qquad \text{for} \quad n \in N(n_1) \,.$$

Hence $\{y_n\}$ is monotonic decreasing, and $\lim\limits_{n\to\infty} y_n = c \geq 0$, where c is constant. The proof for the case y_n is eventually negative is similar.

Thus, any nonoscillatory solution of (1) must belong to either M_c or M_0. This completes the proof of the theorem. □

Remark 1. In the above theorem, we impose no condition on $\{a_n\}$ such as $\sum\limits_{n=n_0+1}^{\infty} \frac{1}{a_n^{1/\alpha}}$ is convergent or divergent.

Remark 2. If $\sum\limits_{n=n_0+1}^{\infty} \frac{1}{a_n^{1/\alpha}}$ is convergent or divergent and $F(n, y_n) = q_n f(y_n)$ and $G(n, u, v) = 0$ for $n \in N(n_0)$, then the possible asymptotic behaviour of nonoscillatory solutions of equation (1) are given in Lemma 1 of [10] and Lemma 1 of [9] respectively.

Remark 3. If $\alpha = 1$, then Theorem 1 reduces to Theorem 1 of [2].

Theorem 2. *Let conditions* (c_1) – (c_3) *and* (2) *hold.*

a) *If $f(u) = |u|^\alpha \operatorname{sgn} u$, then a necessary condition for equation* (1) *to have a nonoscillatory solution $\{y_n\} \in M_c$ is that*

$$\sum_{n=n_1+1}^{\infty} \left(\frac{1}{a_n} \sum_{i=n_1+1}^{n} (q_i - p_i) \right)^{1/\alpha} < \infty \tag{5}$$

where $n_1 \in N(n_0)$ is sufficiently large.

b) *If $f(u) = |u|^m \operatorname{sgn} u$ for $0 < m < \alpha$, $(m > \alpha)$ then a necessary condition for equation* (1) *to have a nonoscillatory solution $\{y_n\} \in M_c$ or $M_0(\{y_n\} \in M_0)$ is also* (5).

Proof. (a) Let $\{y_n\}$ be a nonoscillatory solution of (1) which belongs to M_c. If $c > 0$, then $\{y_n\}$ is eventually positive. From the proof of Theorem 1, we have that Δy_n is eventually negative and from (2), there exists $n_1 \in N(n_0)$ such that $y_n > 0$, $\Delta y_n < 0$ and

$$\sum_{i=n_0+1}^{n} (q_0 - p_0) > 0 \quad \text{for} \quad n \in N(n_1)\,.$$

Summing (3) from $n_1 + 1$ to n, it follows that

$$\begin{aligned} \frac{a_n |\Delta y_n|^{\alpha-1} \Delta y_n}{y_n^\alpha} &\leq \frac{a_{n_1} |\Delta y_{n_1}|^{\alpha-1} \Delta y_{n_1}}{y_{n_1}^\alpha} - \sum_{k=n_1+1}^{n} (q_k - p_k) \\ &\leq - \sum_{k=n_1+1}^{n} (q_k - p_k)\,, \end{aligned}$$

that is,

$$\left(\frac{1}{a_n}\sum_{k=n_1+1}^{n}(q_k-p_k)\right)^{1/\alpha}\le -\frac{\Delta y_n}{y_n}\,.$$

Summing the above inequality from n_1+1 to n we have

$$\sum_{k=n_1+1}^{n}\left(\frac{1}{a_k}\sum_{i=n_1+1}^{k}(q_i-p_i)\right)^{1/\alpha}\le\sum_{k=n_1+1}^{n}\frac{y_k-y_{k+1}}{y_k}\le\frac{y_{n_1+1}-c}{c}\,,$$

from which letting $n\to\infty$, we obtain (5).

