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FROM SASAKIAN 3-STRUCTURES TO QUATERNIONIC GEOMETRY

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Dedicated to the memory of Hitoshi TAKAGI

Abstract. We construct a family of almost quaternionic Hermitian structures from an almost contact metric 3-structure and also do three kinds of quaternionic Kähler structures from a Sasakian 3-structure. In particular we have a generalization of the second main result of Boyer-Galicki-Mann [5].

1. Introduction

By means of warped product there is a one-to-one correspondence between Sasakian 3-structures and hyperkähler structures (see Bär [2]). The fundamental technique used in this paper is simple, but a more natural one from view points of a generalization of the standard examples in quaternionic geometry (see Remark 2). In fact it enables us to construct many examples of almost quaternionic Hermitian and quaternionic Kähler manifolds (see Ejiri [6], Nakashima-Watanabe [12], Watanabe-Mori [15] for the almost Hermitian, Hermitian and Kählerian cases).

Recently almost quaternionic Hermitian, quaternionic Kähler and hyperkähler manifolds have received a great deal of attention, and explicit examples of quaternionic Kähler manifolds and hyperkähler manifolds are already given (see [1], [4], [5] and the references therein).

In the first half of 1970's Sasakian 3-structures were studied by Kuo [11], Tachibana-Yu [13], Kashiwada [9], Konishi [10], Tanno [14] and so on. Unfortunately, in this early period examples of manifolds with a Sasakian 3-structure were only manifolds of constant curvature. This was a weak point in studying them. Recently Boyer-Galicki-Mann [4], [5] have called a Riemannian manifold admitting a Sasakian 3-structure a 3-Sasakian manifold, and have pointed out its importance in contrast with quaternionic Kähler manifolds. They completed the classification of homogeneous 3-Sasakian manifolds, and found countable families

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of strongly inhomogeneous 3-Sasakian manifolds in [5]. Thus, thanks to Boyer-Galicki-Mann's results and due to the technique, we can easily obtain many model spaces in quaternionic geometry.

2. Almost contact metric and Sasakian 3-structures

Let M be a (2m+1)-dimensional differentiable manifold. An almost contact metric structure on M is by definition a pair of a Riemannian metric g and an almost contact structure (ϕ, ξ, η), where ϕ is a tensor field of type (1,1), ξ is a vector field and η is a 1-form, satisfying the following conditions (cf. Blair [3]):

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \quad \phi^2 X = -X + \eta(X)\xi \tag{2.1}$$

$$g(X, \xi) = \eta(X), \quad g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields X, Y on M. An almost contact metric structure (ϕ, ξ, η, g) is called Sasakian if furthermore

$$(\nabla_X \phi)Y = \eta(Y)X - g(X, Y)\xi \tag{2.3}$$

for any vector fields X, Y on M.

Suppose that a differentiable manifold admits three almost contact structures $(\phi_{(\alpha)}, \xi_{(\alpha)}, \eta_{(\alpha)}), \alpha = 1, 2, 3$, satisfying

$$\begin{gathered} \eta_{(\alpha)}(\xi_{(\beta)}) = \delta_{\alpha\beta}, \\ \phi_{(\alpha)}\xi_{(\beta)} = -\phi_{(\beta)}\xi_{(\alpha)} = \xi_{(\gamma)}, \quad \eta_{(\alpha)} \circ \phi_{(\beta)} = -\eta_{(\beta)} \circ \phi_{(\alpha)} = \eta_{(\gamma)}, \\ \phi_{(\alpha)}\phi_{(\beta)} - \xi_{(\alpha)} \otimes \eta_{(\beta)} = -\phi_{(\beta)}\phi_{(\alpha)} + \xi_{(\beta)} \otimes \eta_{(\alpha)} = \phi_{(\gamma)} \end{gathered} \tag{2.4}$$

for $\varepsilon(\alpha, \beta, \gamma) = 1$, where $\varepsilon(\alpha, \beta, \gamma) = 1$ means that (α, β, γ) is a cyclic permutation of $(1, 2, 3)$. Then $(\phi_{(\alpha)}, \xi_{(\alpha)}, \eta_{(\alpha)})$, $\alpha = 1, 2, 3$ is called an almost contact 3-structure. It is well known (cf. Kuo [11]) that the dimension of a manifold with an almost contact 3-structure is $4m + 3$ for some non-negative integer m. A Riemannian metric g is said to be associated to the 3-structure if it satisfies

$$g(\phi_{(\alpha)}X, \phi_{(\alpha)}Y) = g(X, Y) - \eta_{(\alpha)}(X)\eta_{\alpha}(Y), \quad \alpha = 1, 2, 3 \tag{2.5}$$

for any vector fields X, Y on M. In a manifold with an almost contact 3-structure there always exists a Riemannian metric g satisfying (2.5), and $(\phi_{(\alpha)}, \xi_{(\alpha)}, \eta_{(\alpha)}, g)$, $\alpha = 1, 2, 3$ is called an almost contact metric 3-structure.

