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# ON THE OSCILLATION OF A CLASS OF LINEAR HOMOGENEOUS THIRD ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper we have considered completely the equation

$$y''' + a(t)y'' + b(t)y' + c(t)y = 0\,, \tag{*}$$

where $a \in C^2([\sigma,\infty),R)$, $b \in C^1([\sigma,\infty),R)$, $c \in C([\sigma,\infty),R)$ and $\sigma \in R$ such that $a(t) \le 0$, $b(t) \le 0$ and $c(t) \le 0$. It has been shown that the set of all oscillatory solutions of (*) forms a two-dimensional subspace of the solution space of (*) provided that (*) has an oscillatory solution. This answers a question raised by S. Ahmad and A. C. Lazer earlier.

## 1. Introduction

The object of this work is to answer a question raised in [1, p. 683] for the differential equation

$$y''' + a(t)y'' + b(t)y' + c(t)y = 0\,, \tag{1.1}$$

where $a \in C^2([\sigma,\infty),R)$, $b \in C^1([\sigma,\infty),R)$, $c \in C([\sigma,\infty),R)$ and $\sigma \in R$ such that $a(t) \le 0$, $b(t) \le 0$ and $c(t) \le 0$ with $b(t) \not\equiv 0$ and $c(t) \not\equiv 0$ on any sub-interval of $[\sigma,\infty)$. A solution $y(t)$ of (1.1) on $[\sigma,\infty)$ is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if it has an oscillatory solution; otherwise, it is said to be nonoscillatory.

If $a(t)$, $b(t)$ and $c(t)$ are constants, then (1.1) reduces to

$$y''' + ay'' + by' + cy = 0 \tag{1.2}$$

By the rule of signs, the characteristic equation

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 \tag{1.3}$$

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has one and only one positive real root $\gamma$, say, and either two complex conjugate roots $\alpha + i\beta$ and $\alpha - i\beta$, where

$$2\alpha = -(\alpha^2 + \beta^2 - b)/\gamma < 0\,,$$

or two negative real roots, counting multiplicities. In the first case every solution of (1.2) is of the form

$$c_1 e^{\gamma t} + e^{\alpha t}(c_2 \cos \beta t + c_3 \sin \beta t)\,.$$

Thus a real nontrivial solution of (1.2) has arbitrarily large zeros if and only if $c_1 = 0$. If $c_1 \neq 0$, then the solution and all of its derivatives have the same sign from a certain point on. In the second case, all solutions of (1.2) are nonoscillatory and there exist solutions whose derivatives alternate in sign. In fact, (1.3) admits two imaginary roots if

$$\frac{ab}{3} - \frac{2a^3}{27} - c - \frac{2}{3\sqrt{3}}\left(\frac{a^2}{3} - b\right)^{3/2} > 0\,.$$

Otherwise, (1.3) has two negative roots.

In an attempt to obtain results similar to the above observations for (1.1), Ahmad and Lazer [1] have obtained the following theorems.

**Theorem 1.** *Suppose that $a(t) \le 0$, $b(t) \le 0$ and $c(t) < 0$. The following statements are equivalent:*

A) *There exists an oscillatory solution of (1.1)*
B) *If $w$ is a nonoscillatory solution of (1.1), then there exists a $t_0 \ge \sigma$ such that $w(t)w'(t)w''(t) \neq 0$ for $t \ge t_0$ and $\operatorname{sgn} w(t) = \operatorname{sgn} w'(t) = \operatorname{sgn} w''(t)$, $t \ge t_0$.*

**Theorem 2.** *If $a(t) \le 0$, $b(t) \le 0$, $c(t) < 0$ and (1.1) admits an oscillatory solution, then there exist two linearly independent oscillatory solutions $u$ and $v$ of (1.1) such that any nontrivial linear combination of $u$ and $v$ is also oscillatory and the zeros of $u$ and $v$ separate.*

**Theorem 3.** *If $p(t) < 0$, $q(t) \le 0$, $2p(t) - q'(t) \le 0$ for $t \in [\sigma, \infty)$ and*

$$y''' + q(t)y' + p(t)y = 0 \tag{1.4}$$

*has an oscillator solution, then there exist two linearly independent oscillatory solutions $u$ and $v$ of (1.4) whose zeros separate and such that a solution of (1.4) is oscillatory if and only if it is a nontrivial linear combination of $u$ and $v$.*

In [3, Theorem 1], Jones has obtained Theorem 3 above without the condition $2p(t) - q'(t) \le 0$. However, in the proof of his Theorem 1, the identity $y_1'y_2 - y_1y_2' = N$ is not clear. This identity plays the crucial role in the proof of the theorem.