(b) If $0<m<\alpha$, let $\{y_n\}$ be a solution of (1) which belongs to M_0 or M_c, as shown in the proof of case (a), we can obtain

$$\frac{\Delta y_n}{y_n^{m/\alpha}}\le\left(\frac{1}{a_n}\sum_{k=n_1+1}^{n}(q_k-p_k)\right)^{1/\alpha}\,.$$

Summing the above inequality from n_1+1 to n, we have

$$\begin{aligned}\sum_{k=n_1+1}^{n}\left[\frac{1}{a_k}\sum_{i=n_1+1}^{k}(q_i-p_i)\right]^{1/\alpha}&\le\sum_{k=n_1+1}^{n}\frac{y_k-y_{k+1}}{y_k^{m/\alpha}}\le\int_{y_{n+1}}^{y_{n_1+1}}\frac{ds}{s^{m/\alpha}}\\&=\frac{1}{1-\frac{m}{\alpha}}\left[y_{n_1+1}^{1-\frac{m}{\alpha}}-y_{n+1}^{1-\frac{m}{\alpha}}\right]\,,\end{aligned}$$

from which letting $n\to\infty$ and noting that $0<m<\alpha$ and $\lim_{n\to\infty}y_n=0$ or $\lim_{n\to\infty}y_n=c>0$, we obtain (5).

If $\{y_n\}$ is eventually negative, similarly, we can show that (5) holds. Thus the proof of Theorem 2 is complete. □

Remark. If $\alpha=1$, then Theorem 2 reduces to Theorem 2 of [2].

Example. Consider the difference equation

$$\begin{aligned}\Delta(2^{n(\alpha+2)})|\Delta y_{n-1}|^{\alpha-1}\Delta y_{n-1})+y_n(b(n,y_n)+\theta_n)&\\=b(n,y_n)y_n\,,\qquad n\ge 1\,,&\end{aligned}\tag{8}$$

where $\alpha>0$ and $b(n,y_n)$ is any function of n and y_n and $\theta_n=3\cdot(2)^n$. Choosing $f(y_n)=y_n$ we have

$$\frac{F(n,y_n)}{f(y_n)}=b(n,y_n)+\theta_n=b(n,y_n)+3(2^n)=q_n$$

and $\dfrac{G(n,y_n,\Delta y_n)}{f(y_n)}=b(n,y_n)=p_n$. All conditions of Theorem 1 hold, and hence a nonoscillatory solution $\{y_n\}$ of (8) must belong either to M_c or M_0. In fact $\{y_n\}=\{\frac{1}{2^n}\}\in M_0$ is one such solution.

3. Oscillation of solutions

In this section we will establish some oscillation criteria for equation (1).

Theorem 3. *Let $f(u) = |u|^\alpha \operatorname{sgn} u$ and the condition* (c_3) *holds. Assume that function $H(n,s)$ satisfies*

$$\begin{aligned} H(n,n) &= 0 \qquad \text{for} \quad n \geq n_0 \in N(n_0)\,, \\ H(n,s) &> 0 \qquad \text{for} \quad n > s \geq n_0 \quad \text{and} \\ \Delta_2 H(n,s) &\leq 0 \qquad \text{for} \quad n > s \geq n_0 \quad \text{where} \end{aligned}$$

$$\Delta_2 H(n,s) = H(n,s+1) - H(n,s)\,.$$

Suppose there exists a real valued function $h(n,s)$ such that

$$-\Delta_2 H(n,s) = h(n,s)[H(n,s)]^{1/\beta} \qquad \text{for all} \qquad n > s \geq n_0$$

where $\beta = \dfrac{\alpha+1}{\alpha}$. If for sufficiently large $n_1 \in N(n_0)$ such that for $n \in N(n_1)$

$$q_n - p_n \geq 0\,, \tag{9}$$

$$\sum_{n=n_0+1}^{\infty} \frac{1}{a_{n-1}^{1/\alpha}} = \infty\,, \tag{10}$$

and

$$\limsup_{n\to\infty} \frac{1}{H(n,n_0)} \sum_{s=n_0}^{n-1} \left[H(n,s)(q_s - p_s) - a_{s-1} \left(\frac{h(n,s)}{\alpha+1} \right)^{\alpha+1} \right] = \infty \tag{11}$$

then all solutions of equation (1) *are oscillatory.*

Proof. Let $\{y_n\}$ be a nonoscillatory solution of equation (1). Without loss of generality we may assume that $y_n > 0$ for $n \geq N_0 \in N(n_0)$. From condition (9) and the equation (1), we get

$$\Delta(a_{n-1}|\Delta y_{n-1}|^{\alpha-1}\Delta y_{n-1}) \leq 0, \qquad n \geq N_0$$

