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EXTREMAL SOLUTIONS AND RELAXATION FOR SECOND ORDER VECTOR DIFFERENTIAL INCLUSIONS

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Abstract. In this paper we consider periodic and Dirichlet problems for second order vector differential inclusions. First we show the existence of extremal solutions of the periodic problem (i.e. solutions moving through the extreme points of the multifunction). Then for the Dirichlet problem we show that the extremal solutions are dense in the $C^1(T, R^N)$-norm in the set of solutions of the "convex" problem (relaxation theorem).

1. Introduction

Periodic problems for second order differential inclusions were studied recently by Frigon [4]. She considered nonconvex scalar differential inclusions and assuming the existence of an upper φ and of a lower solution ψ such that $\varphi \leq \psi$ proved the existence of at least one periodic solution located in the order interval $[\psi, \varphi]$. Her method of proof based on truncation and penalization techniques. Here we consider vector differential inclusions and we prove the existence of a periodic solution when the multifunction $F(t, x, y)$ is replaced by ext $F(t, x, y)$ (the extreme points of $F(t, x, y)$). Recall that ext $F(t, x, y)$ need not be closed (even if $F(t, x, y)$ is) and need not have any continuity properties (like lower semicontinuity), even if the multifunction $(x, y) \to F(t, x, y)$ is regular enough, (like Hausdorff continuous). So even if we restrict ourselves to the scalar case our results in the present work go beyond those of Frigon [4]. Moreover, in the present paper we also prove for the Dirichlet problem a relaxation theorem. Namely we show that the solutions passing from the extreme points of $F(t, x, y)$ are $C^1(T, R^N)$–dense, in the solution set of the convexified problem. Such a result is important in control problem, in connection with the "bang-bang principle".

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2. Preliminaries

In what follows, by $P_{f(c)}(R^N)$ (resp. $P_{k(c)}(R^N)$), we denote the collection of all nonempty, closed (and convex) (resp. nonempty, compact (and convex)) subsets of R^N. Let $T=[0,b]$. A multifunction $F: T \to P_f(R^N)$ is said to be measurable, if for all $x \in R^N$, $t \to d(x, F(t)) = \inf[\|x-v\| : v \in F(t)]$ is measurable. This definition of measurability of $F(\cdot)$ is equivalent to saying that $GrF = \{(t,v) \in T \times R^N : v \in F(t)\} \in \mathcal{L} \times B(R^N)$, with $\mathcal{L}$ being the Borel $\sigma-$ field of T,and $B(R^N)$ being the Borel $\sigma-$ field of R^N (graph measurability). For details we refer to the survey paper of Wagner [11].

Given $F: T \to P_f(R^N)$, we define the set

$$S_F^1 = \{v \in L^1(T, R^N) : v(t) \in F(t) \text{ a.e. on } T\}.$$

This set may be empty. Using Aumann's selection theorem (see Wagner [11], theorem 5.10) we can verify that for a measurable multifunction $F(\cdot)$, $S_F^1 \neq \oslash$ if and only if $t \to \inf\{\|v\| : v \in F(t)\} \in L^1(T)$. The set S_F^1 is closed in $L^1(T, R^N)$, is convex if and only if $F(t)$ is convex for almost all $t \in T$ and is bounded if and only if $t \to |F(t)| = \sup\{\|v\| : v \in F(t)\} \in L^1(T)$. Finally the set S_F^1 is "decomposable"; i.e. if $(A, f_1, f_2) \in \mathcal{L} \times S_F^1 \times S_F^1$, then $\chi_A f_1 + \chi_{A^c} f_2 \in S_F^1$.

If Y, Z are metric spaces a multifunction $G : Y \to 2^Z \setminus \{\oslash\}$ is said to be "lower semicontinuous "(lsc for short), if for all $z \in Z$, the R_+ - valued function $y \to d_Z(z, G(y))$ is upper semicontinuous.

On $P_f(R^N)$ we can define a generalized metric, known as the "Hausdorff metric", by setting $h(A,B) = \min\left[\inf_{a \in A} d(a,B), \inf_{b \in B} d(b,A)\right]$.