In section 2 of this work we have obtained a result for (1.1) which provides an affirmative answer to the question raised by Ahmad and Lazer in [1]. However, we have additional restrictions *viz*, $a(t)$ is twice and $b(t)$ is once continuously differentiable. This is because we are taking the help of adjoints. The adjoint of (1.1) is given by

$$x''' - (a(t)x)'' + (b(t)x)' - c(t)x = 0\,. \tag{1.5}$$

Equation (1.1) may be written as

$$(r(t)y'')' + q(t)y' + p(t)y = 0\,, \tag{1.6}$$

where $r(t) = \exp\left(\int_\sigma^t a(s)\,ds\right)$, $q(t) = r(t)b(t)$ and $p(t) = r(t)c(t)$. The adjoint of (1.6) is written as

$$(r(t)z')'' + (q(t)z)' - p(t)z = 0\,. \tag{1.7}$$

We may note that the transformation $x = r(t)z$ transforms (1.5) into (1.7) and vice versa. In Section 3 we obtain sufficient conditions for oscillation of (1.1).

Equation (1.1) is said to be of Class I or $C_I$ if any of its solutions $y(t)$ for which $y(t_0) = y'(t_0) = 0$, $y''(t_0) > 0$ $(\sigma < t_0 < \infty)$ satisfies $y(t) > 0$ for $t \in [\sigma, t_0)$. Equation (1.1) is said to be of Class II or $C_{II}$ if any of its solutions $y(t)$ for which $y(t_0) = y'(t_0) = 0$, $y''(t_0) > 0$ $(\sigma \le t_0 < \infty)$ satisfies $y(t) > 0$ for $t > t_0$.

**Remark 4.** It is easy to see that (1.5) is of $C_I$ or $C_{II}$ if and only if (1.7) is of $C_I$ or $C_{II}$. Further, (1.5) is oscillatory if and only if (1.7) is oscillatory.

## 2. Main results

**Theorem 5.** *If (1.1) has an oscillatory solution, then there exist two linearly independent oscillatory solutions $y_1(t)$ and $y_2(t)$ of (1.1) whose zeros separate and such that a solution of (1.1) is oscillatory if and only if it is a nontrivial linear combination of $y_1(t)$ and $y_2(t)$.*

Theorem 5 may be written as follows:

**Theorem 6.** *If (1.1) has an oscillatory solution, then the set of all oscillatory solutions of (1.1) form a two dimensional subspace of the solution space of (1.1).*

In the following we obtain some results which are interesting in themselves and which will be needed for the proof of Theorem 5.

**Lemma 7.** *Equation (1.6) is of Class II.*

**Proof.** Let $y(t)$ be a solution of (1.6) with $y(t_0) = 0 = y'(t_0)$ and $y''(t_0) > 0$, $t_0 > \sigma$. From the continuity of $y''$ it follows that $y(t) > 0$ in a neighbourhood of $t_0$. We claim that $y(t) > 0$ for $t > t_0$. If not, there exists a $t_1 > t_0$ such that $y''(t_1) = 0$ and $y(t) > 0$, $y'(t) > 0$, $y''(t) > 0$ for $t \in (t_0, t_1)$. Now integrating (1.6) from $t_0$ to $t_1$ we obtain

$$0 > -r(t_0)y''(t_0) = -\int_{t_0}^{t_1} [q(t)y'(t) + p(t)y(t)]\,dt > 0\,,$$

a contradiction. Hence our claim holds and this completes the proof of the lemma.□

**Lemma 8.** *Equation (1.7) is of Class I.*

**Proof.** Equation (1.6) is of $C_{II}$ implies that Equation (1.1) is of $C_{II}$. This in turn implies that, by Lemma 2.9 due to Hanan [2], Equation (1.5) is of $C_I$ and hence Equation (1.7) is of $C_I$. □

**Lemma 9.** *Equation (1.6) is oscillatory if and only if Equation (1.7) is oscillatory.*

This follows from Theorem 4.1 due to Hanan [2].