and hence $a_{n-1}|\Delta y_{n-1}|^{\alpha-1}\Delta y_{n-1}$ is nonincreasing. We claim that

$$a_{n-1}|\Delta y_{n-1}|^{\alpha-1}\Delta y_{n-1} > 0\,.$$

If $a_{n-1}|\Delta y_{n-1}|^{\alpha-1}\Delta y_{n-1} \leq 0$, then there exists an integer $N_1 \in N(n_0)$ such that

$$a_{n-1}|\Delta y_{n-1}|^{\alpha-1}\Delta y_{n-1} < a_{N_1-1}|\Delta y_{N_1-1}|^{\alpha-1}\Delta y_{N_1-1} \leq 0$$

for $n > N_1$.
Since

$$a_{n-1}|\Delta y_{n-1}|^{\alpha-1}\Delta y_{n-1} = -a_{n-1}(-\Delta y_{n-1})^\alpha\,,$$

we have

$$-a_{n-1}(-\Delta y_{n-1})^{\alpha} < -a_{N_1-1}(-\Delta y_{N_1-1})^{\alpha}$$

or

$$\Delta y_{n-1} < \frac{a_{N_1-1}^{1/\alpha}(\Delta y_{N_1-1})}{a_{n-1}^{1/\alpha}}, \qquad n > N_1 .$$

Summing the above inequality from N_1+1 to n and letting $n \to \infty$, we obtain, in view of condition (10), $y_n \to -\infty$, a contradiction. Hence there exists an integer $N_0 \in N(n_0)$ such that $a_{n-1}|\Delta y_{n-1}|^{\alpha-1}\Delta y_{n-1} > 0$ or $\Delta y_{n-1} > 0$ for all $n \geq N_0$. Define

$$z_n = \frac{a_{n-1}(\Delta y_{n-1})^{\alpha}}{y_{n-1}^{\alpha}}, \qquad \text{for} \quad n \geq N_0 .$$

Then

$$\Delta z_n \leq -(q_n - p_n) - \frac{a_{n-1}(\Delta y_{n-1})^{\alpha}\Delta y_{n-1}^{\alpha}}{y_{n-1}^{\alpha} y_n^{\alpha}} \qquad \text{for} \quad n \geq N_0 . \tag{12}$$

By the mean value theorem

$$\Delta y_{n-1}^{\alpha} = \alpha\ t_n^{\alpha-1}\ \Delta y_{n-1} \tag{13}$$

where $y_{n-1} < t_n < y_n$. Using (13) in (12), we get

$$\Delta z_n < -(q_n - p_n) - \frac{\alpha a_{n-1}(\Delta y_{n-1})^{\alpha} y_{n-1}^{\alpha-1}\Delta y_{n-1}}{y_{n-1}^{\alpha}\, y_n^{\alpha}} \qquad \text{for} \quad n \geq N_0 . \tag{14}$$

Now using monotonic nature of $\{y_n\}$ and $\{a_n(\Delta y_n)^{\alpha}\}$, in (14), we obtain

$$\Delta z_n \leq -(q_n - p_n) - \frac{\alpha z_{n+1}^{1+1/\alpha}}{a_{n-1}^{1/\alpha}} \qquad \text{for all} \quad n \geq N_0 .$$

Then, for all $n \geq N \geq n_0$

$$\begin{aligned}
\sum_{s=N}^{n-1} H(n,s)\,(q_s - p_s) &= H(n,N)z_N - \sum_{s=N}^{n-1}\Big(-\Delta_2 H(n,s) z_{s+1} + \frac{\alpha H(n,s) z_{s+1}^{\beta}}{a_{s-1}^{1/\alpha}}\Big) \\
&= H(n,N)z_N - \sum_{s=N}^{n-1}\left(h(n,s)H^{1/\beta}(n,s)z_{s+1} + \frac{\alpha H(n,s)}{a_{s-1}^{1/\alpha}} z_{s+1}^{\beta}\right) \\
&= H(n,N)z_N - \sum_{s=N}^{n-1}\Bigg[h(n,s)H^{1/\beta}(n,s)z_{s+1} + \frac{\alpha H(n,s)}{a_{s-1}^{1/\alpha}}\, z_{s+1}^{\beta} \\
&\qquad + \left(\frac{h(n,s)}{\alpha+1}\right)^{\alpha+1} a_{s-1}\Bigg] + \sum_{s=N}^{n-1}\left(\frac{h(n,s)}{\alpha+1}\right)^{\alpha+1} a_{s-1} .
\end{aligned}$$

