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NATURAL TRANSFORMATIONS OF SYMPLECTIC STRUCTURES INTO POISSON'S AND JACOBI'S BRACKETS

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Abstract. A complete classification of natural transformations of symplectic structures into Poisson's brackets as well as into Jacobi's brackets is given.

It is very well-known that on an arbitrary symplectic manifold (M, ω) there is the Poisson bracket $B_M(\omega) : F(M) \times F(M) \longrightarrow F(M)$, where $F(M)$ denotes the set of all smooth functions $M \longrightarrow \mathbf{R}$. From the point of view of natural geometry (see [2]) the operators B_M which map ω to $B_M(\omega)$ form a natural transformation of symplectic structures into Poisson's brackets as well as into Jacobi's brackets. The aim of this paper is to give the full classification of natural transformations of these types. We will formulate and prove two propositions asserting that any such natural transformation is of the form μB, where μ is a real number. Of course, the result in the Poisson case is an immediate corollary from the result in the Jacobi case, but we will handle both problems separately, because the proof in the first case is shorter and less complicated then in the second one.

First we recall the definition of the Poisson and Jacobi brackets (see for instance [5]). Let M be a smooth manifold and let $F(M)$ denote the set of all smooth functions $M \longrightarrow \mathbf{R}$. A Poisson bracket on the manifold M is a map $F(M) \times F(M) \ni (f, g) \longrightarrow \{f, g\} \in F(M)$ such that:
(i) it defines a Lie algebra structure on the vector space $F(M)$, i. e.

$$\{f, \alpha g + \beta h\} = \alpha\{f, g\} + \beta\{f, h\}, \tag{1}$$

$$\{g, f\} = -\{f, g\}, \tag{2}$$

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \tag{3}$$

for all $\alpha, \beta \in \mathbf{R}$, $f, g, h \in F(M)$;

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(ii) it has a natural compatibility with the usual associative product of functions, which is

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \tag{4}$$

for all $f, g, h \in F(M)$. A Jacobi bracket is a generalization of the notion of a Poisson bracket obtained by relaxing condition (4), and asking instead that the bracket be just an operation of the local type, in the sense that

$$\operatorname{support}\{f, g\} \subset \operatorname{support} f \cap \operatorname{support} g \tag{5}$$

for all $f, g \in F(M)$.

We will denote by $P(M)$ the set of all Poisson brackets on M and by $J(M)$ the set of all Jacobi brackets on M. The set of all symplectic structures on M will be denoted by $S(M)$. Let n be a fixed positive integer.

Definition. A family of maps $A_M : S(M) \longrightarrow P(M)$ $(A_M : S(M) \longrightarrow J(M))$, where M is an arbitrary $2n$-dimensional smooth manifold, is called *a natural transformation of symplectic structures into Poisson's brackets (Jacobi's brackets),* if for every embedding $\varphi : L \longrightarrow M$ of a $2n$-dimensional smooth manifold L into a $2n$-dimensional smooth manifold M the following condition holds:

$$A_L(\omega \circ (T \times T)\varphi)(f \circ \varphi, g \circ \varphi) = A_M(\omega)(f, g) \circ \varphi \tag{6}$$

for every $\omega \in S(M)$ and all $f, g \in F(M)$.

Suppose that ω is a symplectic structure on a smooth manifold M. It is well known that for each $f \in F(M)$ there is the unique vector field $V_{\omega,f}$ on M such that

$$\omega \circ (V, V_{\omega,f}) = df \circ V \tag{7}$$

for every vector field V on M. We call $V_{\omega,f}$ the Hamilton vector field. Putting $\{f, g\} = \omega \circ (V_{\omega,f}, V_{\omega,g})$ for all $f, g \in F(M)$, we obtain the standard Poisson bracket on the symplectic manifold (M, ω). We will also use the notation

$$B_M(\omega)(f, g) = \omega \circ (V_{\omega,f}, V_{\omega,g}). \tag{8}$$

Of course, if L and M are two $2n$-dimensional smooth manifold, $\varphi : L \longrightarrow M$ is an embedding and $\omega \in S(M)$, then $T\varphi \circ V_{\omega \circ (T \times T)\varphi, f \circ \varphi} = V_{\omega,f} \circ \varphi$ for every $f \in F(M)$. Therefore, for all $f, g \in F(M)$

$$\begin{aligned} B_L(\omega \circ (T \times T)\varphi)(f \circ \varphi, g \circ \varphi) &= \omega \circ (T \times T)\varphi \circ (V_{\omega \circ (T\times T)\varphi, f\circ\varphi} V_{\omega \circ (T\times T)\varphi, g\circ\varphi}) \\ = \omega \circ (T\varphi \circ V_{\omega \circ (T\times T)\varphi, f\circ\varphi}, T\varphi \circ V_{\omega \circ (T\times T)\varphi, g\circ\varphi}) &= \omega \circ (V_{\omega,f} \circ \varphi, V_{\omega,g} \circ \varphi) \\ = \omega \circ (V_{\omega,f}, V_{\omega,g}) \circ \varphi &= B_M(\omega)(f, g) \circ \varphi\,. \end{aligned}$$

This means that the family of maps $B_M : S(M) \longrightarrow P(M)$ for all $2n$-dimensional smooth manifolds M is a natural transformation of symplectic structures into Poisson's brackets.

That there are no natural transformations of this type essentialy different from the above one is the content of the following proposition.

