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ON TORSION GORENSTEIN INJECTIVE MODULES

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Abstract. In this paper, we define Gorenstein injective rings, Gorenstein injective modules and their envelopes. The main topic of this paper is to show that if D is a Gorenstein integral domain and M is a left D-module, then the torsion submodule tGM of Gorenstein injective envelope GM of M is also Gorenstein injective. We can also show that if M is a torsion D-module of a Gorenstein injective integral domain D, then the Gorenstein injective envelope GM of M is torsion.

1. Gorenstein injective modules and envelopes

Definition 1. (**Iwanaga** [5]) A ring is said to be Gorenstein if it is left and right Noetherian and if it has a finite injective dimension as a module over itself both on the left and on the right. If R is Gorenstein and $n \geq 0$ is an upper bound for those two dimensions, then R is said to be n-Gorenstein.

Proposition 1. (**Iwanaga** [6]) If R is n-Gorenstein and if M is a left R-module the followings are equivalent:

(1) $proj.dim_R M < \infty$;
(2) $proj.dim_R M \leq n$;
(3) $inj.dim_R M < \infty$;
(4) $inj.dim_R M \leq n$;
(5) $flat.dim_R M < \infty$;
(6) $flat.dim_R M \leq n$.

Examples of Gorenstein rings :

(1) $\mathcal{Z}/(n)$ for $n \neq 0$ is 0-Gorenstein.
(2) A Noetherian commutative ring of finite global dimension n is n-Gorenstein.
(3) If R is n-Gorenstein, then $R[x]$ is $(n+1)$-Gorenstein.

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(4) If R is n-Gorenstein, then for any $k \geq 1$, $M_k(R)$ (the $k \times k$ matrices over R) is n-Gorenstein.

(5) If k is a field and $f \in k[x_1, \cdots, x_n]$, then if $f \neq 0$ and f is not a constant, $k[x_1, \cdots, x_n]/(f)$ is $(n-1)$-Gorenstein.

For any ring R, $\mathcal{L}$ denotes the class of left R-modules of finite projective dimension.

Definition 2. For a Gorenstein ring R, K is Gorenstein injective if and only if $Ext^1(L, K) = 0$ whenever $proj.dimL < \infty$. Gorenstein injective right R-modules are defined analogously.

Proposition 2. If R is Gorenstein ring, K is a Gorenstein injective left R-module and $E \longrightarrow K$ is an injective cover of K, then $E \longrightarrow K$ is surjective and is also an $\mathcal{L}$-cover of K. Furthermore $ker(E \longrightarrow K)$ is Gorenstein injective.

Proof. Proposition 4.3 [4]. □

Definition 3. If N is a left R-module then a complex

$$\cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow N \longrightarrow 0$$

is called an injective resolvent of N if all the E_i's are injective and if for any injective module E the functor $\mathrm{Hom}(E, -)$ makes the complex exact. We note this is equivalent to $E_0 \longrightarrow N$, $E_1 \longrightarrow ker(E_0 \longrightarrow N)$, and $E_{i+1} \longrightarrow ker(E_i \longrightarrow E_{i-1})$ for $i \geq 1$ being injective precovers. If in fact these are all injective covers, we call this a minimal injective resolvent of N and write $E_i = E_i(N)$.

If we consider a minimal injective resolvent and a minimal injective resolution of N, say

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & E_1(N) & \longrightarrow & E_0(N) & \longrightarrow & N & \longrightarrow & 0 \\ 0 & \longrightarrow & N & \longrightarrow & E^0(N) & \longrightarrow & E^1(N) & \longrightarrow & \cdots \end{array} \tag{1}$$

then pasting together along N we get a complex

$$\cdots \longrightarrow E_1(N) \longrightarrow E_0(N) \longrightarrow E^0(N) \longrightarrow \cdots$$

Definition 4. The complex above is said to be the complete minimal injective resolution of N.

Proposition 3. The left R-module N is Gorenstein injective if and only if

$$0 = Ext_i(Q, N) = Ext^i(Q, N) = \overline{Ext}_0(Q, N) = \overline{Ext}^0(Q, N),$$

for all $i \geq 1$ and for modules Q which have finite injective or projective dimension.

