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THE PROOF OF THE ISOMORPHISM OF THE n - DIMENSIONAL PROJECTIVE SPACES DEFINED AXIOMATICALLY

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Abstract. The paper gives a proof (without of using of "great" Desargues' axiom) that any two axiomatically defined n - dimensional projective spaces are isomorphic.

1. Axioms and auxiliary theorems.

The *projective space* $\mathbf{P}_n$ of dimension $n \geq 2$ is meant to be a non-empty set with $n-1$ systems of non-empty subsets (so called *subspaces*) fulfilling the generalized Hilbert's axioms of incidence **J1- J5**, the projective axiom **P** and a special Desargues' axiom **GP**, on which separation relation $\nu \subset \mathbf{P}_n \times \mathbf{P}_n \times \mathbf{P}_n \times \mathbf{P}_n$ is defined so that it is fulfilling the separation axioms **N1-N6** and Dedekind's axiom **DN**. The *points* (i.e. the elements) of space $\mathbf{P}_n$ we will note by A , B , C , B', B'', and similarly, or also $\mathbf{P}_0$, $\mathbf{P}'_0$, $\mathbf{P}''{}_0$, ... as one-point sets. The subspaces of dimension $k = 1, 2, \dots, n-1$ (i.e. the subsets of the k-th system) of the space $\mathbf{P}_n$ we will note by $\mathbf{P}_k$, $\mathbf{P}'{}_k$, $\mathbf{P}''{}_k$, ... The empty set we will note also by $\mathbf{P}_{-1}$, $\mathbf{P}'{}_{-1}$, $\mathbf{P}''{}_{-1}$, ...

Definition. $B_0, \dots, B_k \in \mathbf{P}_n$ are *independent* in $\mathbf{P}_n$ (and we write $B_0 \dots B_k$) $\Leftrightarrow$ for every $\mathbf{P}_{k-1} \subset \mathbf{P}_n$ at least one $B_i \notin \mathbf{P}_{k-1}$.

Generalized Hilbert's axioms of incidence ([4], p.69)

- **J1** $\mathbf{P}_1 \subset \mathbf{P}_n \Rightarrow$ there are independent $B_0, B_1 \in \mathbf{P}_1$.
- **J2** For every $k = 0, 1, \dots, n$ there are independent $B_0, \dots, B_k \in \mathbf{P}_n$.
- **J3** Independent $B_0, \dots, B_k \in \mathbf{P}_n \Rightarrow$ there is one and only one $\mathbf{P}_k \subset \mathbf{P}_n$ such that $B_0, \dots, B_k \in \mathbf{P}_k$ (we write $\mathbf{P}_k = B_0 \dots B_k$).
- **J4** $\mathbf{P}_k, \mathbf{P}_{k+1} \subset \mathbf{P}_n$ and independent $B_0, \dots, B_k \in \mathbf{P}_k \cap \mathbf{P}_{k+1} \Rightarrow \mathbf{P}_k \subset \mathbf{P}_{k+1}$.
- **J5** $\mathbf{P}_k$, $\mathbf{P}'_k \subset \mathbf{P}_{k+1} \subset \mathbf{P}_n$ and $\mathbf{P}_k \cap \mathbf{P}'_k \neq \emptyset \Rightarrow$ there are independent $B_0, \dots, B_{k-1} \in \mathbf{P}_k \cap \mathbf{P}'_k$.

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Projective axiom and special Desarguesian axiom

P $\mathbf{P}_1, \mathbf{P}'_1 \subset \mathbf{P}_2 \subset \mathbf{P}_n \Rightarrow \mathbf{P}_1 \cap \mathbf{P}'_1 \neq \emptyset$.

GP Independent $A, B, C \in \mathbf{P}_2 \subset \mathbf{P}_n$, independent $D, E, F \in \mathbf{P}_2$, $AD \cap BE = = AD \cap CF = BE \cap CF = Q \in \mathbf{P}_1 \subset \mathbf{P}_2$, $AB \cap DE = X \in \mathbf{P}_1$, $AC \cap DF = = Y \in \mathbf{P}_1 \Rightarrow BC \cap EF = Z \in \mathbf{P}_1$.

