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ASYMPTOTIC PROPERTIES OF THE SOLUTIONS OF THE SECOND ORDER DIFFERENCE EQUATION

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Abstract. Asymptotic properties of the solutions of the second order nonlinear difference equation (with perturbed arguments) of the form

$$\Delta^2 x_n = a_n \varphi(x_{n+k})$$

are studied.

In this paper we are concerned with the difference equation

$$\Delta^2 x_n = a_n \varphi(x_{n+k}), \qquad n = 1, 2, \dots, \qquad k = 0, 1, 2, \dots \tag{E}$$

where Δ is the forward difference operator, i.e.,

$$\Delta x_n = x_{n+1} - x_n, \qquad \Delta^2 x_n = \Delta(\Delta x_n),$$

(a_n) is a sequence of real numbers and φ is a real function. Throughout this paper N denotes the set of positive integers, R denotes the set of real numbers.

Some qualitative properties of the solutions of second order nonlinear difference equations have been investigated in many papers, for instance, in [4], [6], [7]. In this paper the asymptotic behaviour of solutions will be considered. The results obtained here (Theorems 1,2,3) generalize some results of A. Drozdowicz and J. Popenda [2], [3].

We first mention a useful lemma.

Lemma. *Assume the series* $\sum_{n=1}^{\infty} n|a_n|$ *is convergent and* $r_n = \sum_{j=n}^{\infty} a_j$. *Then the series* $\sum_{n=1}^{\infty} r_n$ *is absolutely convergent and*

$$\sum_{n=1}^{\infty} r_n = \sum_{n=1}^{\infty} n a_n .$$

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Proof. Since the series $\sum_{n=1}^{\infty} na_n$ is absolutely convergent we have

$$a_1 + (a_2 + a_2) + (a_3 + a_3 + a_3) + (a_4 + a_4 + a_4 + a_4) \dots =$$
$$= (a_1 + a_2 + a_3 + \dots) + (a_2 + a_3 + a_4 + \dots) + (a_3 + a_4 + a_5 + \dots) + \dots =$$
$$= r_1 + r_2 + r_3 + \dots .$$

□

Theorem 1. *If the series* $\sum_{n=1}^{\infty} n|a_n|$ *is convergent and* $\varphi : R \to R$ *is a continuous function then for every* $c \in R$ *and for all* $k \in N$ *there exists a solution* (x_n) *of the equation* (E) *such that*

$$\lim x_n = c .$$

Proof. Let $c \in R$ and choose a real number $a > 0$. Then there exists a constant $M > 0$ such that

$$|\varphi(t)| < M \quad \text{for every} \quad t \in [c-a, c+a] . \tag{1}$$

Let us denote

$$r_n = \sum_{j=n}^{\infty} |a_j| \quad \text{for} \quad n \in N . \tag{2}$$

Using Lemma one can see that the series $\sum_{n=1}^{\infty} r_n$ is convergent. Let us denote

$$\varrho_n = \sum_{j=1}^{\infty} r_j \quad \text{for} \quad n \in N . \tag{3}$$

There exists an index $m \in N$ such that $M\varrho_n < a$ for every $n \geq m$. Let ℓ_∞ denote the Banach space of all real bounded sequences equipped with *sup* norm. Let

$$T = \{x \in \ell_\infty : x_1 = \dots = x_m = c \quad \text{and} \quad |x_n - c| \leq M\varrho_n \quad \text{for} \quad n \geq m\} .$$

Obviously, T is a convex and closed subset of the space ℓ_∞. Let $\varepsilon > 0$. It is easy to construct a finite ε-net for the set T. Hence T is compact.

If $x \in T$ then $x_n \in [c-a, c+a]$ for each $n \in N$. Hence $|\varphi(x_n)| < M$ for every $x \in T$, $n \in N$.