Hence, for all $n \geq N \geq N_0$, we have

$$\sum_{s=N}^{n-1}\left[H(n,s)(q_s-p_s)-a_{s-1}\left(\frac{h(n,s)}{\alpha+1}\right)^{\alpha+1}\right]=H(n,N)z_N$$
$$-\sum_{s=N}^{n-1}\left[h(n,s)H^{1/\beta}(n,s)z_{s+1}+\frac{\alpha H(n,s)}{a_{s-1}^{1/\alpha}}\,z_{s+1}^{\beta}+\left(\frac{h(n,s)}{\alpha+1}\right)^{\alpha+1}a_{s-1}\right].$$

Since $\beta > 0$, we have

$$h(n,s)H^{1/\beta}(n,s)z_{s+1}+\frac{\alpha H(n,s)}{a_{s-1}^{1/\alpha}}\,z_{s+1}^{\beta}+\left(\frac{h(n,s)}{\alpha+1}\right)^{\alpha+1}a_{s-1}\geq 0$$

for all $n > s \geq N_0$. This implies that for all $n \geq N_0$

$$\begin{aligned}\sum_{s=N_0}^{n-1}\left[H(n,s)\,(q_s-p_s)-a_{s-1}\left(\frac{h(n,s)}{\alpha+1}\right)^{\alpha+1}\right]&\\ \leq H(n,N_0)\,z_{N_0}&\\ \leq H(n,n_0)\,z_{N_0}\,.&\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{s=n_0}^{n-1}&\left[H(n,s)\,(q_s-p_s)-\,a_{s-1}\left(\frac{h(n,s)}{\alpha+1}\right)^{\alpha+1}\right]\\ &=\sum_{s=n_0}^{N_0-1}\left[H(n,s)\,(q_s-p_s)-a_{s-1}\left(\frac{h(n,s)}{\alpha+1}\right)^{\alpha+1}\right]\\ &\quad+\sum_{s=N_0}^{n-1}\left[H(n,s)\,(q_s-p_s)-a_{s-1}\left(\frac{h(n,s)}{\alpha+1}\right)^{\alpha+1}\right]\\ &\leq H(n,n_0)\sum_{s=n_0}^{N_0-1}(q_s-p_s)+H(n,n_0)\,z_{N_0}\\ &=H(n,n_0)\left[\sum_{s=n_0}^{N_0-1}(q_s-p_s)+z_{N_0}\right]\end{aligned}$$

for all $n \geq N_0$. This gives

$$\limsup_{n\to\infty}\frac{1}{H(n,n_0)}\sum_{s=n_0}^{n-1}\left[H(n,s)\,(q_s-p_s)-a_{s-1}\left(\frac{h(n,s)}{\alpha+1}\right)^{\alpha+1}\right]$$
$$\leq\sum_{s=n_0}^{N_0-1}(q_s-p_s)+z_{N_0}$$

which contradicts (13). This completes the proof of the theorem. □

In the following theorems, we obtain oscillation criterion for equation (1) by using a generalized Riccati transformation due to Yu [14].

Theorem 4. *Let $\alpha = 1$ and the conditions* (c_1) – (c_3), (9) *and* (10), *satisfies with* $g(u,v) \geq \lambda > 0$ *for* $u, v \neq 0$. *Let* $H(n,s)$ *be such that* $H(n,n) = 0$ *for* $n \in N(n_0)$, $H(n,s) > 0$ *for* $n > s \in N(n_0)$ *and* $\Delta_2 H(n,s)$ *is nonpositive for all* $n \geq s \in N(n_0)$. *Suppose there exists a real valued function* $h(n,s)$ *with*

$$-\Delta_2 H(n,s) = h(n,s)\sqrt{H(n,s)} \quad \text{for all} \quad n \geq s \in N(n_0)\,.$$