It is well-known (see for example Kisielewicz [6] or Klein-Thompson [7]), that $(P_f(R^N), h)$ is a complete metric space and $P_{fc}(R^N)$ is a closed subspace of it. A multifunction $F: R^N \to P_f(R^N)$, is said to be "Hausdorff continuous" (h - continuous for short), if it is continuous from R^N into the metric space $(P_f(R^N), h)$.

Finally for $m \in N$, $1 \le r \le \infty$, by $\|\cdot\|_{m,r}$ we denote the norm of the Sobolev space $W^{m,r}(T, R^N)$.

3. Extremal Periodic Solutions

In this section we will be dealing with the following two second order periodic differential inclusions:

$$\left\{ \begin{array}{c} x''(t) - x(t) \in F(t, x(t), x'(t)) \text{ a.e. on } T \\ x(0) = x(b), x'(0) = x'(b) \end{array} \right\} \tag{1}$$

and

$$\left\{ \begin{array}{c} x''(t) - x(t) \in \operatorname{ext} F(t, x(t), x'(t)) \text{ a.e. on } T \\ x(0) = x(b), x'(0) = x'(b) \end{array} \right\} \tag{2}$$

By a solution of (1) (resp(2)), we mean a function $x \in W^{2,1}(T, R^N)$ such that $x''(t) - x(t) = v(t)$ a.e. on T, $x(0) = x(b), x'(0) = x'(b)$, with $v \in S^1_{F(\cdot, x(\cdot), x'(\cdot))}$ (resp. $v \in S^1_{\operatorname{ext} F(\cdot, x(\cdot), x'(\cdot))}$).

In what follows by S_c (resp. S_e) we will denote the set of solution set of (1) (resp. of (2)). Here we prove the nonemptiness of S_e. For this purpose, we need the following hypotheses on the multifunction $F(t,x,y)$.

H_1: $F : T \times R^N \times R^N \to P_{kc}(R^N)$ is a multifunction such that

(i) for every $x, y \in R^N$, $t \to F(t,x,y)$ is measurable;

(ii) for every $t \in T$, $(x,y) \to F(t,x,y)$ is h-continuous;

(iii) $|F(t,x,y)| = \sup\{\|v\| : v \in F(t,x,y)\} \leq \gamma_1(t,\|x\|) + \gamma_2(t,\|x\|)\|y\|$ a.e. on T, with $\sup\{\gamma_1(t,r) : |r| \leq k\} \leq \eta_{1,k}(t)$ a.e. on T, $\eta_{1,k} \in L^1(T)$ and $\sup\{\gamma_2(t,r) : |r| \leq k\} \leq \eta_{2,k}(t)$ a.e. on T, $\eta_{2,k} \in L^\infty(T)$.

(iv) for almost all $t \in T$, all $x, y \in R^N$ and all $v \in F(t,x,y)$, we have

$$(v,x)_{R^N} \geq -\beta\|x\|\|y\| - a(t)\|x\|$$

with $0 \leq \beta < 2$ and $a \in L^1(T)$, $a \geq 0$.

Theorem 1. *If $F : T \times R^N \times R^N \to P_{kc}(R^N)$ is a multifunction satisfying hypotheses H_1, then problem (2) has a solution $x(\cdot) \in W^{2,1}(T,R^N)$ (i.e. $S_e \neq \oslash$).*

Proof. We start by obtaining some a priori bounds for the elements of the set S_c. So let $x \in S_c$. Then by definition we have $x''(t) - x(t) = v(t)$ a.e. on T, $x(0) = x(b)$, $x'(0) = x'(b)$, with $v \in S^1_{F(\cdot,x(\cdot),x'(\cdot))}$.