Let $x \in T$. Since $|\varphi(x_n)| < M$ for every $n \in N$, the series $\sum_{j=1}^{\infty} a_j\varphi(x_{j+k})$ is absolutely convergent. Denoting

$$u_n = \sum_{j=n}^{\infty} a_j\varphi(x_{j+k}), \qquad n \in N \tag{4}$$

by (2) we have

$$|u_n| \leq \sum_{j=n}^{\infty} |a_j| M = M r_n \,. \tag{5}$$

Since the series $\sum\limits_{j=1}^{\infty} |r_j|$ is convergent, the series $\sum\limits_{j=1}^{\infty} |u_j|$ is convergent, too. Now, we define the sequence $A(x)$ by

$$A(x)(n) = \begin{cases} c & \text{for} \quad n < m\,, \\ c + \sum_{j=n}^{\infty} u_j & \text{for} \quad n \geq m\,. \end{cases}$$

If $n \geq m$ then

$$|A(x)(n) - c| = \left| \sum_{j=n}^{\infty} u_j \right| \leq \sum_{j=n}^{\infty} |u_j| \,.$$

By (3), (5) we have

$$|A(x)(n) - c| \leq M \sum_{j=n}^{\infty} r_j = M \varrho_n \,.$$

Hence $A(x) \in T$ for every $x \in T$, and we get a map $a : T \to T$.

Let $x \in T$, $\varepsilon > 0$. The function φ is uniformly continuous on the interval $[c-a, c+a]$. Hence there exists a constant $\delta > 0$ such that if $t, s \in [c-a, c+a]$ and $|t - s| < \delta$ then $|\varphi(t) - \varphi(s)| < \varepsilon$. Let $z \in T$ and let $\|x - z\| < \delta$. Then $|x_n - z_n| < \delta$ for every $n \in N$. Hence

$$|\varphi(x_n) - \varphi(z_n)| < \varepsilon \quad \text{for each} \quad n \in N\,. \tag{6}$$

Let us denote

$$v_n = \sum_{j=n}^{\infty} a_j \varphi(z_{j+k}), \quad \text{for} \quad n \in N\,. \tag{7}$$

Using (4) and (7) we get

$$\|A(x) - A(z)\| = \sup_{n \geq m} \left| \sum_{j=n}^{\infty} u_j - \sum_{j=n}^{\infty} v_j \right| \leq \sum_{j=m}^{\infty} |u_j - v_j| \,.$$

By (6) one yields

$$|u_j - v_j| = \left| \sum_{i=j}^{\infty} a_i \varphi(x_{i+k}) - \sum_{i=j}^{\infty} a_i \varphi(z_{i+k}) \right| \leq$$

$$\leq \sum_{i=j}^{\infty} |a_i| \, |\varphi(x_{i+k}) - \varphi(z_{i+k})| \leq \varepsilon \sum_{i=j}^{\infty} |a_i| = \varepsilon r_j \,.$$

Hence

$$\|A(x) - A(z)\| \le \sum_{j=m}^{\infty} \varepsilon |r_j| = \varepsilon \varrho_m \,.$$

This shows that A is a continuous map.

By Schauder's theorem there exists $z \in T$ such that $A(z) = z$. Then $z_n = c + \sum\limits_{j=n}^{\infty} v_j$ for all $n \ge m$. Hence

$$\Delta z_n = c + \sum_{j=n+1}^{\infty} v_j - c - \sum_{j=n}^{\infty} v_j = -v_n \,.$$

Therefore

$$\Delta^2 z_n = -v_{n+1} + v_n = - \sum_{j=n+1}^{\infty} a_j \varphi(z_{j+k}) + \sum_{j=n}^{\infty} a_j (z_{j+k}) = a_n \varphi(z_{n+k})$$

for every $n \ge m$.

Since the series $\sum\limits_{j=1}^{\infty} v_j$ is convergent and $z_n = c + \sum\limits_{j=n}^{i} nftyv_j$ we get $\lim z_n = c$.