Separation axioms and Dedekind's axiom ([3], p. 262-263 and 278; $(A, B, C, D) \in \nu$ we read: the pair A, C separates the pair B, D)

N1 $(A, B, C, D) \in \nu \Rightarrow A, B, C, D \in \mathbf{P}_1 \subset \mathbf{P}_n$ mutually distinct, $(C, B, A, D) \in \nu$, $(B, A, D, C) \in \nu$.

N2 $A, C \in \mathbf{P}_1 \subset \mathbf{P}_n$, $A \neq C \Rightarrow$ there is $(A, B, C, D) \in \nu$.

N3 Mutually distinct $A, B, C, D \in \mathbf{P}_1 \subset \mathbf{P}_n \Rightarrow$ it holds just one of the following three relations: $(A, B, C, D) \in \nu$, $(A, C, B, D) \in \nu$, $(A, B, D, C) \in \nu$.

N4 $A, B, C, D, E \in \mathbf{P}_1 \subset \mathbf{P}_n$, $A \neq C \neq B$, $(A, C, B, D) \notin \nu$, $(A, C, B, E) \notin \nu \Rightarrow (A, D, B, E) \notin \nu$.

N5 $(A, C, B, D), (A, C, B, E) \in \nu \Rightarrow (A, D, B, E) \notin \nu$.

N6 $A, B, C, D \in \mathbf{P}_1 \subset \mathbf{P}_2 \subset \mathbf{P}_n$, $E, F, G, H \in \mathbf{P}'_1 \subset \mathbf{P}_2$, $Q \in \mathbf{P}_2$, $Q \notin \mathbf{P}_1$, $Q \notin \mathbf{P}'_1$, $E \in AQ$, $F \in BQ$, $G \in CQ$, $H \in DQ$, $(A, B, C, D) \in \nu \Rightarrow (E, F, G, H) \in \nu$.

DN If $A, B, C \in \mathbf{P}_1 \subset \mathbf{P}_n$, $A \neq B \neq C \neq A$, $X \in D \Leftrightarrow (A, X, B, C) \in \nu$, $\emptyset \neq D' \subset D$, $\emptyset \neq D'' \subset D$,

a) $D' \cup D'' = D$, $D' \cap D'' = \emptyset$,

b) $Y \in D'$, $(A, X, Y, C) \in \nu \Rightarrow X \in D'$

c) $Y \in D''$, $(Y, X, B, C) \in \nu \Rightarrow X \in D''$,

then there is $H \in D$ such that

d) $(A, X, H, C) \in \nu \Rightarrow X \in D'$
$(H, X, B, C) \in \nu \Rightarrow X \in D''$.

The *affine space* $\mathbf{A}_n$ of dimension $n \geq 2$ is meant to be non-empty set together with $n-1$ systems of non-empty subsets (so called *subspaces*) fulfilling the generalized Hilbert's axioms of incidence **J1-J5**, (Euklid's) parallel axiom **E** and a special Desargues' axiom **GE**, on which the *betweenness relation* $\mu \subset \mathbf{A}_n \times \mathbf{A}_n \times \mathbf{A}_n$, fulfilling the axioms **M1-M4** and Dedekind's axiom **DM**, is defined. The subspaces of $\mathbf{A}_n$ will be denoted by $\mathbf{A}_k$, $\mathbf{A}'_k$,$\mathbf{A}''_k$, ...

The parallel axiom and the special Desarguesian axiom (see [4], p.70)

E $\mathbf{A}_1 \subset \mathbf{A}_2 \subset \mathbf{A}_n$, $B \in \mathbf{A}_2 - \mathbf{A}_1 \Rightarrow$ there is exactly one $\mathbf{A}'_1 \subset \mathbf{A}_2$ such that $B \in \mathbf{A}'_1$ and $\mathbf{A}_1 \cap \mathbf{A}'_1 = \emptyset$.

GE Independent $A, B, C \in \mathbf{A}_2 \subset \mathbf{A}_n$, independent $D, E, F \in \mathbf{A}_2$, $AD \cap BE = = AD \cap CF = BE \cap CF = \emptyset$, $AB \cap DE = \emptyset$, $AC \cap DF = \emptyset \Rightarrow BC \cap EF = \emptyset$.