Hence $-x''(t) + x(t) + v(t) = 0$ a.e. on T, $x(0) = x(b)$, $x'(0) = x'(b)$. Taking the inner product with $x(t)$ and then integrating over T, we obtain

$$\int_0^b (-x''(t), x(t))_{R^N}\,dt + \int_0^b \|x(t)\|^2\,dt + \int_0^b (v(t), x(t))_{R^N}\,dt = 0. \tag{3}$$

From the integration by parts formula (Green's formula) and the periodic boundary conditions, we obtain

$$\int_0^b (-x''(t), x(t))_{R^N}\,dt = \|x'\|_2^2 \tag{4}$$

Also from hypothesis H_1(iv) and since $W^{2,1}(T,R^N)$ is embedded continuously in $C(T,R^N)$ (see for example Brezis [3]), we have

$$\int_0^b (v(t), x(t))_{R^N}\,dt \geq -\beta\|x\|_2\|x'\|_2 - \|a\|_1\|x\|_\infty. \tag{5}$$

Using (4) and (5) in (3), we have

$$\|x\|_{1,2}^2 = \|x\|_2^2 + \|x'\|_2^2 \leq \beta\|x\|_2\|x'\|_2 + \|a\|_1\|x\|_\infty.$$

Since $2\|x\|_2\|x'\|_2 \leq \|x\|_2^2 + \|x'\|_2^2 = \|x\|_{1,2}^2$, we have $\beta\|x\|_2\|x'\|_2 \leq \frac{\beta}{2}\|x'\|_{1,2}^2$ and so $(1-\frac{\beta}{2})\|x\|_{1,2}^2 \leq \|a\|_1\|x\|_\infty$.

Because $W^{2,1}(T,R^N)$ is continuously embedded in $C(T,R^N)$, there exists $c > 0$ such that $\|x\|_\infty \leq c\|x\|_{1,2}$.

So

$$(1-\tfrac{\beta}{2})\|x\|_{1,2} \leq c\|a\|_1 \Rightarrow \|x\|_{1,2} \leq \frac{c}{1-\frac{\beta}{2}}\|a\|_1 = M$$

for all $x \in S_c$ (since $\beta < 2$; see hypothesis H_1(iv)).

Therefore S_c is bounded in $W^{1,2}(T, R^N)$, thus bounded in $C(T, R^N)$ too. Hence we can find $M_1 > 0$ such that $\|x\|_\infty \leq M_1$ for all $x \in S_c$. Using hypothesis H_1 (iii), we see that for all $x \in S_c$ we have

$$\left\|x''\right\|_1 \leq \|\eta_{1,M_1}\|_1 + \|\eta_{2,M_2}\|_\infty \sqrt{b}M = M_2.$$

Thus we infer that S_c is bounded in $W^{2,1}(T, R^N)$.

Recalling that $W^{2,1}(T, R^N)$ is embedded continuously in $C(T, R^N)$, we can find $M_3 > 0$ such that $\|x\|_{C^1(T,R^N)} \leq M_3$ for all $x \in S_c$. Therefore without any loss of generality we may assume that $|F(t,x,y)| = \sup\{\|v\| : v \in F(t,x,y)\} \leq \varphi(t)$ a.e. on T, with $\varphi \in L^1(T)$. Indeed otherwise we replace $F(t,x,y)$ by $\widehat{F}(t,x,y) = F(t, r_{M_3}(x), r_{M_3}(y))$ with $r_{M_3}(\cdot)$ being the M_3- radial retraction on R^N. Note that $\widehat{F}(t,x,y)$ satisfies hypotheses H_1(i),(ii) and (iv) and also $\left|\widehat{F}(t,x,y)\right| \leq \eta_{1,M_3}(t) + \eta_{2,M_3}(t)M_3 = \varphi(t)$ a.e. on T, with $\varphi \in L^1(T)$ for all $x, y \in R^N$.

Now let $V = \left\{u \in L^1(T, R^N) : \|u(t)\| \leq \varphi(t) \text{ a.e. on } T\right\}$. Given $u \in V$, let $p(u)(\cdot) \in W^{2,1}(T, R^N)$ be the unique solution of the periodic problem $-x''(t) - x(t) = u(t)$ a.e. on T, $x(0) = x(b)$, $x'(0) = x'(b)$. We know that $x(t) = \int_0^b G(t,s)u(s)ds$, $t \in T$, where $G(t,s)$ is the Green's function for this problem (see Šeda [9]).