Proposition 1. *If A is a natural transformation of symplectic structures into Poisson's brackets, then there is one and only one real number μ such that $A_M(\omega)(f,g) = \mu\{f,g\}$ for any $2n$-dimensional smooth manifold M, any $\omega \in S(M)$ and all $f,g \in F(M)$, where $\{,\}$ is the standard Poisson bracket on the symplectic manifold (M,ω).*

Proof. Fix a $2n$-dimensional smooth manifold M, $\omega \in S(M)$ and $f \in F(M)$. The basic remark is the following obvious consequence of (1) and (4): $F(M) \ni g \longrightarrow A_M(\omega)(f,g) \in F(M)$ is a derivation. Hence there exists a well defined vector field $U_{\omega,f}$ on M such that

$$A_M(\omega)(f,g) = U_{\omega,f}(g) \tag{9}$$

for every $g \in F(M)$. Moreover, if L and M are two $2n$-dimensional smooth manifolds, $\varphi : L \longrightarrow M$ is an embedding, $\omega \in S(M)$, $f,g \in F(M)$, then

$$\begin{aligned} U_{\omega\circ(T\times T)\varphi, f\circ\varphi}(g\circ\varphi) &= A_L(\omega\circ(T\times T)\varphi)(f\circ\varphi, g\circ\varphi) \\ &= A_M(\omega)(f,g)\circ\varphi = U_{\omega,f}(g)\circ\varphi\,, \end{aligned}$$

and so $T\varphi \circ U_{\omega\circ(T\times T)\varphi, f\circ\varphi} = U_{\omega,f}\circ\varphi$. This means that the family of maps $K_{(M,\omega)} : F(M) \ni f \longrightarrow U_{\omega,f} \in V(M)$, where (M,ω) is an arbitrary $2n$-dimensional symplectic manifold and $V(M)$ denotes the set of all vector fields on M, is a natural transformation of Hamiltonians into vector fields, according to the definition from [1].The form of all natural transformations of this type is described in [1]. Using this result we can write

$$U_{\omega,f}(g) = (\lambda\circ f)V_{\omega,f}(g) \tag{10}$$

for every $2n$-dimensional smooth manifold M, every $\omega \in S(M)$ and all $f,g \in F(M)$, where $\lambda : \mathbf{R} \longrightarrow \mathbf{R}$ is a smooth function. Clearly, from (1), (2) and (9) we get that

$$U_{\omega,\alpha f}(g) = \alpha U_{\omega,f}(g) \tag{11}$$

for every $\alpha \in \mathbf{R}$. Let (x,y) denote the standard system of coordinates on $\mathbf{R}^n \times \mathbf{R}^{n*}$. Then (10), (11) and a trivial computation show that

$$\begin{aligned} &(\lambda\circ\alpha x^1)V_{dx^i\wedge dy_i,\alpha x^1}(y_1) = U_{dx^i\wedge dy_i,\alpha x^1}(y_1) \\ &= \alpha U_{dx^i\wedge dy_i, x^1}(y_1) = \alpha(\lambda\circ x^1)V_{dx^i\wedge dy_i, x^1}(y_1) = \alpha(\lambda\circ x^1) \end{aligned} \tag{12}$$

for every $\alpha \in \mathbf{R}$. Replacing the function x^1 by 1 in (12) we obtain $\lambda(\alpha) = \lambda(1)$, and so λ is constant. Write $\mu = \lambda(1)$. Finally, from (9), (10), (7), (8) we conclude that

$$\begin{aligned} &A_M(\omega)(f,g) = U_{\omega,f}(g) = \mu V_{\omega,f}(g) \\ &= \mu dg\circ V_{\omega,f} = \mu\omega\circ(V_{\omega,f}, V_{\omega,g}) = \mu B_M(\omega)(f,g). \end{aligned}$$

This proves the proposition.

We now state the analogue of Proposition 1 for Jacobi's brackets.

Proposition 2. *If A is a natural transformation of symplectic structures into Jacobi's brackets, then there is one and only one real number μ such that $A_M(\omega)(f,g) = \mu\{f,g\}$ for any $2n$-dimensional smooth manifold M, any $\omega \in S(M)$ and all $f,g \in F(M)$, where $\{,\}$ is the standard Poisson bracket on the symplectic manifold (M,ω).*

Proof. First we observe that if M is a $2n$-dimensional smooth manifold, $\omega \in S(M)$, $f,g \in F(M)$ and if U is an open subset of M then we have

$$A_U(\omega|_{(T\times T)U})(f|_U, g|_U) = A_M(\omega)(f,g)|_U. \tag{13}$$

To see this, it suffices to replace the embedding φ in (6) by the inclusion $U \longrightarrow M$. The equality (13) makes it obvious that if $e,h \in F(M)$ are such that $e|_U = f|_U$, $h|_U = g|_U$ then

$$A_M(\omega)(e,h)|_U = A_M(\omega)(f,g)|_U. \tag{14}$$

For each $f \in F(\mathbf{R}^n \times \mathbf{R}^{n*})$ we can define the map $\Phi_f : F(\mathbf{R}^n \times \mathbf{R}^{n*}) \ni g \longrightarrow A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i \wedge dy_i)(f,g) \in F(\mathbf{R}^n \times \mathbf{R}^{n*})$. This map is linear, which is clear from (1), and is local, i. e. if U is an open subset of $\mathbf{R}^n \times \mathbf{R}^{n*}$ and $g \in F(\mathbf{R}^n \times \mathbf{R}^{n*})$ is such that $g|_U = 0$, then $\Phi_f(g)|_U = 0$ too, which is clear from (14). It is known that a linear and local operator is a differential operator (see [3], [4]). This means that there are smooth functions $c_{f,(\gamma,\delta)} : \mathbf{R}^n \times \mathbf{R}^{n*} \longrightarrow \mathbf{R}$ for $(\gamma,\delta) \in \mathbf{N}^n \times \mathbf{N}^n$ such that for every $g \in F(\mathbf{R}^n \times \mathbf{R}^{n*})$