**Proposition 10.** *Equation (1.7) admits a nonoscillatory solution $N(t)$ satisfying $N(t) > 0$, $N'(t) < 0$ and $(rN')'(t) + q(t)N(t) > 0$ for $t \in [\sigma, \infty)$.*

The proof is similar to that of Theorem 2 due to Jones [3] and hence is omitted.

**Lemma 11.** *With $N(t)$ as in Proposition 10, the following statements hold:*

(i) $\lim_{t\to\infty} r(t)N'(t) = 0$

(ii) $\lim_{t\to\infty} tr(t)N'(t) = 0$

(iii) $\lim_{t\to\infty} t^2[(rN')'(t) + q(t)N(t)] = 0\,.$

The proof is simple and along the lines of Jones [3] and hence is omitted.

**Theorem 12.** *If (1.1) has an oscillatory solution, then there exist two linearly independent oscillatory solutions $u_1(t)$ and $u_2(t)$ of*

$$\left(\frac{y'}{N(t)}\right)' + \left(\frac{(rN')'(t) + q(t)N(t)}{r(t)N^2(t)}\right)\, y = 0 \tag{2.1}$$

*which satisfy (1.1).*

**Proof.** Since Eq. (1.1) has an oscillatory solution, then Eq. (1.6) has an oscillatory solution. From Lemma 9 it follows that (1.7) is oscillatory. It is clear from a result due to Hanan [2, Theorem 3.4] that a solution of (1.7) which has at least one zero is oscillatory.

Let $z_1(t)$ and $z_2(t)$ be two linearly independent solutions of (1.7) with

$$\begin{gathered} z_1(\sigma) = 0 = z_1'(\sigma)\,, \quad z_1''(\sigma) = 1 \\ z_2(\sigma) = 0\,, \quad z_2'(\sigma) = 1\,, \quad (rz_2')'(\sigma) = 0\,. \end{gathered}$$

So $z_1(t)$ and $z_2(t)$ are oscillatory. It is easy to verify that

$$w_1(t) = N(t)z_1'(t) - N'(t)z_1(t) = N^2(t)\left(\frac{z_1}{N}\right)'(t)$$

and

$$w_2(t) = N(t)z_2'(t) - N'(t)z_2(t) = N^2(t)\left(\frac{z_2}{N}\right)'(t)$$

are oscillatory solutions of

$$\left(\frac{(r(t)x)'}{N(t)}\right)' + \left(\frac{(rN')'(t) + q(t)N(t)}{N^2(t)}\right) x = 0\,. \tag{2.2}$$

Consequently, $u_1(t) = r(t)w_1(t)$ and $u_2(t) = r(t)w_2(t)$ are oscillatory solutions of (2.1). It may be shown easily that $u_1(t)$ and $u_2(t)$ satisfy (1.6) and hence (1.1).

To complete the proof of the theorem, it is to be shown that $u_1(t)$ and $u_2(t)$ are linearly independent. If possible, let $u_1(t)$ and $u_2(t)$ be linearly dependent. So there exist $c_1$ and $c_2$, not both zero, such that $c_1u_1(t) + c_2u_2(t) = 0$ for $t \in [\sigma, \infty)$, that is, $c_1w_1(t) + c_2w_2(t) = 0$ for $t \in [\sigma, \infty)$. Since $w_1(t)$ and $w_2(t)$ are nontrivial solutions of (2.2), then $c_1 = 0$ implies that $c_2 = 0$ and $c_2 = 0$ implies that $c_1 = 0$. Hence $c_1 \neq 0$ and $c_2 \neq 0$. Now $w_1(t) + \lambda w_2(t) = 0$ for $t \in [\sigma, \infty)$, where $\lambda = c_2/c_1$, implies that

$$\frac{z_1'(t) + \lambda z_2'(t)}{z_1(t) + \lambda z_2(t)} = \frac{N'(t)}{N(t)}\,.$$