The axioms of betweenness relation and Dedekind's axiom (see [4], p.70, and [3], pp.44-45; $(A, B, C) \in \mu$ we read: the point B is between the points A,C)

M1 $(A, B, C) \in \mu \Rightarrow A, B, C \in \mathbf{A}_1 \subset \mathbf{A}_n$, $A \neq B \neq C \neq A$, $(C, B, A) \in \mu$.

M2 $A, B \in \mathbf{A}_n$, $A \neq B \Rightarrow$ there is $(A, B, C) \in \mu$.

M3 $(A, B, C) \in \mu \Rightarrow (B, A, C), (A, C, B) \notin \mu$.

M4 If independent $A, B, C \in \mathbf{A}_2 \subset \mathbf{A}_n$, $\mathbf{A}_1 \subset \mathbf{A}_2$, $A, B, C \notin \mathbf{A}_1$ and there is

$D \in \mathbf{A}_1$, $(A, D, B) \in \mu$, then there is either $E \in \mathbf{A}_1$, $(A, E, C) \in \mu$, or $F \in \mathbf{A}_1$, $(B, F, C) \in \mu$.

DM If $A, B \in \mathbf{A}_n$, $A \neq B$, $X \in D \Leftrightarrow (A, X, B) \in \mu$, $\emptyset \neq D' \subset D$, $\emptyset \neq D'' \subset D$,
a) $D' \cup D'' = D$, $D' \cap D'' = \emptyset$,
b) $Y \in D'$, $(A, X, Y) \in \mu \Rightarrow X \in D'$
c) $Y \in D''$, $(Y, X, B) \in \mu \Rightarrow X \in D''$,
then there is $H \in D$ such that
d) $(A, X, H) \in \mu \Rightarrow X \in D'$
$(H, X, B) \in \mu \Rightarrow X \in D''$.

The projective space $\mathbf{P}_n$ is *isomorphic* with the projective space $\mathbf{P}'_n$ if there is a bijective mapping f of the space $\mathbf{P}_n$ onto the space $\mathbf{P}'_n$ (so called *isomorphism*) such that the (isomorphic) image of a subspaces $\mathbf{P}_k \subset \mathbf{P}_n$ are the subspaces $\mathbf{P}'_k \subset \mathbf{P}_n$ and the (isomorphic) image of a separation ν in $\mathbf{P}_n$ is a separation ν' in $\mathbf{P}'_n$.

Given the projective space $\mathbf{P}_n$, let us choose a subspace $\mathbf{P}^{\infty}_{n-1} \subset \mathbf{P}_n$ and let us put $\mathbf{A}_n = \mathbf{P}_n - \mathbf{P}^{\infty}_{n-1}$. For $k = 1, 2, \ldots, n-1$ we define the subsets $\mathbf{A}_k \subset \mathbf{A}_n$ and a subset $\mu \subset \mathbf{A}_n \times \mathbf{A}_n \times \mathbf{A}_n$ as following:

(1) $\mathbf{A}_k = \mathbf{P}_k - \mathbf{P}^{\infty}_{n-1}$ for $\mathbf{P}_k \subset \mathbf{P}_n$, $\mathbf{P}_k \not\subset \mathbf{P}^{\infty}_{n-1}$;
(2) $(B, D, C) \in \mu \Leftrightarrow (B, D, C, Z) \in \nu$ and $Z \in \mathbf{P}^{\infty}_{n-1}$ for the separation ν in $\mathbf{P}_n$.

Now $\mathbf{A}_n$ is the affine space of dimension n, $\mathbf{A}_k$ are its subspaces of dimension k and μ is the betweenness relation in $\mathbf{A}_n$. Axioms **J1-J5** and **P** imply the following statements:

(3) $\mathbf{P}_1, \mathbf{P}'_k \subset \mathbf{P}_{k+1}$, $\mathbf{P}_1 \not\subset \mathbf{P}'_k$, $1 \leq k \leq n-1 \Rightarrow \mathbf{P}_1 \cap \mathbf{P}'_k = B$.
(4) $\mathbf{P}_h, \mathbf{P}'_k \subset \mathbf{P}_{k+1}$, $\mathbf{P}_h \not\subset \mathbf{P}'_k$, $1 \leq h \leq k \leq n-1 \Rightarrow \mathbf{P}_h \cap \mathbf{P}'_k = \mathbf{P}''_{h-1}$.