$$\Phi_f(g) = \sum_{(\gamma,\delta)\in\mathbf{N}^n\times\mathbf{N}^n} c_{f,(\gamma,\delta)} \frac{\partial^{|\gamma+\delta|} g}{\partial x^\gamma \partial y^\delta}, \tag{15}$$

where the family of functions $(c_{f,(\gamma,\delta)})_{(\gamma,\delta)\in\mathbf{N}^n\times\mathbf{N}^n}$ is locally finite, i. e. for every point of $\mathbf{R}^n \times \mathbf{R}^{n*}$ there is a neighbourhood U of this point such that the number of non-vanishing functions $c_{f,(\gamma,\delta)}|_U$ for $(\gamma,\delta) \in \mathbf{N}^n \times \mathbf{N}^n$ is finite. It is easy to check that the functions $c_{f,(\gamma,\delta)}$ for $(\gamma,\delta) \in \mathbf{N}^n \times \mathbf{N}^n$ in (15) are uniquely determined. Therefore for each $(\gamma,\delta) \in \mathbf{N}^n \times \mathbf{N}^n$ we can define the map $\Psi_{(\gamma,\delta)} : F(\mathbf{R}^n \times \mathbf{R}^{n*}) \ni f \longrightarrow c_{f,(\gamma,\delta)} \in F(\mathbf{R}^n \times \mathbf{R}^{n*})$, which is linear and local from (1), (2), (14), (15). Hence there are smooth functions $d_{(\alpha,\beta),(\gamma,\delta)} : \mathbf{R}^n \times \mathbf{R}^{n*} \longrightarrow \mathbf{R}$ for $(\alpha,\beta) \in \mathbf{N}^n \times \mathbf{N}^n$ such that for every $f \in F(M)$

$$\Psi_{(\gamma,\delta)}(f) = \sum_{(\alpha,\beta)\in\mathbf{N}^n\times\mathbf{N}^n} d_{(\alpha,\beta),(\gamma,\delta)} \frac{\partial^{|\alpha+\beta|} f}{\partial x^\alpha \partial y^\beta}, \tag{16}$$

where the family of functions $(d_{(\alpha,\beta),(\gamma,\delta)})_{(\alpha,\beta)\in\mathbf{N}^n\times\mathbf{N}^n}$ is locally finite. Combining (15) and (16) we get for all $f,g \in F(\mathbf{R}^n \times \mathbf{R}^{n*})$

$$\begin{aligned} &A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i \wedge dy_i)(f,g) \\ &\qquad = \sum_{(\alpha,\beta),(\gamma,\delta)\in\mathbf{N}^n\times\mathbf{N}^n} d_{(\alpha,\beta),(\gamma,\delta)} \frac{\partial^{|\alpha+\beta|} f}{\partial x^\alpha \partial y^\beta} \frac{\partial^{|\gamma+\delta|} g}{\partial x^\gamma \partial y^\delta}, \end{aligned} \tag{17}$$

where the family of functions $(d_{(\alpha,\beta),(\gamma,\delta)})_{(\alpha,\beta),(\gamma,\delta)\in\mathbf{N}^n\times\mathbf{N}^n}$ is locally finite.

Let $(s,t) \in \mathbf{R}^n \times \mathbf{R}^{n*}$. We define the embedding $\varphi_{(s,t)} : \mathbf{R}^n \times \mathbf{R}^{n*} \longrightarrow \mathbf{R}^n \times \mathbf{R}^{n*}$ to be such that $\varphi_{(s,t)}(x,y) = (x+s, y+t)$ for $(x,y) \in \mathbf{R}^n \times \mathbf{R}^{n*}$. We check at once that $dx^i \wedge dy_i \circ (T \times T)\varphi_{(s,t)} = dx^i \wedge dy_i$. From (17) and (6) it follows that for all $f, g \in F(\mathbf{R}^n \times \mathbf{R}^{n*})$

$$\begin{aligned}
&\sum_{(\alpha,\beta),(\gamma,\delta)\in \mathbf{N}^n\times\mathbf{N}^n} d_{(\alpha,\beta),(\gamma,\delta)} \left(\frac{\partial^{|\alpha+\beta|} f}{\partial x^\alpha \partial y^\beta} \circ \varphi_{(s,t)}\right)\left(\frac{\partial^{|\gamma+\delta|} g}{\partial x^\gamma \partial y^\delta} \circ \varphi_{(s,t)}\right) \\
&\quad = \sum_{(\alpha,\beta),(\gamma,\delta)\in \mathbf{N}^n\times\mathbf{N}^n} d_{(\alpha,\beta),(\gamma,\delta)} \frac{\partial^{|\alpha+\beta|} (f\circ\varphi_{(s,t)})}{\partial x^\alpha \partial y^\beta} \frac{\partial^{|\gamma+\delta|} (g\circ\varphi_{(s,t)})}{\partial x^\gamma \partial y^\delta} \\
&\quad = A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i \wedge dy_i)(f\circ\varphi_{(s,t)}, g\circ\varphi_{(s,t)}) \\
&\quad = A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i \wedge dy_i \circ (T\times T)\varphi_{(s,t)})(f\circ\varphi_{(s,t)}, g\circ\varphi_{(s,t)}) \\
&\quad = A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i \wedge dy_i)(f,g)\circ\varphi_{(s,t)} \\
&\quad = \left(\sum_{(\alpha,\beta),(\gamma,\delta)\in \mathbf{N}^n\times\mathbf{N}^n} d_{(\alpha,\beta),(\gamma,\delta)} \frac{\partial^{|\alpha+\beta|} f}{\partial x^\alpha \partial y^\beta} \frac{\partial^{|\gamma+\delta|} g}{\partial x^\gamma \partial y^\delta}\right) \circ \varphi_{(s,t)}.
\end{aligned}$$