Thus $N(t) = c(z_1(t) + \lambda z_2(t))$, $c \neq 0$. Consequently, $z_1(t) + \lambda z_2(t)$ is nonoscillatory. Hence there exists a $t_1 > \sigma$ such that $z_1(t) + \lambda z_2(t)$ has one sign for $t \geq t_1$. Let $t_2$ and $t_3 (t_1 < t_2 < t_3)$ be successive zeros of $z_1$. From a result in [5] it follows that the function $(z_1(t) + \lambda z_2(t)) + \mu z_1(t)$ has a double zero in $(t_2, t_3)$, where $\mu$ is a constant, that is, $(1 + \mu)z_1(t) + \lambda z_2(t)$ has a double zero in $(t_2, t_3)$. Clearly, $(1 + \mu)z_1(t) + \lambda z_2(t)$ is a solution of (1.7) with a zero at $t = \sigma$. This contradicts the fact that (1.7) is of $C_I$. Thus $u_1(t)$ and $u_2(t)$ are linearly independent.

This completes the proof of the theorem. □

**Remark 13.** Any solution of (2.1) is a solution of (1.1). It is possible to choose two linearly independent solutions $y_1(t)$ and $y_2(t)$ of (2.1) such that $w(t) > 0$, where $w(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$. Since (2.1) is oscillatory, $y_1(t)$ and $y_2(t)$ are oscillatory. Moreover, $y_1(t)$ and $y_2(t)$ are solutions of (1.1).

**Proposition 14.** *$N$ and $W$ are linearly dependent. In fact, $W(t) = \lambda N(t)$, where $\lambda > 0$ is a constant.*

**Proof.** As $y_1(t)$ and $y_2(t)$ are linearly independent solutions of (2.1), then $W(t) \neq 0$ for $t \geq \sigma$. Clearly, $y_1(t)$ and $y_2(t)$ are solutions of

$$\begin{vmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ r(t)y'' & r(t)y_1'' & r(t)y_2'' \end{vmatrix} = 0\,,$$

$t \in [\sigma, \infty)$, that is, of

$$y'' - \left(\frac{W'(t)}{W(t)}\right) y' + \left(\frac{(rW')'(t) + q(t)W(t)}{r(t)W(t)}\right) y = 0\,. \tag{2.3}$$

Equation (2.1) may be written as

$$y'' - \left(\frac{N'(t)}{N(t)}\right) y' + \left(\frac{(rN')'(t) + q(t)N(t)}{r(t)N(t)}\right) y = 0\,.$$

Clearly, equations (2.1) and (2.3) have the same solution space. If $u(t)$ is a solution of (2.1), then it is a solution of (2.3) and hence $u(t)$ is a solution of the first order equation

$$a_1(t)y' + b_1(t)y = 0\,,$$

where

$$a_1(t) = \frac{N'(t)}{N(t)} - \frac{W'(t)}{W(t)}$$

and

$$b_1(t) = \frac{(rW')'(t) + q(t)W(t)}{r(t)W(t)} - \frac{(rN')'(t) + q(t)N(t)}{r(t)N(t)}\,.$$

Hence, in particular,

$$\begin{aligned} a_1(t)y_1'(t) + b_1(t)y_1(t) &= 0 \\ a_1(t)y_2'(t) + b_1(t)y_2(t) &= 0\,. \end{aligned}$$

Since $W(t) \neq 0$ for $t \geq \sigma$, then $a_1(t) = 0$ and $b_1(t) = 0$ for $t \geq \sigma$. But $a_1(t) = 0$, $t \geq \sigma$ implies that $W(t) = \lambda N(t)$, where $\lambda \neq 0$ is a constant. Further, $W(t) > 0$ and $N(t) > 0$ implies that $\lambda > 0$.

Hence the proposition is proved. □

**Remark 15.** In view of Proposition 14, Proposition 10, Lemma 11 and Theorem 12 will hold when $N(t)$ is replaced by $W(t)$.