If there exists a real positive sequence $\{\Phi_n\}$ *with* $\Phi_{n+1} < \dfrac{1}{2\lambda}$ *and*

$$\limsup_{n\to\infty} \frac{1}{H(n,n_0)} \sum_{s=n_0}^{n-1} \Big[H(n,s)\Psi_s - \frac{1}{4\lambda}\,\frac{a_s\Phi_{s+1}^2}{\Phi_s} h^2(n,s)\Big] = \infty$$

where $\Phi_n = \dfrac{\Phi_{n_0}}{1 + 2\lambda(n-n_0)\Phi_{n_0}}$ *and*

$$\Psi_n = \Phi_n\Big[(q_n - p_n) + \lambda a_n \Phi_{n+1}^2 - \Delta(a_{n-1}\Phi_n)\Big]$$

then all solutions of equation (1) *are oscillatory.*

Proof. Let $\{y_n\}$ be a nonoscillatory solution of equation (1). Without loss of generality we may assume that $y_n > 0$ for $n \geq N_0 \in N(n_0)$. From the proof of Theorem 3 and from equation (1), we have $\Delta y_{n-1} > 0$ and $\{a_{n-1}\Delta y_{n-1}\}$ is nonincreasing for all $n \geq N_1 \geq N_0 + 1$. Define

$$z_n = \Phi_n\Big[\frac{a_{n-1}\Delta y_{n-1}}{f(y_n)} + a_{n-1}\Phi_n\Big] \qquad \text{for} \quad n \geq N_1\,.$$

Then

$$\begin{aligned}\Delta z_n = \Delta\Phi_n\Big[\frac{a_n\Delta y_n}{f(y_{n-1})} + a_n\Phi_{n+1}\Big] + \Phi_n\Big[&\frac{\Delta(a_{n-1}\Delta y_{n-1})}{f(y_n)} \\ &- \frac{a_n\Delta y_n\, g(y_n, y_{n-1})\Delta y_n}{f(y_n)f(y_{n-1})} + \Delta(a_{n-1}\Phi_n)\Big]\,, \\ &\qquad\qquad n \geq N_1\,.\end{aligned}$$

In view of condition (c_3) and from the monotonic nature of $\{y_n\}$ and $\{a_{n-1}\Delta y_{n-1}\}$, we obtain

$$\begin{aligned}\Delta z_n &\leq \frac{\Delta\Phi_n z_{n+1}}{\Phi_{n+1}} + \Phi_n\Big[-(q_n - p_n) - \frac{\lambda}{a_n}\big(\frac{z_{n+1}}{\Phi_{n+1}} - a_n\Phi_{n+1}\big)^2 + \Delta(a_{n-1}\Phi_n)\Big] \\ &= -\,\Psi_n - \frac{\lambda}{a_n}\,\frac{\Phi_n}{\Phi_{n+1}^2}\, z_{n+1}^2\,.\end{aligned}$$

Hence for all $n \geq N \geq N_1$, we have

$$\begin{aligned}\sum_{s=N}^{n-1} H(n,s)\Psi_s &\leq H(n,N)\Psi_N - \sum_{s=N}^{n-1}\Big(-\Delta_2 H(n,s)\, z_{s+1} + \frac{\lambda H(t,s)}{a_s\Phi_{s+1}^2}\Phi_s z_{s+1}^2\Big)\\ &= H(n,N)\Psi_N - \sum_{s=N}^{n-1}\Big(h(n,s)\sqrt{H(n,s)} z_{s+1} + \frac{\lambda}{a_s}\,\frac{\Phi_s}{\Phi_{s+1}} H(n,s) z_{s+1}^2\Big)\\ &= H(n,N)\Psi_N - \sum_{s=N}^{n-1}\left(\sqrt{\frac{\lambda\Phi_s H(n,s)}{a_s\Phi_{s+1}^2}} z_{s+1} + \frac{1}{2}\,\frac{h(n,s)\Phi_{s+1}\sqrt{a_s}}{\sqrt{\lambda\Phi_s}}\right)^2\\ &\qquad + \frac{1}{4\lambda}\sum_{s=N}^{n-1}\frac{h^2(n,s)\Phi_{s+1}^2 a_s}{\Phi_s}\,.\end{aligned}$$