Composing with $\varphi_{(s,t)}^{-1}$ we can rewrite the above equality as

$$\begin{aligned}
&\sum_{(\alpha,\beta),(\gamma,\delta)\in \mathbf{N}^n\times\mathbf{N}^n} (d_{(\alpha,\beta),(\gamma,\delta)} \circ \varphi_{(s,t)}^{-1}) \frac{\partial^{|\alpha+\beta|} f}{\partial x^\alpha \partial y^\beta} \frac{\partial^{|\gamma+\delta|} g}{\partial x^\gamma \partial y^\delta} \\
&\quad = \sum_{(\alpha,\beta),(\gamma,\delta)\in \mathbf{N}^n\times\mathbf{N}^n} d_{(\alpha,\beta),(\gamma,\delta)} \frac{\partial^{|\alpha+\beta|} f}{\partial x^\alpha \partial y^\beta} \frac{\partial^{|\gamma+\delta|} g}{\partial x^\gamma \partial y^\delta}.
\end{aligned}$$

Consequently

$$d_{(\alpha,\beta),(\gamma,\delta)} \circ \varphi_{(s,t)}^{-1} = d_{(\alpha,\beta),(\gamma,\delta)} \tag{18}$$

for all $(\alpha,\beta),(\gamma,\delta) \in \mathbf{N}^n \times \mathbf{N}^n$. Taking (18) at the point (s,t) yields $d_{(\alpha,\beta),(\gamma,\delta)}(0,0) = d_{(\alpha,\beta),(\gamma,\delta)}(s,t)$. Since (s,t) is an arbitrary point of $\mathbf{R}^n \times \mathbf{R}^{n*}$, the functions $d_{(\alpha,\beta),(\gamma,\delta)}$ for $(\alpha,\beta),(\gamma,\delta) \in \mathbf{N}^n \times \mathbf{N}^n$ are constant. Put $e_{(\alpha,\beta),(\gamma,\delta)} = d_{(\alpha,\beta),(\gamma,\delta)}(0,0)$ for $(\alpha,\beta),(\gamma,\delta) \in \mathbf{N}^n \times \mathbf{N}^n$. Thus (17) becames

$$\begin{aligned}
&A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i \wedge dy_i)(f,g) \\
&\qquad = \sum_{(\alpha,\beta),(\gamma,\delta)\in \mathbf{N}^n\times\mathbf{N}^n} e_{(\alpha,\beta),(\gamma,\delta)} \frac{\partial^{|\alpha+\beta|} f}{\partial x^\alpha \partial y^\beta} \frac{\partial^{|\gamma+\delta|} g}{\partial x^\gamma \partial y^\delta},
\end{aligned} \tag{19}$$

where $e_{(\alpha,\beta),(\gamma,\delta)} \in \mathbf{R}$ for all $(\alpha,\beta),(\gamma,\delta) \in \mathbf{N}^n \times \mathbf{N}^n$ and $e_{(\alpha,\beta),(\gamma,\delta)} = 0$ for all $(\alpha,\beta),(\gamma,\delta) \in \mathbf{N}^n \times \mathbf{N}^n$ but a finite number.

Let $p \in \mathbf{R} \setminus \{0\}$. We define the embedding $\varphi_p : \mathbf{R}^n \times \mathbf{R}^{n*} \longrightarrow \mathbf{R}^n \times \mathbf{R}^{n*}$ to be such that $\varphi_p(x,y) = (px^1, x^2, \dots, x^n, \frac{1}{p}y_1, y_2, \dots, y_n)$ for $(x,y) \in \mathbf{R}^n \times \mathbf{R}^{n*}$. We check at once that $dx^i \wedge dy_i \circ (T \times T)\varphi_p = dx^i \wedge dy_i$. From (19) and (6) it follows that for all $f, g \in F(\mathbf{R}^n \times \mathbf{R}^{n*})$

$$\begin{aligned}
&\sum_{(\alpha,\beta),(\gamma,\delta)\in\mathbf{N}^n\times\mathbf{N}^n} e_{(\alpha,\beta),(\gamma,\delta)} p^{\alpha^1-\beta^1+\gamma^1-\delta^1} \left(\frac{\partial^{|\alpha+\beta|} f}{\partial x^\alpha \partial y^\beta} \circ \varphi_p\right) \left(\frac{\partial^{|\gamma+\delta|} g}{\partial x^\gamma \partial y^\delta} \circ \varphi_p\right) \\
&\qquad = \sum_{(\alpha,\beta),(\gamma,\delta)\in\mathbf{N}^n\times\mathbf{N}^n} e_{(\alpha,\beta),(\gamma,\delta)} \frac{\partial^{|\alpha+\beta|} (f\circ\varphi_p)}{\partial x^\alpha \partial y^\beta} \frac{\partial^{|\gamma+\delta|} (g\circ\varphi_p)}{\partial x^\gamma \partial y^\delta} \\
&\qquad = A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i \wedge dy_i)(f\circ\varphi_p, g\circ\varphi_p) \\
&\qquad = A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i \wedge dy_i \circ (T\times T)\varphi_p)(f\circ\varphi_p, g\circ\varphi_p) \\
&\qquad = A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i \wedge dy_i)(f, g)\circ\varphi_p \\
&\qquad = \left(\sum_{(\alpha,\beta),(\gamma,\delta)\in\mathbf{N}^n\times\mathbf{N}^n} e_{(\alpha,\beta),(\gamma,\delta)} \frac{\partial^{|\alpha+\beta|} f}{\partial x^\alpha \partial y^\beta} \frac{\partial^{|\gamma+\delta|} g}{\partial x^\gamma \partial y^\delta}\right) \circ \varphi_p\,.
\end{aligned}$$