For every sequence (x_n) the following equality holds

$$\Delta^2 x_n = x_{n+2} - 2x_{n+1} + x_n \,.$$

Hence

$$z_{n+2} - 2z_{n+1} + z_n = a_n \varphi(z_{n+k})$$

for every $n \ge m$. Therefore

$$z_n = 2z_{n+1} + a_n \varphi(z_{n+k}) - z_{n+2} \tag{8}$$

for each $n \ge m$. Using (8) one can change successively all the terms $z_{m-1}, z_{m-2}, \ldots, z_1$ of the sequence (z_n) to obtain a solution (z_n) of the equation (E) such that $\lim z_n = c$. □

Theorem 2. *Assume that $k = 0$, the series $\sum\limits_{n=1}^{\infty} n|a_n|$ is convergent and $\varphi : R \to R$ is a continuous function such that*

(*) *for every $\alpha, x \in R$ there exists a constant $t \in R$ such that*

$$t - \alpha\varphi(t) = x \,.$$

Then for every $c \in R$ there exists a solution (x_n) of the equation (E) *with* $\lim x_n = c$.

Proof. If $c \in R$ then, as in Theorem 1, we can show that there exist $m \in N$ and a sequence (x_n) such that

$$\lim x_n = c \quad \text{and} \quad \Delta^2 x_n = a_n \varphi(x_n)$$

for every $n \geq m$.

The equation (E) can be rewritten in the form

$$x_{n+2} - 2x_{n+1} + x_n = a_n \varphi(x_n) .$$

Hence

$$x_n - a_n \varphi(x_n) = x_{n+2} - 2x_{n+1} .$$

Using (*) we can calculate successively all the terms $x_{m-1}, x_{m-2}, \ldots, x_1$ to obtain a solution (x_n) of the equation (E) which satisfy the condition $\lim x_n = c$. $\square$

Remark 1. It is easy to show that if $\varphi : R \to R$ is a continuous and bounded function or if it is a polynomial of degree $2k+1$, $k \in N$ then φ satisfies the condition (*).

Theorem 3. *It the series* $\sum\limits_{n=1}^{\infty} n|a_n|$ *is convergent and* $\varphi : R \to R$ *is a bounded and uniformly continuous function then for every* $c \in R$ *and for any* $k = 0, 1, 2, \ldots$ *there exists a solution* (x_n) *of the equation* (E) *which possesses the asymptotic behaviour*

$$x_n = cn + o(1) .$$

Proof. Let $c \in R$. Let us choose a constant $M > 0$ such that $|\varphi(t)| < M$ for each $t \in R$. Similarly as in the proof of Theorem 1 for $n \in N$ we denote r_n, ϱ_n by (2), (3).

Let ℓ be the space of all sequences $x : N \to R$ and let

$$T = \{x \in \ell_\infty : |x_n| \leq M\varrho_n \quad \text{for all} \quad n \in N\} ,$$
$$S = \{x \in \ell : |x_n - nc| \leq M\varrho_n \quad \text{for all} \quad n \in N\} .$$

We define the map $F : T \to S$ by

$$F(x)(n) = nc + x_n .$$

Obviously, the formula $d(x, z) - \sup\{|x_n - z_n| : n \in N\}$ defines a metric on the set S such that F is an isometry of the set T onto S. The set T, similarly as in the proof of Theorem 1, is a compact and convex subset of the space ℓ_∞. The space S is homeomorphic to T. Hence by Schauder's theorem every continuous map $A : S \to S$ has a fixed point.

For $x \in S$ and $n \in N$ we define u_n by (4) and

$$A(x)(n) = nc + \sum_{j=n}^{\infty} u_j \, .$$

Then

$$|A(x)(n) - nc| = \left| \sum_{j=n}^{\infty} u_j \right| \le \sum_{j=n}^{\infty} |u_j| \le M \varrho_n$$

for every $n \in N$. Hence $A(x) \in S$ and we get a map $A : S \to S$. Let $x \in S$, $\varepsilon > 0$. Since the function φ is uniformly continuous there exists a $\delta > 0$ such that if $|t - s| < \delta$ then $|\varphi(t) - \varphi(s)| < \varepsilon$. If $z \in S$ and $d(x, z) < \delta$ then $|x_n - z_n| < \delta$ for every $n \in N$. Hence $|\varphi(x_n) - \varphi(z_n)| < \varepsilon$ for every $n \in N$.