Proof. This is just the usual dimension shifting argument. □

Proposition 4. The left R-module N is Gorenstein injective if and only if the complex

$$\cdots \to E_1(N) \to E_0(N) \to E^0(N) \to E^1(N) \to \cdots$$

is exact and remains exact when $\mathrm{Hom}(E, -)$ is applied to it for any injective module E.

Proof. Since this complex can be used to compute $\overline{Ext}_0(-, N)$, $\overline{Ext}^0(-, N)$, $Ext_i(-, N)$ for $i \geq 1$ we see that exactness is equivalent to the vanishing of these functors when applied to projective modules. The hypothesis guarantees they vanish when applied to injective modules.

Corollary 1. N is Gorenstein injective if and only if its minimal injective resolvent

$$\cdots \to E_1(N) \to E_0(N) \to N \to 0$$

is exact and if $Ext^i(E, N) = 0$ for all $i \geq 1$ and injective left R-module E.

Proof. This is just a reformation of the preceeding result. □

Proposition 5. If $0 \to A \to N \to B \to 0$ is exact sequence of left R-module such that

$$0 \to Hom(E, A) \to Hom(E, N) \to Hom(E, B) \to 0$$

is exact for all injective modules E then if A and B are Gorenstein injective, so is N.

Proof. Use the extended long exact sequence associated with

$$0 \longrightarrow A \longrightarrow N \longrightarrow B \longrightarrow 0\,.$$ □

Definition 5. A left R-module N is said to be reduced if it has no injective submodules other than 0.

Given a left R-module N, let

$$\bullet \cdots \to E_1(N) \to E_0(N) \to E^0(N) \to E^1(N) \to \cdots$$

be its complete minimal injective resolution. Let $C_i = ker(E_i(N) \to E_{i-1}(N))$ for $i \geq 1$, $C_0 = ker(E_0(N) \to E^0(N))$ and $C^i = ker(E^i(N) \to E^{i+1}(N)$ for $i \geq 0$.

Proposition 6. If N is a reduced, Gorenstein injective left R-module, then the complex $\bullet$ is the complete minimal injective resolution of any of the terms C_i and C^i for $i \geq 0$. And each C_i and C^i is reduced and Gorenstein injective.

Proof. This is true for $C^0 = N$ by hypothesis. If we can argue that it is true for C_0 and C^1, then by repeating the argument we can get it true for any of C_i's and C^i's. We first argue for C^1. We have $0 \to N \to E^0(N) \to C^1 \to 0$ is exact. Also $Ext^1(E, N) = 0$ for any injective module E so $E^0(N) \to C^1$ is an arrow injective precover. If $E \subset C^1$ is an injective submodule, then

$$\begin{tikzcd}
E \arrow[d, "f"'] \arrow[dr] & \\
E^0(N) \arrow[r] & C^1
\end{tikzcd}$$

can be completed to a commutative diagram. But then $f(E) \bigcap N = 0$, so $f(E) = 0$ and hence $E = 0$. So C^1 is reduced. If $E^0(N) \to C^1$ were not a cover, then $ker(E^0(N) \to C^1) = N$ would contain a non-zero injective module, contradicting the fact that N is reduced. Hence $(\bullet)$ is the complete minimal injective resolution of C^1. Proposition 4 shows that C^1 is Gorenstein injective. The proof for C_0 is similar. $\square$

If $f : M \to N$ is a linear map then it is easy to see that the following are equivalent:

(1) f can be factorized through some injective module E;

(2) f can be factorized by $M \to E^0(M)$;

(3) f can be factorized by $E_0(N) \to N$.

Then we have

Corollary 2. If N is a reduced Gorenstein injective left R-module and $0 \to K \to E_0(N) \to N \to 0$ is exact, then the set of $f \in End(N)$ which can be factorized through an injective module form a two-sided ideal of $End(N)$ contained in the jacobson radical of $End(N)$. If $A \subset End(N)$ is this ideal and $B \subset End(K)$, is the corresponding ideal for K, then $End(N/A) \cong End(K/B)$.