Then, for all $n \geq N \geq N_1$

$$\begin{aligned}\sum_{s=N}^{n-1}\Big[H(n,s)\Psi_s - \frac{1}{4\lambda}\,\frac{a_s\Phi_{s+1}^2 h^2(n,s)}{\Phi_s}\Big] &\leq H(n,N)\Psi_N\\ &\quad - \sum_{s=N}^{n-1}\left(\sqrt{\frac{\lambda\Phi_s H(n,s)}{a_s\Phi_{s+1}^2}} z_{s+1} + \frac{1}{2}\,\frac{h(n,s)\Phi_{s+1}\sqrt{a_s}}{\sqrt{\lambda\Phi_s}}\right)^2 .\end{aligned}$$

This implies that for every $n \geq N_1$

$$\begin{aligned}\sum_{s=N_1}^{n-1}\left[H(n,s)\Psi_s - \frac{1}{4\lambda}\,\frac{a_s\Phi_{s+1}^2 h^2(n,s)}{\Phi_s}\right] &\leq H(n,N_1)\Psi_{N_1}\\ &\leq H(n,N_1)|\Psi_{N_1}|\\ &\leq H(n,n_0)|\Psi_{N_1}|\,.\end{aligned}$$

The rest of the proof is similar to that of Theorem 3 and hence the details are omitted. □

To prove the next result, we need the following well-known inequality which is due to Hardy, Littlewood and Polya [5, Theorem 41].

Lemma 5. *If X and Y are nonnegative, then*

$$X^\beta + (\beta-1)Y^\beta - \beta X Y^{\beta-1} \geq 0, \qquad \beta > 1\,,$$

where equality holds if and only if $X = Y$.

Theorem 6. *Let conditions* (c_1) – (c_3) *with $g(u,v) \geq 0$ for $u, v \neq 0$, and condition* (9) *and* (10) *hold. If there exists a positive (nondecreasing for $\alpha \neq 1$) real sequence $\{c_n\}$ such that*

$$\sum_{k=n_1+1}^{\infty} c_k(q_k - p_k) = \infty \tag{15}$$

and

$$\sum_{k=n_1+1}^{\infty} \frac{a_{k-1}(\Delta c_{k-1})^{\alpha+1}}{c_k^{\alpha}\big[\sum\limits_{s=k}^{\infty}(q_s-p_s)\big]^{\alpha}} < \infty \tag{16}$$

then all solutions of equation (1) *are oscillatory.*

Proof. Suppose there exists a nonoscillatory solution $\{y_n\}$. Without loss of generality, we may assume that $y_n > 0$ for $n \in N(n_0)$. From the proof of Theorem 3, we have

$$\Delta y_{n-1} > 0 \quad \text{and} \quad a_{n-1}(\Delta y_{n-1})^{\alpha} \le a_{n_1}(\Delta y_{n_1})^{\alpha} = M_0 \,. \tag{17}$$

for all $n \ge n_1 \in N(n_0)$.

Similar to (4), we have

$$\Delta\left(\frac{a_{n-1}(\Delta y_{n-1})^{\alpha}}{f(y_n)}\right) < -(q_n-p_n) \quad \text{for} \quad n \ge n_1 \,.$$

Summing the last inequality from $n \in N(n_1)$, to $N+1$ and letting $N \to \infty$, we have

$$0 \le \lim_{N\to\infty} \frac{a_N(\Delta y_N)^{\alpha}}{f(y_N)} \le \frac{a_{n-1}(\Delta y_{n-1})^{\alpha}}{f(y_n)} - \sum_{s=n}^{\infty}(q_s-p_s)\,.$$

Thus

$$\frac{1}{a_{n-1}}\sum_{s=n}^{\infty}(q_s-p_s) \le \frac{(\Delta y_{n-1})^{\alpha}}{f(y_n)}, \quad \text{for} \quad n \in N(n_1)\,. \tag{18}$$