2. The mean statement and its proof.

Theorem 1. *Any two projective spaces $\mathbf{P}_n$ and $\mathbf{P}'_n$ are isomorphic.*

Proof. Let us choose a subspace $\mathbf{P}^{\infty}_{n-1} \subset \mathbf{P}_n$ and a subspace $\mathbf{P'}^{\infty}_{n-1} \subset \mathbf{P}'_n$ and construct the affine spaces $\mathbf{A}_n = \mathbf{P}_n - \mathbf{P}^{\infty}_{n-1}$ and $\mathbf{A}'_n = \mathbf{P'}_n - \mathbf{P'}^{\infty}_{n-1}$. By [4] there is an isomorphic mapping f of space $\mathbf{A}_n$ onto the space $\mathbf{A}'_n$ such that $f(\mathbf{A}_k) = \mathbf{A}'_k$ and $f(\mu) = \mu'$. Let us define the mapping $\bar{f}$ as follows:

(5) $\bar{f}(X) = f(X) = X' \in \mathbf{A}'_n$ for every $X \in \mathbf{A}_n$.

For $Y \in \mathbf{P}^{\infty}_{n-1}$ let us choose $B \in \mathbf{A}_n$ and let us put $\mathbf{P}_1 = BY$, $\mathbf{A}_1 = \mathbf{P}_1 - \mathbf{P}^{\infty}_{n-1}$, $f(\mathbf{A}_1) = \mathbf{A}'_1 \subset \mathbf{P}'_1$, $\bar{f}(Y) = Y' = \mathbf{P}'_1 \cap \mathbf{P'}^{\infty}_{n-1}$. We are going to show that in (5) it does not depend on the choosing of the point B; therefore let us choose $C \in \mathbf{A}_n$, $\bar{\mathbf{P}}_1 = CY \neq BY$, $\bar{\mathbf{A}}_1 = \bar{\mathbf{P}}_1 - \mathbf{P}^{\infty}_{n-1}$, $f(\bar{\mathbf{A}}_1) = \bar{\mathbf{A}}'_1 \subset \bar{\mathbf{P}}'_1$, $\bar{f}(Y) = \bar{Y}' = \bar{\mathbf{P}}'_1 \cap \mathbf{P'}^{\infty}_{n-1}$. Because the points B, C, Y are independent, there is only one $\mathbf{P}_2 = BCY \supset \mathbf{P}_1$, $\bar{\mathbf{P}}_1$ and therefore also only one $\mathbf{A}_2 = \mathbf{P}_2 - \mathbf{P}^{\infty}_{n-1} \supset \mathbf{A}_1, \bar{\mathbf{A}}_1$ and it is true that $\mathbf{A}_1 \cap \bar{\mathbf{A}}_1 = \emptyset$ (because $\mathbf{P}_1 \cap \bar{\mathbf{P}}_1 = Y \in \mathbf{P}^{\infty}_{n-1}$). For the images in the isomorphism f we have $\mathbf{A}'_2 = f(\mathbf{A}_2) \supset \mathbf{A}'_1$, $\bar{\mathbf{A}}'_1$ and $\mathbf{A}'_1 \cap \bar{\mathbf{A}}'_1 = \emptyset$. For $\mathbf{P}'_2 \supset \mathbf{A}'_2$ we have

$\mathbf{P}'_2 \supset \mathbf{P}'_1, \bar{\mathbf{P}}'_1$ and by the axiom $\mathbf{P}$ the point $Z' = \mathbf{P}'_1 \cap \bar{\mathbf{P}}'_1$ is unique (if $\mathbf{P}'_1 = \bar{\mathbf{P}}'_1$, then also $\mathbf{A}'_1 = \bar{\mathbf{A}}'_1$) and in consequence of $\mathbf{A}'_1 \cap \bar{\mathbf{A}}'_1 = \emptyset$ it must be $Z' \in \mathbf{P'}^{\infty}_{n-1}$ and $Y' = Z' = \bar{Y}'$, too.

Further we have the following statement:

(6) a) $Y, Z \in \mathbf{P}_n$, $Y \neq Z \Rightarrow \bar{f}(Y) \neq \bar{f}(Z)$,
b) $Y' \in \mathbf{P}'_n \Rightarrow$ there is $Y \in \mathbf{P}_n$ such that $\bar{f}(Y) = Y'$.