Proof. A is clearly a two-sided ideal. Suppose $f : N \to N$ can be factorized $N \overset{g}{\to} E_0(N) \overset{\phi}{\to} N$. Then

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & E_0(N) & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow id & & \downarrow id + g\phi & & \downarrow id + f & & \\
0 & \longrightarrow & K & \longrightarrow & E_0(N) & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

is commutative. Since $K \to E_0(N)$ is an injective envelope, $id + g\phi$ is an isomorphism, and hence so id $id + f$. We see that any map $K \to K$ is part of a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & E_0(N) & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & E_0(N) & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

and similarly for any map $N \to N$. It is a simple matter of chasing diagrams to check that this (not one to one) correspondence between maps $K \to K$ and maps $N \to N$ induces an isomorphism $End(K/B) \cong End(N/A)$. $\square$

Using the notation above we have

Corollary 3. If N is a reduced Gorenstein injective module and $0 \to N \to N \to E^0(N) \to C \to 0$ is exact, then if $\mathcal{D} \subset End(C)$ is the set of $g \in End(C)$ that can be factorized through injective modules then

$$End(N)/A \cong End(C)/\mathcal{D}\,.$$

Proof. Similar to that of the above Corollary. □

Proposition 7. If $0 \to A \to N \to B \to 0$ is an exact sequence of left R-module, then if A and N are Gorenstein injective then so is B. If N and B are Gorenstein injective, then A is Gorenstein injective if and only if $Ext^1(E, A) = 0$ for all injective left R-modules E. If A and B are Gorenstein injective, then so is N.

Proof. For the first claim we note that since A is Gorenstein injective $Ext^1(E, A) = 0$ for all injective module E. Hence

$$0 \to Hom(E, A) \to Hom(E, N) \to Hom(E, B) \to 0$$

is exact for such E and so we can appeal to the long exact sequence. The proof of the remaining claims is similar. □

Corollary 4. If N_1 and N_2 are Gorenstein injective then so is $N_1 \oplus N_2$.

Proposition 8. A reduced Gorenstein injective module $N \neq 0$ has infinite injective and projective dimension.

Proof. If any $C^i = 0$ for $i \geq 0$, then all the terms of the complete minimal injective resolution of C_i will be 0, hence $C^0 = N$ will be 0. Hence inj. dim $N = \infty$. Since $\bullet$ is a minimal injective resolution of C_i for $i \geq 0$, we see that $Ext^{i+1}(N, C_i) = Ext^1(N, C_0)$. Since $0 \to C_0 \to E_0(N) \to N \to 0$ doesn't split, $Ext^1(N, C_0) \neq 0$. So proj. dim $N \geq i + 1$ for all i. □

Proposition 9. If R is an n-Gorenstein ring and a left R-module K is Gorenstein injective if and only if K is an n-th cosyzygy, i.e., there is an exact sequence

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^n \longrightarrow K \longrightarrow 0$$

of injective resolution of M.

Proof. Given such an exact sequence of modules over the n-Gorenstein ring R, let proj. dim ${}_R L < \infty$. Then by Proposition 1, proj. dim $L \leq n$. Hence $Ext^1(L, K) = Ext^{n+1}(L, M) = 0$. Hence K is Gorenstein injective. Conversely, suppose K is Gorenstein injective. Let $E \to K$ be an injective cover of K. If $\bar{K} = ker(E \to K)$ then $\bar{K}$ is Gorenstein injective by Proposition 2. Then if $\bar{E} \to \bar{K}$ is an injective cover of $\bar{K}$, we have the exact sequence $\bar{E} \to E \to K \to 0$.
□

Proposition 10. If R is a Gorenstein ring, then every left R-module M has a Gorenstein injective envelope GM (or we denote G(M) if it is necessary).

Proof. Theorem 6.1 [4]. □

2. Torsion Gorenstein injective modules

Definition 6. The torsion submodule tA of a module A is defined by

$$tA = \{a \in A | ra = 0 \text{ for some non-zero } r \in R\}$$

Remark. If R is not a domain, then tA might not be a submodule. If D is an integral domain and M is a torsion D-module, then the injective envelope $E(M)$ of M is torsion.