**Theorem 16.** *For any solution $y(t)$ of*

$$\left(\frac{y'}{W(t)}\right)' + \left(\frac{(rW')'(t) + q(t)W(t)}{r(t)W^2(t)}\right) y = 0\,, \tag{2.4}$$

*the function $G(y(t))$ is a decreasing function of $t$, where*

$$G(y(t)) = r(t)W(t)(y'(t))^2 + ((rW')'(t) + q(t)W(t))y^2(t)\,.$$

The proof is similar to that of Theorem 4 due to Jones [3] and hence is omitted.

The proof of Theorem 5 of this paper proceeds along the lines of the proof of Theorem 1 in [3]. However, for completeness and clarity the proof is given here.

**Proof of Theorem 5.** From Remark 13 it follows that there exist two linearly independent oscillatory solutions $y_1(t)$ and $y_2(t)$ of (1.1) whose zeros separate. To complete the proof of the theorem it is enough to show that any oscillatory solution of (1.1) can be expressed as a linear combination of $y_1(t)$ and $y_2(t)$.

Let $y_3(t)$ be a solution of (1.1) with $y_3(\sigma) = 0 = y_3'(\sigma)$, $y_3''(\sigma) > 0$. From Lemma 7 it follows that $y_3(t) > 0$ for $t > \sigma$. Consequently, $y_3'(t) > 0$ and $y_3''(t) > 0$ for $t > \sigma$. Clearly $\{y_1, y_2, y_3\}$ is linearly independent. Hence

$$\begin{vmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1'(t) & y_2'(t) & y_3'(t) \\ r(t)y_1''(t) & r(t)y_2''(t) & r(t)y_3''(t) \end{vmatrix} = k\,,$$

where $k \neq 0$ is a constant. Thus

$$\begin{vmatrix} y_1(t) & y_2(t) & u(t) \\ y_1'(t) & y_2'(t) & u'(t) \\ r(t)y_1''(t) & r(t)y_2''(t) & r(t)u''(t) \end{vmatrix} = 1\,,$$

where $u(t) = y_3(t)/k$. Expanding the determinant we get

$$r(t)W(t)u''(t) - r(t)W'(t)u'(t) + ((rW')'(t) + q(t)W(t))u(t) = 1\,. \tag{2.5}$$

Clearly, $k < 0$ implies that $u(t) < 0$, $u'(t) < 0$ and $u''(t) < 0$. This in turn leads to contradiction in (2.5) where the left hand side becomes negative. Thus $k > 0$.

Let $z(t)$ be an oscillatory solution of (1.1). We claim that $z(t)$ can be expressed as a linear combination of $y_1(t)$ and $y_2(t)$. If not, there exist $c_1$, $c_2$ and $c_3 \neq 0$, such that $z(t) = c_1y_1(t) + c_2y_2(t) + c_3u(t)$. We may note that $c_1$ and $c_2$ cannot be zero simultaneously. Writing

$$z_1(t) = z(t)/c_3 \quad \text{and} \quad y(t) = -(c_1y_1(t) + c_2y_2(t))/c_3\,,$$

we get

$$z_1(t) = u(t) - y(t)\,.$$

Clearly, $y(t)$ is an oscillatory solution (nontrivial) of (2.4) and (1.1). Thus $z_1(t)$ is a solution of (2.5). Consequently,

$$\begin{aligned} r(t)W(t)(u(t) - y(t))'' - r(t)W'(t)(u(t) - y(t))' \\ + ((rW')'(t) + q(t)W(t))(u(t) - y(t)) = 1\,. \end{aligned} \tag{2.6}$$

Since $z(t)$ is oscillatory, $u(t) - y(t)$ is oscillatory. From Theorem 16 it follows that

$$((rW')'(t) + q(t)W(t))y^2(t)$$

is bounded. As

$$\begin{aligned} &[((rW')'(t) + q(t)W(t))y(t)]^2 \\ &\qquad = ((rW')'(t) + q(t)W(t))y^2(t) \cdot t^2((rW')'(t) + q(t)W(t)) \cdot t^{-2}\,, \end{aligned}$$