Note that

$$-G(t,s) = \tfrac{1}{2(e-1)} \left\{ \begin{array}{ll} (e^{\frac{-t+s}{b}} + e^{\frac{t-s+b}{b}})I & \text{if} \quad 0 \leq t \leq s \leq b \\ (e^{\frac{-t+s+b}{b}} + e^{\frac{t-s}{b}})I & \text{if} \quad 0 \leq s \leq t \leq b \end{array} \right\}$$

Using the fact that $x(t) = \int_0^b G(t,s)u(s)ds$, $t \in T$, we can easily check that the sets $\{x = p(u) : u \in V\}$ and $\{x' = p(u)' : u \in V\}$, are both bounded and equicontinuous in $C(T, R^N)$ and of course closed. Therefore by the Arzela-Ascoli theorem we can conclude that $K = p(V)$ is a compact and of course convex subset of $C^1(T, R^N)$.

Then let $G : K \to P_{fc}\left(L^1(T, R^N)\right)$ be the multivalued Nemitsky operator

$$G(x) = \left\{v \in L^1(T, R^N) : v(t) \in F\left(t, x(t), x'(t)\right) \text{ a.e. on } T\right\} = S^1_{F(\cdot,x(\cdot),x'(\cdot))},$$

$x \in K$.

Invoking theorem 1.1. of Tolstonogov [10], we can find a continuous map $r : K \to L^1_w(T, R^N)$ such that $r(x) \in \operatorname{ext} G(x)$ for all $x \in K$.

Here by $L^1_w(T, R^N)$ we mean the space $L^1(T, R^N)$ furnished with the weak norm

$$\|v\|_w = \sup\left[\left\|\int_{t_1}^{t_2} v(s)ds\right\| : 0 \leq t_1 \leq t_2 \leq b\right].$$

From Benamara [1] we know that

$$\operatorname{ext} G(x) = \operatorname{ext} S^1_{F(\cdot,x(\cdot),x'(\cdot))} = S^1_{\operatorname{ext} F(\cdot,x(\cdot),x'(\cdot))}$$

for all $x \in K$.

Then let $q = p \circ r$. Recalling that $|F(t,x,y)| \leq \varphi(t)$ a.e. on T, we see that $q : K \to K$. We claim that $q(\cdot)$ is continuous. Indeed let $x_n \to x$ in K as $n \longrightarrow \infty$.

Then $r(x_n) \overset{\|\cdot\|_w}{\to} r(x)$ as $n \to \infty$. But note that $r(x_n)(t) \in F(t, \overline{B}_{M_3}, \overline{B}_{M_3}) \in P_k(R^N)$ a.e. on T, with $\overline{B}_{M_3} = \{z \in R^N : \|z\| \leq M_3\}$. So we can apply the theorem of Gutman [5] and obtain that $r(x_n) \overset{w}{\to} r(x)$ in $L^1(T, R^N)$ as $n \to \infty$. Using the fact that $q(x_n)(t) = \int_0^b G(t,s) r(x_n)(s) ds$ and $q(x)(t) = \int_0^b G(t,s) r(x)(s) ds$ for all $t \in T$, we see that $q(x_n)(t) \to q(x)(t)$ as $n \to \infty$ for all $t \in T$. Since $\{q(x_n)(\cdot)\}_{n \geq 1} \subseteq K$ and the latter is compact in $C^1(T, R^N)$, we have $q(x_n) \to q(x)$ in $C^1(T, R^N)$ as $n \to \infty$. This proves the continuity of $q(\cdot)$. We apply Schauder's fixed point theorem and obtain $x \in K$ such that $x = q(x)$. Evidently $x \in S_e \neq \emptyset$. □

4. Relaxation theorem

In this section we show that every solution of the Dirichlet problem $x''(t) - x(t) \in F(t, x(t), x'(t))$ a.e. on T, $x(0) = x(b) = 0$ can be obtained as the $C^1(T, R^N)$- limit of a sequence of solutions of the "extremal" Dirichlet problem $x''(t) - x(t) \in \text{ext}\, F(t, x(t), x'(t))$ a.e. on T, $x(0) = x(b) = 0$. Such a result is known as "relaxation theorem". To prove such a result, we strengthen our hypotheses on the multifunction $F(t,x,y)$. To simplify our calculations we assume $b = 1$; i.e. $T = [0,1]$.