An almost contact metric 3-structure $(\phi_{(\alpha)}, \xi_{(\alpha)}, \eta_{(\alpha)}, g)$, $\alpha = 1, 2, 3$ is called a Sasakian 3-structure if each $(\phi_{(\alpha)}, \xi_{(\alpha)}, \eta_{(\alpha)}, g)$ is a Sasakian structure . Then $\{\xi_{(1)}, \xi_{(2)}, \xi_{(3)}\}$ are orthonormal vector fields, satisfying

$$[\xi_{(\alpha)}, \xi_{(\beta)}] = 2\xi_{(\gamma)}$$

for $\varepsilon(\alpha, \beta, \gamma) = 1$ (cf. Tachibana-Yu [13], Tanno [14]). A manifold with a Sasakian 3-structure is called a 3-Sasakian manifold.

Remark that a 3-Sasakian manifold is an Einstein manifold (see Kashiwada [9]).

3. Almost quaternionic Hermitian and quaternionic Kähler strucures

Following Alekseevsky-Marchiafava [1] and Ishihara [8], we recall the definitions of almost quaternionic Hermitian, quaternionic Kähler and hyperkähler structures.

An almost hypercomplex structure on a manifold M of dimension 4m is by definition a triple $H = (J_{(\alpha)}), \alpha = 1, 2, 3$ of almost complex structures, satisfying

$$J_{(\alpha)}J_{(\beta)} = J_{(\gamma)} \tag{3.1}$$

for $\varepsilon(\alpha, \beta, \gamma) = 1$. By TM we denote the tangent bundle of M. It generates a subbundle $Q =< H >$ of the bundle $End(TM)$ of endomorphisms whose fiber $Q_x = \mathbb{R}J_{(1)}|_x + \mathbb{R}J_{(2)}|_x + \mathbb{R}J_{(3)}|_x$ in a point $x \in M$ is isomorphic to the Lie algebra $\mathfrak{sp}_1$ of the symplectic group $Sp(1)$. Such a subbundle is called an almost quaternionic structure generated by H. More generally, an almost quaternionic structure on a manifold M is defined as a subbundle $Q \subset End(TM)$ of the bundle of endomorphisms which is locally generated by an almost hypercomplex structure H. We shall refer to such H as an almost hypercomplex structure compatible with Q. Let Q be an almost quaternionic structure on M with a Riemannian metric. Then M can be equipped with a Q-Hermitian metric g, that is, all endomorphisms from Q are skew-symmetric with respect to g. An almost quaternionic structure Q together with a Q-Hermitian metric g is called an almost quaternionic Hermitian structure and a manifold with such a structure is called an almost quaternionic Hermitian manifold.

An almost quaternionic Hermitian manifold is called a quaternionic Kähler manifold if an almost hypercomplex structure $(J_{(\alpha)}), \alpha = 1, 2, 3$ in any local coordinate neighbourhood U satisfies

$$\begin{aligned} \nabla_X J_{(1)} &= && r(X)J_{(2)} - q(X)J_{(3)}, \\ \nabla_X J_{(2)} &= -r(X)J_{(1)} && + p(X)J_{(3)}, \\ \nabla_X J_{(3)} &= q(X)J_{(1)} - p(X)J_{(2)} && \end{aligned} \tag{3.2}$$

for any vector field X on U, where ∇ is the Levi-Civita connection of the Riemannian metric, and p, q, r are certain local 1-forms defined in U. In particular, if all p, q, r for each U are vanishing, then the structure is called hyperkähler.

Remark that if $m > 1$, a quaternionic Kähler manifold is an Einstein manifold (cf. Alekseevsky-Marchiafava [2], Ishihara [8]).