Composing with φ_p^{-1} we can rewrite the above equality as

$$\begin{aligned}
&\sum_{(\alpha,\beta),(\gamma,\delta)\in\mathbf{N}^n\times\mathbf{N}^n} e_{(\alpha,\beta),(\gamma,\delta)} p^{\alpha^1-\beta^1+\gamma^1-\delta^1} \frac{\partial^{|\alpha+\beta|} f}{\partial x^\alpha \partial y^\beta} \frac{\partial^{|\gamma+\delta|} g}{\partial x^\gamma \partial y^\delta} \\
&\qquad = \sum_{(\alpha,\beta),(\gamma,\delta)\in\mathbf{N}^n\times\mathbf{N}^n} e_{(\alpha,\beta),(\gamma,\delta)} \frac{\partial^{|\alpha+\beta|} f}{\partial x^\alpha \partial y^\beta} \frac{\partial^{|\gamma+\delta|} g}{\partial x^\gamma \partial y^\delta}\,.
\end{aligned}$$

Consequently

$$\alpha^1 - \beta^1 + \gamma^1 - \delta^1 \neq 0 \Longrightarrow e_{(\alpha,\beta),(\gamma,\delta)} = 0, \tag{20}$$

because $p \in \mathbf{R} \setminus \{0\}$ is arbitrary.

According to (19) and (2), we have for all $f, g \in F(\mathbf{R}^n \times \mathbf{R}^{n*})$

$$\begin{aligned}
&\sum_{(\alpha,\beta),(\gamma,\delta)\in\mathbf{N}^n\times\mathbf{N}^n} e_{(\alpha,\beta),(\gamma,\delta)} \frac{\partial^{|\alpha+\beta|} g}{\partial x^\alpha \partial y^\beta} \frac{\partial^{|\gamma+\delta|} f}{\partial x^\gamma \partial y^\delta} \\
&\qquad = A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i \wedge dy_i)(g, f) = -A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i \wedge dy_i)(f, g) \\
&\qquad = -\sum_{(\alpha,\beta),(\gamma,\delta)\in\mathbf{N}^n\times\mathbf{N}^n} e_{(\alpha,\beta),(\gamma,\delta)} \frac{\partial^{|\alpha+\beta|} f}{\partial x^\alpha \partial y^\beta} \frac{\partial^{|\gamma+\delta|} g}{\partial x^\gamma \partial y^\delta}\,.
\end{aligned}$$

Consequently $e_{(\gamma,\delta),(\alpha,\beta)} = -e_{(\alpha,\beta),(\gamma,\delta)}$ for all $(\alpha,\beta),(\gamma,\delta) \in \mathbf{N}^n \times \mathbf{N}^n$, and so

$$e_{(0,0),(0,0)} = 0. \tag{21}$$

Taking $g = x^1$ in (19) and writing $e_1, \dots, e_n$ for the canonical basis of the module $\mathbf{Z}^n$ we get for every $f \in F(\mathbf{R}^n \times \mathbf{R}^{n*})$

$$\begin{aligned}(22)\qquad & A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i \wedge dy_i)(f, x^1) \\ &= \sum_{(\alpha,\beta)\in\mathbf{N}^n\times\mathbf{N}^n} \left(e_{(\alpha,\beta),(0,0)}x^1 + e_{(\alpha,\beta),(e_1,0)}\right)\frac{\partial^{|\alpha+\beta|}f}{\partial x^\alpha \partial y^\beta}.\end{aligned}$$

Let $k \in \{1, \dots, n\}$. We define the embedding $\varphi_k : \mathbf{R}^n \times \mathbf{R}^{n*} \longrightarrow \mathbf{R}^n \times \mathbf{R}^{n*}$ to be such that $\varphi_k(x, y) = (x, y_1, \dots, y_{k-1}, y_k + x^k, y_{k+1}, \dots, y_n)$ for $(x, y) \in \mathbf{R}^n \times \mathbf{R}^{n*}$. We check at once that $dx^i \wedge dy_i \circ (T \times T)\varphi_k = dx^i \wedge dy_i$ and $x^1 \circ \varphi_k = x^1$. From (22) and (6) it follows that for every $f \in F(\mathbf{R}^n \times \mathbf{R}^{n*})$