H_2: $F : T \times R^N \times R^N \to P_{kc}(R^N)$ is a multifunction such that

(i) for every $x, y \in R^N$, $t \to F(t,x,y)$ is measurable;

(ii) $h\left(F(t,x,y), F(t,x',y')\right) \leq k(t)\left[\|x - x'\| + \|y - y'\|\right]$ a.e. on T for all $x, x', y, y' \in R^N$; with $k \in L^\infty(T)$, $\|k\|_\infty < 1$;

(iii) $|F(t,x,y)| \leq \gamma_1(t, \|x\|) + \gamma_2(t, \|x\|)\, \|y\|$ a.e. on T, with $\sup\{\gamma_1(t,r) : 0 \leq r \leq k\} \leq \eta_{1,k}(t)$ a.e. on T, $\eta_{1,k} \in L^1(T)$ and $\sup\{\gamma_2(t,r) : 0 \leq r \leq k\} \leq \eta_{2,k}(t)$ a.e. on T, $\eta_{2,k} \in L^\infty(T)$;

(iv) for almost all $t \in T$, all $x, y \in R^N$ and all $v \in F(t,x,y)$

$$(v, x)_{R^N} \geq -\beta \|x\| \|y\| - a(t) \|x\|$$

with $0 \leq \beta < 2$ and $a \in L^1(T)$, $a \geq 0$.

As we did before in section 2, by $S_c \subseteq W^{2,1}(T, R^N)$ we denote the solution set of the "convexified problem" $x''(t) - x(t) \in F(t, x(t), x'(t))$ a.e. on T, $x(0) = x(1) = 0$ and by $S_e \subseteq W^{2,1}(T, R^N)$ we denote the solution set of $x''(t) - x(t) \in \text{ext}\, F(t, x(t), x'(t))$ a.e. $x(0) = x(1) = 0$.

Theorem 2. *If $F : T \times R^N \times R^N \to P_{kc}(R^N)$ is a multifunction satisfying hypotheses H_2, then $\overline{S}_e^{C^1(T,R^N)} = S_c$.*

Proof. Let $x \in S_c$. Then by definition we have that $x''(t) - x(t) = v(t)$ a.e. on T, with $x(0) = x(b) = 0$ and $v \in S^1_{F(\cdot, x(\cdot), x'(\cdot))}$. Arguing as in the proof of theorem 1, we know that without any loss of generality, we may assume that for almost all $t \in T$ and all $x, y \in R^N$, $|F(t,x,y)| \leq \varphi(t)$ with $\varphi \in L^1(T)$.

As in the proof of theorem 1, a nonempty, compact and convex set $K \subseteq C^1(T, R^N)$ can be constructed such that $S_c \subseteq K$ (note that because of the Dirichlet boundary conditions, equation (4) holds and so the estimation which led to the derivation of K is still valid here).

Given $y \in K$ and $\varepsilon > 0$, we define the multifunction $U_\varepsilon : T \to 2^{R^N} \setminus \{\emptyset\}$ by

$$U_\varepsilon(t) =$$
$$= \left\{u \in R^N : \|v(t) - u\| < \varepsilon + d\left(v(t), F\left(t, y(t), y'(t)\right)\right), u \in F\left(t, y(t), y'(t)\right)\right\}.$$

Because of hypotheses H_2(i) and (ii), $t \to d\left(v(t), F\left(t, y(t), y'(t)\right)\right)$ is measurable and so the multifunction $t \to F\left(t, y(t), y'(t)\right)$ is graph measurable (see Papageorgiou [8]). Therefore $GrU_\varepsilon \in \mathcal{L} \times B(R^N)$.