4. Examples of almost quaternionic Hermitian structures

Let $(\phi_{(\alpha)}, \xi_{(\alpha)}, \eta_{(\alpha)}, g)$, $\alpha = 1, 2, 3$ be an almost contact metric 3-structure on a manifold M of dimension 4m+3. By $\mathbb{I}$ we denote $\mathbb{R}$ or some open interval in $\mathbb{R}$. For a positive function λ on $\mathbb{I}$, we define an almost hypercomplex structure

$(\tilde{J}_{(\alpha)})$, $\alpha = 1,2,3$ on $M \times \mathbb{I}$ by

$$\tilde{J}_{(\alpha)} = \begin{pmatrix} \phi_{(\alpha)} & \frac{\xi_{(\alpha)}}{\lambda} \\ -\lambda\eta_{(\alpha)} & 0 \end{pmatrix}. \tag{4.1}$$

Let a, b be real valued functions on $\mathbb{I}$, satisfying

$$a(t) > 0, \quad a(t) + b(t) > 0. \tag{4.2}$$

Then, we define a Riemannian metric on $M \times \mathbb{I}$ by

$$\tilde{g} = a(t)g + b(t)\sum_{\alpha=1}^{3} \eta_{(\alpha)} \otimes \eta_{(\alpha)} + dt^2, \tag{4.3}$$

where dt^2 is the usual metric on $\mathbb{I}$. Thus by (4.1) and (4.3) we have the following (see Nakashima-Watanabe [12]).

Proposition 4.1. *Let $(\phi_{(\alpha)}, \xi_{(\alpha)}, \eta_{(\alpha)}, g)$, $\alpha = 1,2,3$ be an almost contact metric 3-structure on a manifold M of dimension 4m+3 and λ a positive function on $\mathbb{I}$. Let a, b be real valued functions on $\mathbb{I}$, satisfying (4.2). Then $(\tilde{J}_{(\alpha)}, \tilde{g})$, $\alpha = 1,2,3$ is an almost quaternionic Hermitian structure on $M \times \mathbb{I}$ if and only if $\lambda = \sqrt{a+b}$.*

In this section, capital Latin indices run on the range $1, 2, ..., 4m+4$, while small ones run on the range $1, ..., 4m+3$ and $\Delta = 4m+4$. Then the components $\tilde{g}_{BC}$ of $\tilde{g}$ in (4.3) with respect to a natural local coordinate of $M \times \mathbb{I}$ are given by

$$(\tilde{g}_{BC}) = \begin{pmatrix} ag_{ij} + b\sum \eta_{(\alpha)i}\eta_{(\alpha)j} & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.4}$$

The inverse matrix $(\tilde{g}^{AB})$ of $(\tilde{g}_{BC})$ is given by

$$(\tilde{g}^{AB}) = \begin{pmatrix} \frac{g^{hi}}{a} - \frac{b}{a(a+b)}\sum \xi^h_{(\alpha)}\xi^i_{(\alpha)} & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.5}$$

Let $(\phi_{(\alpha)}, \xi_{(\alpha)}, \eta_{(\alpha)}, g)$, $\alpha = 1,2,3$ be an almost contact metric 3-structure. By Γ^k_{ij} we denote the Christoffel symbols of g. Then, using (4.4) and (4.5), the

Christoffel symbols $\tilde{\Gamma}^{A}_{BC}$ of $\tilde{g}$ are computed as follows:

$$
\begin{aligned}
\tilde{\Gamma}^{\Delta}_{ij} =& -\frac{1}{2}(a' g_{ij} + b' \sum \eta_{(\alpha)i}\eta_{(\alpha)j}),\\
\Gamma^{\ell}_{i\Delta} =& \frac{1}{2a}[a'\delta^{\ell}_{i} + \frac{ab' - a'b}{a+b}\sum \xi^{\ell}_{(\alpha)}\eta_{(\alpha)i}],\\
\tilde{\Gamma}^{\ell}_{ij} =& \Gamma^{\ell}_{ij} + \frac{b}{2a}[g^{\ell k} - \frac{b}{a+b}\sum \xi^{\ell}_{(\alpha)}\xi^{k}_{\alpha)}]\\
&\times[\sum \eta_{(\alpha)k}(\nabla_i \eta_{(\alpha)j} + \nabla_j \eta_{(\alpha)i})\\
&+\sum \eta_{(\alpha)i}(\nabla_j \eta_{(\alpha)k} - \nabla_k \eta_{(\alpha)j})\\
&+\sum \eta_{(\alpha)j}(\nabla_i \eta_{(\alpha)k} - \nabla_k \eta_{(\alpha)i})],\\
\text{others} =& \ 0,
\end{aligned}
\tag{4.6}
$$