$$\begin{aligned}(23)\qquad & \sum_{(\alpha,\beta)\in\mathbf{N}^n\times\mathbf{N}^n} \left(e_{(\alpha,\beta),(0,0)}(x^1\circ\varphi_k) + e_{(\alpha,\beta),(e_1,0)}\right)\frac{\partial^{|\alpha+\beta|}(f\circ\varphi_k)}{\partial x^\alpha \partial y^\beta} \\ &= \sum_{(\alpha,\beta)\in\mathbf{N}^n\times\mathbf{N}^n} \left(e_{(\alpha,\beta),(0,0)}x^1 + e_{(\alpha,\beta),(e_1,0)}\right)\frac{\partial^{|\alpha+\beta|}(f\circ\varphi_k)}{\partial x^\alpha \partial y^\beta} \\ &= A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i \wedge dy_i)(f\circ\varphi_k, x^1) \\ &= A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i \wedge dy_i \circ (T\times T)\varphi_k)(f\circ\varphi_k, x^1\circ\varphi_k) \\ &= A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i \wedge dy_i)(f, x^1)\circ\varphi_k \\ &= \sum_{(\alpha,\beta)\in\mathbf{N}^n\times\mathbf{N}^n} \left(e_{(\alpha,\beta),(0,0)}x^1 + e_{(\alpha,\beta),(e_1,0)}\right)\frac{\partial^{|\alpha+\beta|}f}{\partial x^\alpha \partial y^\beta}\circ\varphi_k .\end{aligned}$$

It is easy to prove by induction on α^k that

$$(24)\qquad \frac{\partial^{|\alpha+\beta|}(f\circ\varphi_k)}{\partial x^\alpha \partial y^\beta} = \sum_{j=0}^{\alpha^k}\binom{\alpha^k}{j}\frac{\partial^{|\alpha+\beta|}f}{\partial x^{\alpha-je_k}\partial y^{\beta+je_k}}\circ\varphi_k .$$

Applying (24) and composing with φ_k^{-1} we can rewrite (23) as

$$\begin{aligned}(25)\qquad & \sum_{(\alpha,\beta)\in\mathbf{N}^n\times\mathbf{N}^n} \left(e_{(\alpha,\beta),(0,0)}x^1 + e_{(\alpha,\beta),(e_1,0)}\right)\sum_{j=0}^{\alpha^k}\binom{\alpha^k}{j}\frac{\partial^{|\alpha+\beta|}f}{\partial x^{\alpha-je_k}\partial y^{\beta+je_k}} \\ &\quad = \sum_{(\alpha,\beta)\in\mathbf{N}^n\times\mathbf{N}^n} \left(e_{(\alpha,\beta),(0,0)}x^1 + e_{(\alpha,\beta),(e_1,0)}\right)\frac{\partial^{|\alpha+\beta|}f}{\partial x^\alpha \partial y^\beta}.\end{aligned}$$

Now we are going to prove that for every $(\alpha, \beta) \in \mathbf{N}^n \times \mathbf{N}^n$

$$(26)\qquad \alpha^k \neq 0 \Longrightarrow (e_{(\alpha,\beta),(0,0)}, e_{(\alpha,\beta),(e_1,0)}) = (0, 0).$$

Clearly, (26) holds provided that $(e_{(\alpha,\beta),(0,0)}, e_{(\alpha,\beta),(e_1,0)}) = (0, 0)$ for every $(\alpha, \beta) \in \mathbf{N}^n \times \mathbf{N}^n$. In another case let us denote by (p, q) the element of $\mathbf{N}^n \times \mathbf{N}^n$ with the properties that

$$(e_{(p,q),(0,0)}, e_{(p,q),(e_1,0)}) \neq (0,0) \tag{27}$$

and

$$(e_{(\alpha,\beta),(0,0)}, e_{(\alpha,\beta),(e_1,0)}) \neq (0,0) \Longrightarrow \alpha^k \leq p^k \tag{28}$$

for every $(\alpha,\beta) \in \mathbf{N}^n \times \mathbf{N}^n$. Suppose, contrary to our claim, that $p^k \geq 1$. Under this assumption we will compute the coefficients of

$$\frac{\partial^{|p+q|} f}{\partial x^{p-e_k} \partial y^{q+e_k}}$$

on both sides of (25). In order to do this, we take $j \in \mathbf{N}$ such that there is $(\alpha,\beta) \in \mathbf{N}^n \times \mathbf{N}^n$ satisfying $(\alpha - je_k, \beta + je_k) = (p-e_k, q+e_k)$ and $(e_{(\alpha,\beta),(0,0)}, e_{(\alpha,\beta),(e_1,0)}) \neq (0,0)$. Thus $\alpha^k - j = p^k - 1$, and so $j = \alpha^k - p^k + 1$. It follows that $j = 0$ or $j = 1$, as according to (28) $\alpha^k \leq p^k$. Therefore from (25) we deduce that

$$\begin{aligned}
&(e_{(p-e_k,q+e_k),(0,0)} x^1 + e_{(p-e_k,q+e_k),(e_1,0)}) \binom{p^k - 1}{0} \\
&\quad + (e_{(p,q),(0,0)} x^1 + e_{(p,q),(e_1,0)}) \binom{p^k}{1} \\
&\quad = e_{(p-e_k,q+e_k),(0,0)} x^1 + e_{(p-e_k,q+e_k),(e_1,0)},
\end{aligned}$$

and so $e_{(p,q),(0,0)} x^1 + e_{(p,q),(e_1,0)} = 0$, contrary to (27). Hence our assumption is false and thus $p^k = 0$. This means, by (28), that (26) holds.