Now,

$$\begin{aligned}
\Delta\Big(\frac{a_{n-1}c_{n-1}(\Delta y_{n-1})^{\alpha}}{f(y_{n-1})}\Big) &= c_n\frac{G(n,y_n,\Delta x_n)-F(n,y_n)}{f(y_n)} \\
&\quad - \frac{c_n a_{n-1}(\Delta y_{n-1})^{\alpha+1}g(y_n,y_{n-1})}{f(y_n)f(y_{n-1})} + \frac{a_{n-1}(\Delta y_{n-1})^{\alpha}\Delta c_{n-1}}{f(y_{n-1})} \\
&\le -c_n(q_n-p_n) - \frac{\lambda c_n a_{n-1}(\Delta y_{n-1})^{\alpha+1}}{f(y_n)f(y_{n-1})} + \frac{a_{n-1}(\Delta y_{n-1})^{\alpha}\Delta c_{n-1}}{f(y_{n-1})} \\
&= -c_n(q_n-p_n) - \frac{a_{n-1}f^{\alpha}(y_n)}{f(y_{n-1})}\Big[\frac{\lambda c_n(\Delta y_{n-1})^{\alpha+1}}{f(y_n)^{\alpha+1}} \\
&\quad - \frac{(\Delta y_{n-1})^{\alpha}\Delta c_{n-1}}{f^{\alpha}(y_n)} + \frac{\big(\frac{\alpha\Delta c_{n-1}}{\alpha+1}\big)^{\alpha+1}}{\alpha\lambda^{\alpha}c_n^{\alpha}}\Big] + \frac{a_{n-1}\big(\frac{\alpha}{\alpha+1}\Delta c_{n-1}\big)^{\alpha+1}}{\alpha\lambda^{\alpha}c_n^{\alpha}}\,\frac{f^{\alpha}(y_n)}{f(y_{n-1})}\,.
\end{aligned}$$

Since by lemma

$$\frac{\lambda c_n(\Delta y_{n-1})^{\alpha+1}}{f(y_n)^{\alpha+1}} - \frac{(\Delta y_{n-1})^{\alpha}\Delta c_{n-1}}{f^{\alpha}(y_n)} + \frac{\big(\frac{\alpha}{\alpha+1}\Delta c_{n-1}\big)^{\alpha+1}}{\alpha\lambda^{\alpha}c_n^{\alpha}} \ge 0$$

for all $n \geq n_1$. Then we have
(19)

$$\Delta\left(\frac{a_{n-1}c_{n-1}(\Delta y_{n-1})^\alpha}{f(y_{n-1})}\right) \leq -c_n(q_n - p_n) + \frac{a_{n-1}\left(\frac{\alpha}{\alpha+1}\Delta c_{n-1}\right)^{\alpha+1}}{\alpha\lambda^\alpha c_n^\alpha}\frac{f^\alpha(y_n)}{f(y_{n-1})}$$

$$= -c_n(q_n - p_n) + \frac{a_{n-1}\left(\frac{\alpha}{\alpha+1}\Delta c_{n-1}\right)^{\alpha+1}}{\alpha\lambda^\alpha c_n^\alpha}\frac{(\Delta y_{n-1})^{\alpha^2}}{(\Delta y_{n-1})^{\alpha^2}}\frac{f^\alpha(y_n)}{f(y_{n-1})}$$

$$= -c_n(q_n - p_n) + \frac{a_{n-1}^{1-\alpha}\left(\frac{\alpha}{\alpha+1}\Delta c_{n-1}\right)^{\alpha+1}(a_{n-1}(\Delta y_{n-1})^\alpha)^\alpha}{\alpha\lambda^\alpha c_n^\alpha\left[\frac{(\Delta y_{n-1})^\alpha}{f(y_n)}\right]^\alpha f(y_{n_1-1})}.$$

Using (17), (18) in (19), we obtain

$$\Delta\left(\frac{a_{n-1}c_{n-1}(\Delta y_{n-1})^\alpha}{f(y_{n-1})}\right) \leq -c_n(q_n - p_n) + \frac{Ma_{n-1}(\Delta c_{n-1})^{\alpha+1}}{c_n^\alpha\left[\sum\limits_{s=n}^{\infty}(q_s - p_s)\right]^\alpha} \tag{20}$$

where $M = \dfrac{M_0\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}}{\alpha\lambda^\alpha f(y_{n_1-1})}$. Summing (20) from $n_1 + 1$ to n, we have

$$\frac{a_n c_n(\Delta y_n)^\alpha}{f(y_n)} \leq \frac{a_{n_1}c_{n_1}(\Delta y_{n_1})^\alpha}{f(y_{n_1})} - \sum_{k=n_1+1}^{n} c_k(q_k - p_k)$$

$$+ M\sum_{k=n_1+1}^{n}\frac{a_{k-1}(\Delta c_{k-1})^{\alpha+1}}{\left[c_k\sum\limits_{i=k}^{\infty}(q_i - p_i)\right]^\alpha}.$$