Apply Aumann's selection theorem (see Wagner [11], theorem 5.10), to obtain $u : T \to R^N$ measurable such that $u(t) \in U_\varepsilon(t)$ for all $t \in T$. Thus if we define $G_\varepsilon : K \to 2^{L^1(T,R^N)}$ by

$$G_\varepsilon(y) =$$
$$= \left\{u \in S^1_{F(\cdot,y(\cdot),y'(\cdot))} : \|v(t) - u(t)\| < \varepsilon + d\left(v(t), F\left(t, y(t), y'(t)\right)\right) \text{ a.e. on } T\right\}$$

we have shown that $G_\varepsilon(y) \neq \emptyset$ for all $y \in K$. Moreover proposition 4 of Bressan-Colombo [2], tells us that $G_\varepsilon(\cdot)$ is lsc. Therefore $y \to \overline{G_\varepsilon(y)}$ is lsc and clearly has decomposable values (i.e. if $(A, u_1, u_2) \in \mathcal{L} \times \overline{G_\varepsilon(y)} \times \overline{G_\varepsilon(y)}$, then $\chi_A u_1 + \chi_{A^c} u_2 \in G_\varepsilon(y)$). Thus we can apply theorem 3 of Bressan Colombo [2] and obtain $g_\varepsilon : K \to L^1(T, R^N)$ a continuous map such that $g_\varepsilon(y) \in \overline{G_\varepsilon(y)}$ for all $y \in K$. In addition theorem 1.1 of Tolstonogov [10], gives us a continuous map $r_\varepsilon : K \to L^1_w(T, R^N)$ such that $r_\varepsilon(y) \in \operatorname{ext} G(y) = S^1_{\operatorname{ext} F(\cdot,y(\cdot),y'(\cdot))}$ and $\|r_\varepsilon(y) - g_\varepsilon(y)\|_w < \varepsilon$ for all $y \in K$.

Now let $\varepsilon_n \downarrow 0$ and set $g_n = g_{\varepsilon_n}$, $r_n = r_{\varepsilon_n}$, $n \geq 1$.

Also let

$$V = \left\{u \in L^1(T, R^N) : \|u(t)\| \leq \varphi(t) \text{ a.e. on } T\right\}$$

and let $p : V \to C^1(T, R^N)$ be the map which to each $u \in V$ assigns the unique solution of the Dirichlet problem $y''(t) - y(t) = u(t)$ a.e. on T, $y(0) = y(1) = 0$.

We claim that $p(V)$ is compact in $C^1(T, R^N)$.

To this end let $y_n \in p(V)$, $n \geq 1$. Then $y_n = p(u_n)$ with $u_n \in V$, $n \geq 1$. We have

$$y_n''(t) - y_n(t) = u_n(t) \text{ a.e. on } T,\ y(0) = y(1) = 0.$$

Take the inner product with $-y_n(t)$ and then integrate over T. We obtain

$$\|y_n\|^2_{1,2} = \|y_n\|_2^2 + \|y_n'\|_2^2 \leq \|u_n\|_1 \|y_n\|_\infty.$$

Since $W^{1,2}(T, R^N)$ is continuously embedded in $C(T, R^N)$, we can find $c > 0$ such that $\|y_n\|^2_{1,2} \leq c\|u_n\|_1 \|y_n\|_{1,2}$, hence $\|y_n\|_{1,2} \leq c\|\varphi\|_1$ for all $n \geq 1$. So $\{y_n\}_{n\geq 1}$ is bounded in $W^{2,1}(T, R^N)$.

Since $y_n'' = u_n + y_n$, we infer that $\{y_n''\}_{n\geq 1} \subseteq L^1(T, R^N)$ is uniformly integrable.

Since V is weakly compact (Dunford-Pettis theorem) by passing to a subsequence if necessary, we may assume that $u_n \xrightarrow{w} u$ in $L^1(T, R^N)$, $u \in V$. Then it is easy to see that $y_n = p(u_n) \to p(u) = y$ in $W^{2,1}(T, R^N)$ and so $y_n \xrightarrow{w} y$ in $C^1(T, R^N)$. But $\{y_n\}_{n\geq 1} \subseteq K$ and the latter is compact in $C^1(T, R^N)$.