Let $l \in \{2, \ldots, n\}$. We define the embedding $\psi_l : \mathbf{R}^n \times \mathbf{R}^{n*} \longrightarrow \mathbf{R}^n \times \mathbf{R}^{n*}$ to be such that $\psi_l(x,y) = (x^1, \ldots, x^{l-1}, x^l + y_l, x^{l+1}, \ldots, x^n, y)$ for $(x,y) \in \mathbf{R}^n \times \mathbf{R}^{n*}$. We check at once that $dx^i \wedge dy_i \circ (T \times T)\psi_l = dx^i \wedge dy_i$ and $x^1 \circ \psi_l = x^1$, because $l \neq 1$. From (22) and (6) it follows that for every $f \in F(\mathbf{R}^n \times \mathbf{R}^{n*})$

(29)

$$\begin{aligned}
&\sum_{(\alpha,\beta) \in \mathbf{N}^n \times \mathbf{N}^n} (e_{(\alpha,\beta),(0,0)} (x^1 \circ \psi_l) + e_{(\alpha,\beta),(e_1,0)}) \frac{\partial^{|\alpha+\beta|}(f \circ \psi_l)}{\partial x^\alpha \partial y^\beta} \\
&\quad = \sum_{(\alpha,\beta) \in \mathbf{N}^n \times \mathbf{N}^n} (e_{(\alpha,\beta),(0,0)} x^1 + e_{(\alpha,\beta),(e_1,0)}) \frac{\partial^{|\alpha+\beta|}(f \circ \psi_l)}{\partial x^\alpha \partial y^\beta} \\
&\quad = A_{\mathbf{R}^n \times \mathbf{R}^{n*}}(dx^i \wedge dy_i)(f \circ \psi_l, x^1) \\
&\quad = A_{\mathbf{R}^n \times \mathbf{R}^{n*}}(dx^i \wedge dy_i \circ (T \times T)\psi_l)(f \circ \psi_l, x^1 \circ \psi_l) \\
&\quad = A_{\mathbf{R}^n \times \mathbf{R}^{n*}}(dx^i \wedge dy_i)(f, x^1) \circ \psi_l \\
&\quad = \sum_{(\alpha,\beta) \in \mathbf{N}^n \times \mathbf{N}^n} (e_{(\alpha,\beta),(0,0)} x^1 + e_{(\alpha,\beta),(e_1,0)}) \frac{\partial^{|\alpha+\beta|} f}{\partial x^\alpha \partial y^\beta} \circ \psi_l .
\end{aligned}$$

It is easy to prove by induction on β^l that

$$\frac{\partial^{|\alpha+\beta|}(f\circ\psi_l)}{\partial x^\alpha \partial y^\beta} = \sum_{j=0}^{\beta^l}\binom{\beta^l}{j}\frac{\partial^{|\alpha+\beta|}f}{\partial x^{\alpha+je_l}\partial y^{\beta-je_l}}\circ\psi_l. \tag{30}$$

Applying (30) and composing with ψ_l^{-1} we can rewrite (29) as

$$\begin{aligned}\sum_{(\alpha,\beta)\in\mathbf{N}^n\times\mathbf{N}^n}&\big(e_{(\alpha,\beta),(0,0)}x^1+e_{(\alpha,\beta),(e_1,0)}\big)\sum_{j=0}^{\beta^l}\binom{\beta^l}{j}\frac{\partial^{|\alpha+\beta|}f}{\partial x^{\alpha+je_l}\partial y^{\beta-je_l}}\\ &=\sum_{(\alpha,\beta)\in\mathbf{N}^n\times\mathbf{N}^n}\big(e_{(\alpha,\beta),(0,0)}x^1+e_{(\alpha,\beta),(e_1,0)}\big)\frac{\partial^{|\alpha+\beta|}f}{\partial x^\alpha\partial y^\beta}.\end{aligned} \tag{31}$$

Now we are going to prove that for every $(\alpha,\beta)\in\mathbf{N}^n\times\mathbf{N}^n$

$$\beta^l\neq 0\Longrightarrow\big(e_{(\alpha,\beta),(0,0)},e_{(\alpha,\beta),(e_1,0)}\big)=(0,0). \tag{32}$$

Clearly, (32) holds provided that $\big(e_{(\alpha,\beta),(0,0)},e_{(\alpha,\beta),(e_1,0)}\big)=(0,0)$ for every $(\alpha,\beta)\in\mathbf{N}^n\times\mathbf{N}^n$. In another case let us denote by (r,s) the element of $\mathbf{N}^n\times\mathbf{N}^n$ with the properties that

$$\big(e_{(r,s),(0,0)},e_{(r,s),(e_1,0)}\big)\neq(0,0) \tag{33}$$

and

$$\big(e_{(\alpha,\beta),(0,0)},e_{(\alpha,\beta),(e_1,0)}\big)\neq(0,0)\Longrightarrow\beta^l\leq s^l \tag{34}$$

for every $(\alpha,\beta)\in\mathbf{N}^n\times\mathbf{N}^n$. Suppose, contrary to our claim, that $s^l\geq 1$. Under this assumption we will compute the coefficients of

$$\frac{\partial^{|r+s|}f}{\partial x^{r+e_l}\partial y^{s-e_l}}$$

on both sides of (31). In order to do this, we take $j\in\mathbf{N}$ such that there is $(\alpha,\beta)\in\mathbf{N}^n\times\mathbf{N}^n$ satisfying $(\alpha+je_l,\beta-je_l)=(r+e_l,s-e_l)$ and $\big(e_{(\alpha,\beta),(0,0)},e_{(\alpha,\beta),(e_1,0)}\big)\neq(0,0)$. Thus $\beta^l-j=s^l-1$, and so $j=\beta^l-s^l+1$. It follows that $j=0$ or $j=1$, as according to (34) $\beta^l\leq s^l$. Therefore from (31) we deduce that

$$\begin{aligned}\big(e_{(r+e_l,s-e_l),(0,0)}x^1&+e_{(r+e_l,s-e_l),(e_1,0)}\big)\binom{s^l-1}{0}\\ &+\big(e_{(r,s),(0,0)}x^1+e_{(r,s),(e_1,0)}\big)\binom{s^l}{1}\\ &=e_{(r+e_l,s-e_l),(0,0)}x^1+e_{(r+e_l,s-e_l),(e_1,0)},\end{aligned}$$

and so $e_{(r,s),(0,0)}x^1+e_{(r,s),(e_1,0)}=0$, contrary to (33). Hence our assumption is false and thus $s^l=0$. This means, by (34), that (32) holds.