Theorem 1. If D is a Gorenstein integral domain and M is a left D-module, then the torsion submodule tGM of the Gorenstein injective envelope GM of M is Gorenstein injective.

Proof. Since D is integral domain, tGM is a submodule of GM. Since D is Gorenstein, say n-Gorenstein for some n, then by the Proposition 9, GM is an n-th cosyzygy for some N. So there is an exact sequence

$$0 \longrightarrow N \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^{n-1} \overset{d^{n-1}}{\longrightarrow} GM \longrightarrow 0$$

where E^i's are injective modules. Consider $\overline{d^{n-1}}(tGM)$ = the inverse image of tGM of $d^{n-1} = (d^{n-1})^{-1}(tGM)$. Let I be a nonzero ideal of D and consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "f"'] & D \\
& \overline{d^{n-1}}(tGM) \arrow[d] & \\
& E^{n-1} &
\end{tikzcd}$$

Since E^{n-1} is injective there exists $g : D \longrightarrow E^{n-1}$ such that $g|_I = f$. Let x be a nonzero element in I, then $g(x) = f(x) \in \overline{d^{n-1}}(tGM)$, i.e., $(d^{n-1}g)(x) \in tGM$. So there exists $\alpha \neq 0$ in D such that $\alpha[(d^{n-1}g)(x)] = \alpha x \cdot [(d^{n-1}g)(1)] = 0$ with $\alpha x \neq 0$. So $(d^{n-1}g)(1) \in tGM$ and $g(1) \in \overline{d^{n-1}}(tGM)$. So $g(D) \subset \overline{d^{n-1}}(tGM)$ since for all $\alpha \in D$, $g(\alpha) = \alpha g(1) \in tGM$. So

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "f"] & D \arrow[dl, dotted, "g"] \\ & \overline{d^{n-1}}(tGM) & \end{tikzcd}$$

So $\overline{d^{n-1}}(tGM)$ is an injective D-module. Now consider the sequence

$$0 \longrightarrow N \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^{n-2} \xrightarrow{d^{n-2}} \overline{d^{n-1}}(tGM) \xrightarrow{d'^{n-1}} tGM \longrightarrow 0$$

where d'^{n-1} is a restricted map. Obviously $E^0, E^1, \cdots, E^{n-2}, \overline{d^{n-1}}(tGM)$ are injective modules and the sequence is exact at $N, E^0, E^1, \cdots, E^{n-2}$, and $\overline{d^{n-1}}(tGM)$. And since $imd^{n-2} \subset kerd^{n-1}$, $imd^{n-2} \subseteq kerd'^{n-1}$ and if $x \in ker(d'^{n-1})$, then $d^{n-1}(x) = d'^{n-1}(x) = 0 \in tGM$. So $x \in \overline{d^{n-1}}(tGM)$. So the sequence is exact at $\overline{d^{n-1}}(tGM)$. Hence, by the Propositon 9, tGM is Gorenstein injective. $\square$

Theorem 2. If D is a Gorenstein injective integral domain and M is a torsion D-module then the Gorenstein injective envelope GM of M is also torsion.

Proof. Consider the diagram:

$$\begin{tikzcd} tM \arrow[r, "\phi_t"] \arrow[d, "i"] & G(tM) \\ M \arrow[r, "\phi"] & GM \end{tikzcd}$$

Let $x \in tM$. Then there exists a nonzero $r \in D$ such that $rx = 0$. Then $r \cdot \phi(x) = \phi(rx) = \phi(0) = 0$, so $\phi(x) \in tGM$ i.e., $\phi \circ i : tM \longrightarrow tGM$. So we get

$$\begin{tikzcd}
tM \arrow[r, "\phi_t"] \arrow[d, "i"] \arrow[dr, dotted, "{}_{tGM}|\phi|_{tM}"] & G(tM) \\
M \arrow[dr, "\phi"'] & tGM \arrow[d, "j"] \\
 & GM
\end{tikzcd}$$

with the bottom parallelogram commutative. Since tGM is Gorenstein injective there exists a map $f : G(tM) \longrightarrow tGM$ such that

$$\begin{tikzcd}
tM \arrow[r, "\phi_t"] \arrow[d, "i"] \arrow[dr, dotted, "{}_{tGM}|\phi|_{tM}"] & G(tM) \arrow[d, dotted, "f"] \\
M \arrow[dr, "\phi"'] & tGM \arrow[d, "j"] \\
 & GM
\end{tikzcd}$$

the upper triangle is commutative. Since M is torsion $tM = M$. So $G(tM) = G(M)$ and we get the following diagram.