from Lemma 11 (iii) we obtain

$$\lim_{t\to\infty}[((rW')'(t)+q(t)W(t))y(t)]=0\,.$$

Hence there exists a $T>\sigma$ such that

$$|((rW')'(t)+q(t)W(t))y(t)|<1/4$$

for $t\geq T$. From (2.5) we get, for $t\geq\sigma$,

$$0[((rW')'(t)+q(t)W(t))u(t)]<1\,.$$

Let $t_0>T$ be a maximum of $u-y$. So $u(t_0)-y(t_0)\geq 0$ and $u'(t_0)-y'(t_0)=0$. Now multiplying (2.6) through by $u'(t)-y'(t)$ and integrating the resulting identity from $t_0$ to $t$, we obtain

$$\begin{aligned}
&\frac{1}{2}r(t)W(t)(u'(t)-y'(t))^2\\
&\quad-\frac{1}{2}\int_{t_0}^{t}(rW)'(s)(u'(s)-y'(s))^2\,ds\\
&\quad-\int_{t_0}^{t}r(s)W'(s)(u'(s)-y'(s))^2\,ds\\
&\quad+\frac{1}{2}((rW')'(t)+q(t)W(t))(u(t)-y(t))^2\\
&\quad-\frac{1}{2}((rW')'(t_0)+q(t_0)W(t_0))(u(t_0)-y(t_0))^2\\
&\quad-\frac{1}{2}\int_{t_0}^{t}p(s)W(s)(u(s)-y(s))^2\,ds\\
&=(u-y)(t)-(u-y)(t_0)\,,
\end{aligned}$$

since $W(t)$ is a solution of (1.7). As $(rW)'(t)=r'(t)W(t)+r(t)W'(t)<0$, we have

$$(u-y)(t_0)[1-\frac{1}{2}((rW')'(t_0)+q(t_0)W(t_0))(u-y)(t_0)]<(u-y)(t)\,.$$

But

$$\begin{aligned}
((rW')'(t_0)&+q(t_0)W(t_0))(u-y)(t_0)\\
&=((rW')'(t_0)+q(t_0)W(t_0))u(t_0)\\
&\quad-((rW')'(t_0)+q(t_0)W(t_0))y(t_0)\\
&<1+\frac{1}{4}=\frac{5}{4}\,.
\end{aligned}$$

So, for $t>t_0$,

$$(u-y)(t)>\frac{3}{8}(u-y)(t_0)\geq 0\,,$$

which contradicts the fact that $u-y$ is oscillatory. Hence our claim holds.

This completes the proof of the theorem. □

## 3. Sufficient conditions for oscillation

In this section we have obtained sufficient conditions for oscillation of (1.1).

**Theorem 17.** *Suppose that $a(t) \le 0$, $a'(t) \ge 0$, $b(t) - 2a'(t) \le 0$ and $c(t) - b'(t) + a''(t) < 0$. If*

$$\begin{aligned} \int^{\infty} \Big[ &-\frac{2}{27}a^3(t) + \frac{1}{3}a(t)b(t) - c(t) - \frac{2}{3}a(t)a'(t) + b'(t) - a''(t) \\ &- \frac{2}{3\sqrt{3}}\left(\frac{1}{3}a^2(t) - b(t) + 2a'(t)\right)^{3/2}\Big]\, dt = \infty\,, \end{aligned} \tag{3.1}$$

*then (1.1) is oscillatory.*

**Proof.** Since (1.1) is of $C_{II}$, then from Theorem 4.7 due to Hanan [2] it follows that (1.1) is oscillatory if and only if (1.5) is oscillatory, that is, if

$$\begin{gathered} x''' - a(t)x'' + (b(t) - 2a''(t))x' \\ -(c(t) - b'(t) + a''(t))x = 0 \end{gathered}$$

is oscillatory. From Theorem 8 [6] it follows that (1.5) is oscillatory and hence (1.1) is oscillatory.

Hence the theorem is proved. □

**Remark 18.** The above theorem generalizes Theorem 2.6 due to Lazer [4].

**Remark 19.** We may note that no sign restriction has been imposed on $b(t)$ and $c(t)$ in the above theorem.

**Theorem 20.** *Suppose that $a(t) \le 0$, $b(t) \le 0$, $c(t) < 0$, $a'(t) \ge 0$, $a''(t) \le 0$ and $b'(t) \ge 0$. If (3.1) holds, then (1.1) is oscillatory.*

This follows from Theorem 17.

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