Proof of a). It is true $\bar{f}(Y) = f(Y) \neq f(Z) = \bar{f}(Z)$ for $Y, Z \in \mathbf{A}_n$; for $Z \in \mathbf{A}_n$, $Y \in \mathbf{P}^{\infty}_{n-1}$ we have $\bar{f}(Z) = f(Z) \in \mathbf{A}'_n$ and $\bar{f}(Y) \in \mathbf{P'}^{\infty}_{n-1}$, and so $\bar{f}(Z) \neq \bar{f}(Y)$. Let us $\bar{\mathbf{P}}_1 = BZ$, $\bar{\mathbf{A}}_1 = \bar{\mathbf{P}}_1 - \mathbf{P}^{\infty}_{n-1}$, $f(\bar{\mathbf{A}}_1) = \bar{\mathbf{A}}'_1 \subset \bar{\mathbf{P}}'_1$, $\bar{f}(Z) = Z' = \bar{\mathbf{P}}'_1 \cap \mathbf{P'}^{\infty}_{n-1}$ for $Y, Z \in \mathbf{P}^{\infty}_{n-1}$; if now $Y' = Z'$, then $\mathbf{P}'_1 = \bar{\mathbf{P}}'_1$, $\mathbf{A}'_1 = \bar{\mathbf{A}}'_1$, $\mathbf{A}_1 = \bar{\mathbf{A}}_1$, $\mathbf{P}_1 = \bar{\mathbf{P}}_1$ and by (3) also $Y = Z$, what is a contradiction.

Proof of b). For $Y' \in \mathbf{A}'_n$ there is $Y \in \mathbf{A}_n$ such that $\bar{f}(Y) = f(Y) = Y'$. Let us $Y' \in \mathbf{P'}^{\infty}_{n-1}$; let us choose $B' \in \mathbf{A}'_n$ and let us put $\mathbf{P}'_1 = B'Y'$, $\mathbf{A}'_1 = \mathbf{P}'_1 - \mathbf{P'}^{\infty}_{n-1}$, $f(B) = B'$, $f(\mathbf{A}_1) = \mathbf{A}'_1$, $\mathbf{P}_1 \supset \mathbf{A}_1 \ni B$, $Y = \mathbf{P}_1 \cap \mathbf{P}^{\infty}_{n-1}$; then evidently $\bar{f}(Y) = Y'$.

By (5) and (6) $\bar{f}$ is a bijective mapping of the space $\mathbf{P}_n$ onto the space $\mathbf{P}'_n$. Now we show

(7) $\mathbf{P}_k \subset \mathbf{P}_n \Rightarrow \bar{f}(\mathbf{P}_k) = \mathbf{P}'_k \subset \mathbf{P}'_n$.

First of all we have $\bar{f}(\mathbf{P}^{\infty}_{n-1}) = \mathbf{P'}^{\infty}_{n-1}$ for $\mathbf{P}^{\infty}_{n-1}$. Let us $\mathbf{P}_k \not\subset \mathbf{P}^{\infty}_{n-1}$. Let us take $\mathbf{A}_k = \mathbf{P}_k - \mathbf{P}^{\infty}_{n-1}$, $f(\mathbf{A}_k) = \mathbf{A}'_k = \mathbf{P}'_k - \mathbf{P'}^{\infty}_{n-1}$, and we show that $\bar{f}(\mathbf{P}_k) = \mathbf{P}'_k$. For $X \in \mathbf{A}_k$ we have $\bar{f}(X) = f(X) = X' \in \mathbf{A}'_k \subset \mathbf{P}'_k$ and, the other way round, $\bar{f}^{-1}(X') = X \in \mathbf{A}_k$ for $X' \in \mathbf{A}'_k$; for $Y \in \mathbf{P}_k \cap \mathbf{P}^{\infty}_{n-1}$ there is $B \in \mathbf{A}_k$ and $\mathbf{P}_1 = BY \subset \mathbf{P}_k$, $\mathbf{A}_1 = \mathbf{P}_1 - \mathbf{P}^{\infty}_{n-1}$, $B' = f(B) \in f(\mathbf{A}_1) = \mathbf{A}'_1 \subset \mathbf{P}'_1 = B'Y' \subset \mathbf{P}'_k$, $Y' = \mathbf{P}'_1 \cap \mathbf{P'}^{\infty}_{n-1}$ and so $Y' \in \mathbf{P}'_k$, and the other way round $\bar{f}^{-1}(Y') = Y \in \mathbf{P}_k \cap \mathbf{P}^{\infty}_{n-1}$ for $Y' \in \mathbf{P'}_k \cap \mathbf{P'}^{\infty}_{n-1}$. Simultaneously we have proved $\bar{f}(\mathbf{P}_{k-1}) = \mathbf{P}'_{k-1} = \mathbf{P}'_k \cap \mathbf{P'}^{\infty}_{n-1}$ for $\mathbf{P}_{k-1} = \mathbf{P}_k \cap \mathbf{P}^{\infty}_{n-1}$ and therefore $\bar{f}(\mathbf{P}_{k-1}) = \mathbf{P}'_{k-1} \subset \mathbf{P'}^{\infty}_{n-1}$ for $\mathbf{P}_{k-1} \subset \mathbf{P}^{\infty}_{n-1}$.