Taking u_n, v_n from (4), (7) we get

$$d(A(x), A(z)) = \sup_n \left| \sum_{j=n}^{\infty} u_j - \sum_{j=n}^{\infty} v_j \right| \le \sum_{j=1}^{\infty} |u_j - v_j| \, .$$

Since

$$\begin{aligned} |u_j - v_j| &= \left| \sum_{i=j}^{\infty} a_i \varphi(x_{i+k}) - \sum_{i=j}^{\infty} a_i \varphi(z_{i+k}) \right| \le \\ &\le \sum_{i=j}^{\infty} |a_i| \, |\varphi(x_{i+k}) - \varphi(z_{i+k})| \le \varepsilon \sum_{i=j}^{\infty} |a_i| = \varepsilon r_j \, , \end{aligned}$$

it follows that

$$d(A(x), A(z)) \le \sum_{j=1}^{\infty} \varepsilon |r_j| = \varepsilon \varrho_1 \, .$$

This shows that A s a continuous map. Hence there exists a sequence $x \in S$ such that $A(x) = x$. Then for every $n \in N$ we have

$$x_n = nc + \sum_{j=n}^{\infty} v_j \, .$$

Hence

$$\Delta x_n = (n+1)c + \sum_{j=n+1}^{\infty} v_j - nc - \sum_{j=n}^{\infty} v_j = c - v_n \,.$$

Therefore

$$\Delta^2 x_n = c - v_{n+1} - c + v_n =$$
$$= - \sum_{j=n+1}^{\infty} a_j \varphi(x_{j+k}) + \sum_{j=n}^{\infty} a_j \varphi(x_{j+k}) = a_n \varphi(x_{n+k})$$

for each $n \in N$. Hence x is a solution of the equation (E). Since the series $\sum\limits_{j=1}^{\infty} v_j$ is convergent, we obtain the asymptotic relation

$$x_n = cn + o(1) \,.$$

□

Theorem 4. *Suppose that* $\sum\limits_{n=1}^{\infty} |a_n| < \infty$ *and that* $\varphi : R \to R$ *is a bounded function. If* (x_n) *is a solution of the equation* (E) *then the sequence* (x_n/n) *is convergent in* R.

Proof. Assume that $|\varphi(t)| < M$ for every $t \in R$. If $m > n$ then

$$\Delta x_m - \Delta x_n = \sum_{j=n}^{m-1} \Delta^2 x_j = \sum_{j=n}^{m-1} a_j \varphi(x_{j+k}) \,.$$

Hence

$$|\Delta x_m - \Delta x_n| \le M \sum_{j=n}^{m-1} |a_j| \,.$$

Therefore, the sequence (Δx_n) is convergent. By virtue of Stolz's theorem (see [1], Theorem 1.7.9), $\lim x_n/n = \lim \Delta x_n$. □

Example. The sequence $x_n = 3^n$ which is a solution of the equation $\Delta^2 x_n = \frac{4}{3^{n+4}} x_{n+2}$ possesses the property $\lim x_n/n = \infty$. Hence we see the assumption of the boundedness of the function φ in Theorem 4 can not be omitted.

Theorem 5. *If $\varphi : R \to [\varepsilon, \infty)$ is a nondecreasing function, $\varepsilon > 0$, $a_n > 0$ for every $n \in N$, $\sum\limits_{n=1}^{\infty} a_n = \infty$, $k \in N$ then every solution (x_n) of the equation* (E) *possesses the asymptotic behaviour*

$$\lim x_n/n = \infty\,.$$

Proof. Suppose that (x_n) is a solution of the equation (E). Since $a_n\varphi(x_{n+k}) > 0$ for every $n \in N$, (Δx_n) is an increasing sequence. By assumption we have $\varphi(x_n) \geq \varepsilon$ for every $n \in N$. Summation of (E) over n gives

$$\sum_{j=1}^{n-1} \Delta^2 x_j = \sum_{j=1}^{n-1} a_j \varphi(x_{j+k})\,.$$

Hence

$$\Delta x_n = \Delta x_1 + \sum_{j=1}^{n-1} a_j \varphi(x_{j+k}) \geq \Delta x_1 + \varepsilon \sum_{j=1}^{n-1} a_j\,.$$