Letting $n \to \infty$ and noting (15) and (16), we get

$$\lim_{n\to\infty}\frac{a_n c_n(\Delta y_n)^\alpha}{f(y_n)} = -\infty$$

which is a contradiction. This completes the proof of the theorem. □

Remark. When $\alpha = 1$ and $f(u) = u$ Theorem 6 reduces to Theorem 4 of B. Liu and J. Yan [2].

Example. Consider the difference equation

$$\Delta\left(\frac{1}{n^{1+\sigma}}|\Delta y_{n-1}|^{\alpha-1}\Delta y_n\right) + \frac{2^\alpha}{n^{1+\delta}}y_n = \frac{-2^\alpha y_n}{4(n+1)^{1+\delta}}(\Delta y_n)^2, \qquad n \in N(1) \tag{21}$$

where $\alpha > 0$, $0 < \delta < 1$. Let $c_n = n$, $q(n) = \frac{1}{n^{1+\delta}}$ and $p_n = 0$, $n \in N(1)$ and $f(y_n) = y_n$, then we find that all conditions of Theorem 6 are satisfied. Hence all solutions of (21) are oscillatory. In fact $\{y_n\} = \{(-1)^n\}$ is such a solution. We believe that the conclusion is not deducible from the oscillation criteria in [2, 3, 7, 8, 12, 13] and other known results.

References

[1] Agarwal, R.P., *Difference Equations and Inequalities*, Marcel Dekker, New York, 1992.

[2] Bing Liu, Yan, J., *Oscillatory and asymptotic behaviour of second order nonlinear difference equations*, Proc. Edin. Math. Soc. **39**(1996), 525-533.

[3] Bing Liu, Cheng, S.S., *Positive solutions of second order nonlinear difference equation*, J. Math. Anal. Appl. **204**(1996), 482-493.

[4] Cheng, S.,S., Li, H.J., *Bounded and zero convergent solutions of second order difference equations*, J. Math. Anal. Appl. **14**(1989), 141-149.

[5] Hardy, G.H., Littlewood, J.E., Polya, G., *Inequalities*, 2nd Edition, Cambridge University Press, 1988.

[6] He, H.Z., *Oscillatory and asymptotic behaviour of second order nonlinear difference equations*, J. Math. Anal. Appl. **175**(1993), 482-498.

[7] Szmanda, B., *Nonoscillation, oscillation and growth of solutions of nonlinear difference equations of second order*, J. Math. Anal. Appl. **109**(1985), 22-30.

[8] Thandapani, E., *Oscillation theorems for perturbed nonlinear second order difference equations*, Computers Math. Appl. **28**(1994), 309-316.

[9] Thandapani, E., Arul, R., *Oscillation and nonoscillation theorems for a class of second order quasilinear difference equations*, ZAA, **16** (1997), 749-759.

[10] Thandapani, E., Arul, R., *Oscillation theory for a class of second order quasilinear difference equations*, Tamkang J. Math. **28** (1997), 229-238.

[11] Trench, W.F., *Asymptotic behaviour of solutions of Emden-Fowler difference equations with oscillating coefficients*, J. Math. Anal. Appl. **179**(1993), 135-153.

[12] Wong, P.J.Y., Agarwal, R.P., *Oscillation theorems and existence of positive monotone solutions for second order nonlinear difference equations*, Math. Comput. Modelling **21**(1995), 63-84.

[13] Wong, P.J.Y., Agarwal, R.P., *Oscillation and monotone solutions of second order quasilinear difference equations*, Funk. Ekva. **39**(1996), 491-517.

[14] Yu, Y.H., *Higher type oscillation criterion and Sturm type comparison theorem*, Math. Nachr. **153**(1991), 485-496.

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