The statement (7) is proved.

Now we show, that for the separation ν in $\mathbf{P}_n$ and ν' in $\mathbf{P}'_n$ it is true the following statement:

(8) $(B, D, C, F) \in \nu \Rightarrow \bar{f}(B, D, C, F) = (B', D', C', F') \in \nu'$.

Let us $(B, D, C, F) \in \nu$. By **N1** the mutually distinct points $B, D, C, F \in \mathbf{P}_1 \subset \mathbf{P}_n$ and there are two possibilities:

I. $\mathbf{P}_1 \not\subset \mathbf{P}^{\infty}_{n-1}$. By (3) we have exactly one point $Z = \mathbf{P}_1 \cap \mathbf{P}^{\infty}_{n-1}$ and we can suppose that $B \neq Z \neq C$, $Z \neq D$ (otherwise in **N1** we change either the points B, C and D, F or the points D and F). Hence the mutually distinct points $B, D, C \in \mathbf{A}_1 = \mathbf{P}_1 - \mathbf{P}^{\infty}_{n-1}$ and by (2) we get

(9) $(B, D, C, Z) \in \nu$ and $(B, D, C) \in \mu$.

By [3], §89, Thm.7, p.264, there are the subsets $\mathbf{K}_1, \mathbf{K}_2 \subset \mathbf{P}_1 - B - C$ such that

(10) $\mathbf{K}_1 \cup \mathbf{K}_2 = \mathbf{P}_1 - B - C$, $\mathbf{K}_1 \cap \mathbf{K}_2 = \emptyset$

$X \in \mathbf{K}_1$ and $Y \in \mathbf{K}_2 \Rightarrow (B, X, C, Y) \in \nu$
$X, Y \in \mathbf{K}_1$ or $X, Y \in \mathbf{K}_2 \Rightarrow (B, X, C, Y) \notin \nu$

and we can suppose that $D \in \mathbf{K}_1$ and $F, Z \in \mathbf{K}_2$ (otherwise we change the indexes of the sets $\mathbf{K}_1$, $\mathbf{K}_2$). Let us denote by B', D', C', F', Z' the images of points B, D, C, F, Z at the mapping $\bar{f}$, so $f(B, D, C) = (B', D', C') \in \mu'$, $B', D', C' \in f(\mathbf{A}_1) = \mathbf{A}'_1 = \mathbf{P}'_1 - \mathbf{P}'^{\infty}_{n-1}$, $Z' = \mathbf{P}'_1 \cap \mathbf{P}'^{\infty}_{n-1}$ and by (2) - which is true also for μ' and ν' - we have $(B', D', C', Z') \in \nu'$.