So $y_n \to y$ in $C^1(T, R^N)$ as $n \to \infty$, which proves the compactness of $p(V)$ in $C^1(T, R^N)$.

Hence $q_n = p \circ r_n : K \to K$, $n \geq 1$ and by Schauder's fixed point theorem, we can find $x_n = q(x_n)$ $n \geq 1$. Since $\{x_n\}_{n\geq 1} \subseteq K$ by passing to a subsequence if necessary, we may assume that $x_n \to z$ in $C^1(T, R^N)$ as $n \to \infty$.

Then for almost $t \in T$ we have

$$\begin{aligned}
&(x_n''(t) - x''(t), x_n'(t) - x'(t))_{R^N} - (x(t) - x_n(t), x_n(t) - x(t))_{R^N} \\
&= (r_n(x_n)(t) - v(t), x_n(t) - x(t))_{R^N} = \\
&= (v(t) - r_n(x_n)(t), x_n(t) - x(t))_{R^N} + (g_n(x_n)(t) - r_n(x_n)(t), x_n(t) - x(t))_{R^N} \\
&\quad \Longrightarrow \|x_n' - x'\|^2 - \|x_n - x\|^2 \leq \\
&\leq \int_0^1 \varepsilon_n + h(F(t, x(t), x'(t)), F(t, x_n(t), x_n'(t))) \, \|x_n(t) - x(t)\| \, dt \\
&\quad + \int_0^1 (g_n(x_n)(t) - r_n(x_n)(t), x_n(t) - x(t))_{R^N} \, dt \leq \\
&\leq \int_0^1 \varepsilon_n + k(t)(\|x_n(t) - x(t)\| + \|x_n'(t) - x'(t)\|) \, \|x_n(t) - x(t)\| \, dt + \\
&\quad + \int_0^1 (g_n(x_n)(t) - r_n(x_n)(t), x_n(t) - x(t))_{R^N} \, dt
\end{aligned}$$

Note that for all $t \in T$

$$\|x_n(t) - x(t)\| \leq \int_0^t \|x_n'(s) - x'(s)\| \, ds \leq \|x_n' - x'\|_1 .$$

So we have

$$\begin{aligned}
&\|x_n' - x'\|_2^2 + \|x_n - x\|_2^2 \leq \\
&\qquad \leq \varepsilon_n \|x_n - x\|_\infty + \|k\|_\infty \|x_n - x\|_2^2 + \|k\|_\infty \|x_n' - x'\|_2^2 + \\
&\qquad\qquad + \int_0^1 (g_n(x_n)(t) - r_n(x_n)(t), x_n(t) - x(t))_{R^N} \, dt .
\end{aligned}$$

By construction $\|g_n(x_n) - r_n(x_n)\|_w \to 0$ as $n \to \infty$ and so as in the proof of theorem 1 via Gutman's theorem we can have that $(g_n(x_n) - r_n(x_n)) \xrightarrow{w} 0$ in $L^1(T, R^N)$ as $n \to \infty$.

So we have $\int_0^1 (g_n(x_n)(t) - r_n(x_n)(t), x_n(t) - x(t))_{R^N} \, dt \to 0$ as $n \to \infty$.

Therefore in the limit as $n \to \infty$ we obtain $\|z - x\|_{1,2} \leq \|k\|_\infty \|z - x\|_{1,2}$

Since by hypothesis $H_2(ii)$ $\|k\|_\infty < 1$, we deduce that $z = x$. Therefore $x_n \to x$ in $C^1(T, R^N)$. But $x_n \in S_e$. Hence $S_c \subseteq \overline{S_e}^{C^1(T,R^N)}$. Since we gan easily check that S_c is closed in $C^1(T, R^N)$ we conclude that $S_c = \overline{S_e}^{C^1(T,R^N)}$. □

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