Since $\sum\limits_{n=1}^{\infty} a_n = \infty$ there exists $m \in N$ such that

$$\Delta x_1 + \varepsilon \sum_{j=1}^{n-1} a_j > 0 \quad \text{for all} \quad n \geq m\,.$$

Therefore $\Delta x_n > 0$ for each $n \geq m$. Hence the sequence (x_n) is increasing for $n \geq m$. Suppose $n \geq m$. Then

$$\Delta x_n = \Delta x_1 + \sum_{j=1}^{n-1} a_j \varphi(x_{j+k}) \geq \Delta x_1 + \sum_{j=1}^{m-1} a_j \varphi(x_{j+k}) + \sum_{j=m}^{n-1} a_j \varphi(x_{j+k}) \geq$$
$$\geq \Delta x_1 + \sum_{j=1}^{m-1} a_j \varphi(x_{j+k}) + \varphi(x_{m+k}) \sum_{j=m}^{n-1} a_j\,.$$

It follows that $\lim \Delta x_n = \infty$. By virtue of Stolz's theorem

$$\lim x_n/n = \lim \Delta x_n\,. \qquad \square$$

Example. Let

$$\varphi(t)=\begin{cases}0 & \text{for} \quad t\le 0\\ t^2 & \text{for} \quad t>0\end{cases}.$$

Then the sequence $x_n=2^{-n}$ is a solution of the equation

$$\Delta^2 x_n = 2^{n-2}\varphi(x_n)$$

such that $\lim x_n/n=0$. All assumptions of the Theorem 5 are satisfied except $\varepsilon>0$. Hence we can see that in Theorem 5 we cannot put $\varepsilon=0$.

The proofs of the following two theorems are similar to the proof of Theorem 5 and they will be omitted.

Theorem 6. *If $\varphi : R\to(-\infty,-\varepsilon]$ is nonincreasing, $a_n<0$ for every $n\in N$, $\sum\limits_{n=1}^{\infty} a_n=-\infty$, $k\in N$ then every solution (x_n) of the equation* (E) *fulfills the condition*

$$\lim x_n/n=\infty\,.$$ □

Theorem 7. *Assume $\sum\limits_{n=1}^{\infty}|a_n|=\infty$. If a function $\varphi : R\to[\varepsilon,\infty)$ is nondecreasing and $a_n<0$ for all $n\in N$ or $\varphi : R\to(-\infty,\varepsilon]$ is nonincreasing and $a_n>0$ for every $n\in N$ then every solution (x_n) of the equation* (E) *fulfills the condition*

$$\lim x_n/n=-\infty\,.$$ □

Remark 2. Theorem 2 of this paper (in the case $a_n\ge 0$ for any $n\in N$ and $\varphi(c)\ne 0$) is similar to Theorem 2 of [2]. Theorem 3 (in the case when φ is a periodic function and $k=0$, $c=1$) have been proved by A. Drozdowicz and J. Popenda (see [3] Theorem 4.1).

Remark 3. It is easy to see that the result of Theorem 3 can be extended to the *delay* difference equation

$$\text{(D)} \qquad \Delta^2 x_n = a_n\varphi(x_{n-k}), \qquad k\in N\,.$$

Moreover if we assume $\varphi(R)=R$ and $a_n\ne 0$ for every $n\in N$ then the result of Theorem 1 can also be extended to the equation (D).

Remark 4. If we assume that the series $\sum n^{m-1}a_n$ is absolutely convergent, for some fixed $m\in N$, $m\ge 2$, then the results of Theorems 1, 2 and probably also of Theorem 3 can be generalized to the case of the *higher order* difference equation

$$\Delta^m x_n = a_n\varphi(x_{n+k})\,.$$

Details concerning this will appear in [5].

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