According to [3], §89, Thm.7, p.264, there are the subsets $\mathbf{K}'_1, \mathbf{K}'_2 \subset \mathbf{P}'_1 - B' - C'$ such that

(11) $\mathbf{K}'_1 \cup \mathbf{K}'_2 = \mathbf{P}'_1 - B' - C'$, $\mathbf{K}'_1 \cap \mathbf{K}'_2 = \emptyset$
$X' \in \mathbf{K}'_1$ and $Y' \in \mathbf{K}'_2 \Rightarrow (B', X', C', Y') \in \nu'$
$X', Y' \in \mathbf{K}'_1$ or $X', Y' \in \mathbf{K}'_2 \Rightarrow (B', X', C', Y') \notin \nu'$

and we can supppose that $D' \in \mathbf{K}'_1$ and $Z' \in \mathbf{K}'_2$ (otherwise we change the indexes of the sets $\mathbf{K}'_1$, $\mathbf{K}'_2$). By (2) - which is true also for μ' and ν' - we have $X \in \mathbf{K}_1 \Leftrightarrow (B, X, C, Z) \in \nu \Leftrightarrow (B, X, C) \in \mu \Leftrightarrow f(B, X, C) = (B', X', C') \in \mu'$ $\Leftrightarrow (B', X', C', Z') \in \nu' \Leftrightarrow X' \in \mathbf{K}'_1$ and so $\bar{f}(\mathbf{K}_1) = f(\mathbf{K}_1) = \mathbf{K}'_1$. Because $\bar{f}(\mathbf{P}_1) = \mathbf{P}'_1$ by (7), we have - in accordance with (10) and (11) - also $\bar{f}(\mathbf{K}_2) = \mathbf{K}'_2$. It is $D \in \mathbf{K}_1$ and $F \in \mathbf{K}_2$, hence for the images at $\bar{f}$ we have $D' \in \mathbf{K}'_1$ and $F' \in \mathbf{K}'_2$ and by (11) we conclude $(B', D', C', F') \in \nu'$.

II. $\mathbf{P}_1 \subset \mathbf{P}^{\infty}_{n-1}$. There is a point $Q \in \mathbf{P}_n - \mathbf{P}^{\infty}_{n-1}$ and by **N2** there is a point $\bar{B} \in BQ$, $\bar{B} \neq B$, $B \neq Q$ so that $\bar{B} \notin \mathbf{P}^{\infty}_{n-1}$ (otherwise it would be $Q \in B\bar{B} \subset$ $\subset \mathbf{P}^{\infty}_{n-1}$) and for $\bar{\mathbf{P}}_1 = \bar{B}F$ it is true $Q \notin \bar{\mathbf{P}}_1$ (otherwise it would be $Q \in Q\bar{B} =$ $= B\bar{B} = \bar{B}F = BF = \mathbf{P}_1 \subset \mathbf{P}^{\infty}_{n-1}$). So there is a projection g from $\mathbf{P}_1$ onto $\bar{\mathbf{P}}_1$ from the point Q such that $g(B) = \bar{B} = \bar{\mathbf{P}}_1 \cap BQ$, $g(D) = \bar{D} = \bar{\mathbf{P}}_1 \cap DQ$, $g(C) =$ $= \bar{C} = \bar{\mathbf{P}}_1 \cap CQ$, $g(F) = \bar{F} = F = \bar{\mathbf{P}}_1 \cap FQ$, and $g(B, D, C, F) = (\bar{B}, \bar{D}, \bar{C}, \bar{F}) \in \nu$ by **N6.** Let us Q', B', D', C', F', $\bar{B}'$, $\bar{D}'$, $\bar{C}'$, $\bar{F}' = F'$ the images of the points Q, B, D, C, F, $\bar{B}$, $\bar{D}$, $\bar{C}$, $\bar{F} = F$ at the mapping $\bar{f}$ so that $\bar{B}$, $\bar{D}$, $\bar{C}$, $\bar{F} \in \bar{\mathbf{P}}_1 \not\subset \mathbf{P}^{\infty}_{n-1}$ and - according to the above proved point I - we get $(\bar{B}', \bar{D}', \bar{C}', \bar{F}') \in \nu'$. However by (7) there is a projection g'^{-1} which is the image of the inverse projection g^{-1} at the mapping $\bar{f}$ and g'^{-1} is a projection from $\bar{\mathbf{P}}'_1 = \bar{f}(\bar{\mathbf{P}}_1)$ onto $\mathbf{P}'_1 = \bar{f}(\mathbf{P}_1)$ from the such point Q' that $g'^{-1}(\bar{B}', \bar{D}', \bar{C}', \bar{F}') = (B', D', C', F')$ and $(B', D', C', F') \in \nu'$ according to **N6**.

From (7) and (8) we conclude that $\bar{f}$ is an isomorphism from $\mathbf{P}_n$ onto $\mathbf{P}'_n$.

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