$$\begin{tikzcd}
tM \arrow[r, "\phi_t"] \arrow[d, "i"'] \arrow[dr, dotted, "{}_{tGM}|\phi|_{tM}"] & G(tM) = GM \arrow[d, dotted, "f"] \\
M \arrow[dr, "\phi"'] & tGM \arrow[d, "j"] \\
 & GM
\end{tikzcd}$$

where j is an inclusion map. Since $j \circ f$ ia an automorphism of GM, j is a surjection. Hence j is an isomorphism between tGM and $G(M)$. Therefore $GM = G(M)$ is torsion. $\square$

Theorem 3. Let I be a directed quasi-ordered set. Suppose there are morphisms of direct systems over I

$$\{A_i, \phi_j^i\} \xrightarrow{t} \{B_i, \psi_j^i\} \xrightarrow{s} \{C_i, \theta_j^i\}$$

such that

$$0 \longrightarrow A_i \xrightarrow{t_i} B_i \xrightarrow{s_i} C_i \longrightarrow 0$$

is exact for each $i \in I$. Then there is an exact sequence of modules

$$0 \longrightarrow \varinjlim A_i \xrightarrow{\bar{t}} \varinjlim B_i \xrightarrow{\bar{s}} \varinjlim C_i \longrightarrow 0\,.$$

Proof. We prove that $\bar{t}$ is monic. Assume $x \in \varinjlim A_i$ and $\bar{t}x = 0$ in $\varinjlim B_i$. Let us set $\varinjlim A_i = (\oplus A_i)/S$ and set $\lambda_i : A_i \longrightarrow \oplus A_i$ the i^{th} injection. Let us set $\varinjlim B_i = (\oplus B_i)/T$ and set $\mu_i : B_i \longrightarrow \oplus B_i$ the i^{th} injection. Thus, $x = \lambda_i a_i + S$ and $\bar{t}x = \mu_i t_i a_i + T$. Since $\bar{t}x = 0$, there is some $j \geq i$ with $\psi_j^i t_i a_i = 0$. Since t is a morphism between direct systems, we have $t_j \phi_j^i a_i = 0$. But t_j is monic, whence $\phi_j^i a_i = 0$, and this gives $x = \lambda_i a_i + S = 0$. $\square$

Theorem 4. The following are equivalent for a ring R:

(1) R is left Noetherian,

(2) every direct limit(directed index set) of injective modules is injective,

(3) every sum of injective modules is injective.

Proof. Theorem 4.10 [7] $\square$

Theorem 5. Let R be an n-Gorenstein ring and (G_i) be a directed system of Gorenstein injective R-modules. Then $\varinjlim G_i$ is Gorenstein injective.

Proof. Since each G_i is Gorenstein injective, there exists an exact sequence ξ_i

$$0 \longrightarrow N_i \longrightarrow E_i^0 \longrightarrow E_i^1 \longrightarrow \cdots \longrightarrow E_i^{n-1} \longrightarrow G_i \longrightarrow 0\,.$$

This sequence can be constructed functorially. By Theorem 3

$$0 \longrightarrow \varinjlim N_i \longrightarrow \varinjlim E_i^0 \longrightarrow \cdots \longrightarrow \varinjlim E_i^{n-1} \longrightarrow \varinjlim G_i \longrightarrow 0$$

is also exact for each i. Since the ring R is Gorenstein each $\varinjlim E_i^j$ is injective module by the Theorem 4. So by the Theorem 9 $\varinjlim G_i$ is Gorenstein injective. $\square$

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