Using (26) and (32) we can rewrite (22) as

$$A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i\wedge dy_i)(f,x^1)=\sum_{j=0}^{\infty}(e_{(0,je_1),(0,0)}x^1+e_{(0,je_1),(e_1,0)})\frac{\partial^j f}{\partial y_1^j},$$

and next, using (20), as

$$A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i\wedge dy_i)(f,x^1)=e_{(0,0),(0,0)}x^1f+e_{(0,e_1),(e_1,0)}\frac{\partial f}{\partial y_1},$$

and finally, using (21), as

$$A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i\wedge dy_i)(f,x^1)=e_{(0,e_1),(e_1,0)}\frac{\partial f}{\partial y_1}. \tag{35}$$

A trivial computation shows that

$$B_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i\wedge dy_i)(f,x^1)=-\frac{\partial f}{\partial y_1}. \tag{36}$$

Hence denoting $\mu=-e_{(0,e_1),(e_1,0)}$ and combining (35) with (36) we obtain

$$A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i\wedge dy_i)(f,x^1)=\mu B_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i\wedge dy_i)(f,x^1) \tag{37}$$

for every $f\in F(\mathbf{R}^n\times\mathbf{R}^{n*})$.

Fix a $2n$-dimensional smooth manifold M, $\omega\in S(M)$, $m\in M$ and $f,g\in F(M)$. If $d_mg\neq 0$ then it is possible to prove that there exists a chart $\varphi:U\longrightarrow\mathbf{R}^n\times\mathbf{R}^{n*}$ on M such that $m\in U$, $dx^i\wedge dy_i\circ(T\times T)\varphi=\omega|_{(T\times T)U}$, $\varphi(m)=(g(m)e_1,0)$ and $x^1\circ\varphi=g|_U$ (see the lemma in [1]). Let $e\in F(\mathbf{R}^n\times\mathbf{R}^{n*})$ be such that there is a neighbourhood V of $(g(m)e_1,0)$ such that $V\subset\varphi(U)$ and $e|_V=(f\circ\varphi^{-1})|_V$. From (13), (14), (6), (37) we have

$$\begin{aligned}(A_M(\omega)(f,g))(m)&=(A_U(\omega|_{(T\times T)U})(f|_U,g|_U))(m)\\&=(A_U(dx^i\wedge dy_i\circ(T\times T)\varphi)(f\circ\varphi^{-1}\circ\varphi,x^1\circ\varphi))(m)\\&=(A_U(dx^i\wedge dy_i\circ(T\times T)\varphi)(e\circ\varphi,x^1\circ\varphi))(m)\\&=(A_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i\wedge dy_i)(e,x^1)\circ\varphi)(m)\\&=(\mu B_{\mathbf{R}^n\times\mathbf{R}^{n*}}(dx^i\wedge dy_i)(e,x^1)\circ\varphi)(m)\\&=(\mu B_U(dx^i\wedge dy_i\circ(T\times T)\varphi)(e\circ\varphi,x^1\circ\varphi))(m)\\&=(\mu B_U(dx^i\wedge dy_i\circ(T\times T)\varphi)(f\circ\varphi^{-1}\circ\varphi,x^1\circ\varphi))(m)\\&=(\mu B_U(\omega|_{(T\times T)U})(f|_U,g|_U))(m)\\&=(\mu B_M(\omega)(f,g))(m)\,,\end{aligned}$$

and so

$$(A_M(\omega)(f,g))(m)=(\mu B_M(\omega)(f,g))(m). \tag{38}$$

If $d_mg=0$, then (38) also holds. In order to prove this, we take an open $U\subset M$, $m_i\in M$ for $i\in\mathbf{N}$, and $h\in F(M)$ such that $m\in\overline{U}$, $\lim_{i\to\infty}m_i=m$, $h|_U=g|_U$

and $d_{m_i}h \neq 0$ for $i \in \mathbf{N}$. By (14) and the continuity of the functions $A_M(\omega)(f,g)$, $A_M(\omega)(f,h)$, we have $(A_M(\omega)(f,g))(m) = (A_M(\omega)(f,h))(m)$. Moreover, (38) implies

$$(A_M(\omega)(f,h))(m) = \lim_{i\to\infty}(A_M(\omega)(f,h))(m_i)$$
$$= \lim_{i\to\infty}(\mu B_M(\omega)(f,h))(m_i) = (\mu B_M(\omega)(f,h))(m)\,.$$

By (14) and the continuity of the functions $B_M(\omega)(f,h)$, $B_M(\omega)(f,g)$, we have $(\mu B_M(\omega)(f,h))(m) = (\mu B_M(\omega)(f,g))(m)$. Combining these we obtain (38).

The uniqueness of μ is clear, which completes the proof.

Remark. We have not used (3) and (5) in the proof of Proposition 2.

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