where ($'$) denotes the differentiation with respect to t. In particular, if $(\phi_{(\alpha)}, \xi_{(\alpha)}, \eta_{(\alpha)}, g), \alpha = 1, 2, 3$ is a Sasakian 3-structure, then we have

$$
\tilde{\Gamma}^{\ell}_{ij} = \Gamma^{\ell}_{ij} + \frac{b}{a}\sum(\eta_{(\alpha)i}\phi_{(\alpha)j}{}^{\ell} + \eta_{(\alpha)j}\phi_{(\alpha)i}{}^{\ell}).
$$

5. Examples of quaternionic Kähler manifolds

Let $(M, \phi_{(\alpha)}, \xi_{(\alpha)}, \eta_{(\alpha)}, g), \alpha = 1, 2, 3$ be a 3-Sasakian manifold and a, b be real valued functions on some open interval $\mathbb{I}$, satisfying (4.2). Then, by Proposition 4.1 we can construct almost quaternionic Hermitian structures $(\tilde{J}_{(\alpha)}, \tilde{g}), \alpha = 1, 2, 3$ on $M \times \mathbb{I}$. Then, by using (2.1), (2.2), (2.3), (2.4) and (4.6), we can compute $\tilde{\nabla}\tilde{J}_{(\alpha)}$ as follows:

$$
\begin{aligned}
\tilde{\nabla}_i \tilde{J}^{\Delta}_{(\alpha)j} =& \left(\frac{a'}{2} - \sqrt{a+b}\right)\phi_{(\alpha)ij} + \frac{b'}{2}(\eta_{(\beta)i}\eta_{(\gamma)j} - \eta_{(\beta)j}\eta_{(\gamma)i}),\\
\tilde{\nabla}_i \tilde{J}^{j}_{(\alpha)\Delta} =& \frac{2\sqrt{a+b} - a'}{2a}\phi^{j}_{(\alpha)i}\\
&-\left\{\frac{b(2\sqrt{a+b} - a')}{2a(a+b)} + \frac{b'}{2(a+b)}\right\}(\eta_{(\beta)i}\xi_{(\gamma)}{}^{j} - \xi_{(\beta)}{}^{j}\eta_{(\gamma)i}),
\end{aligned}
$$

$$\begin{aligned}\tilde{\nabla}_i \tilde{J}_{(\alpha)j}{}^h = \; & \frac{\sqrt{a+b}(2\sqrt{a+b}-a')}{2a}\eta_{(\alpha)j}\delta_i^h + \frac{-2\sqrt{a+b}+a'}{2\sqrt{a+b}}\xi_{(\alpha)}^h g_{ij} \\ & -\frac{2b}{a}(\eta_{(\beta)i}\phi_{(\gamma)j}{}^h - \eta_{(\gamma)i}\phi_{(\beta)j}{}^h) + \frac{b(a'-2\sqrt{a+b})}{2a\sqrt{a+b}}\eta_{(\alpha)i}\eta_{(\alpha)j}\xi_{(\alpha)}^h \\ & +\frac{2b\sqrt{a+b}-ab'+a'b}{2a\sqrt{a+b}}\eta_{(\alpha)j}(\eta_{(\beta)i}\xi_{(\beta)}^h + \eta_{(\gamma)i}\xi_{(\gamma)}^h) \\ & +\frac{ab'-4b\sqrt{a+b}}{2a\sqrt{a+b}}(\eta_{(\beta)i}\eta_{(\beta)j} + \eta_{(\gamma)i}\eta_{(\gamma)j})\xi_{(\alpha)}^h,\end{aligned}$$

others $= 0$.

In this place, suppose that the following equations hold:

$$2\sqrt{a+b} = a', \qquad ab' = 4b\sqrt{a+b}. \tag{5.1}$$

Then we can easily see that $(\tilde{J}_{(\alpha)}, \tilde{g}), \alpha = 1, 2, 3$ is a quaternionic Kähler structure. Putting $a = f^2$ and hence $b = f^2(f'^2 - 1)$, we see that the metric (4.3) reduces to

$$\tilde{g} = dt^2 + f^2 g + f^2(f'^2 - 1)\sum \eta_{(\alpha)} \otimes \eta_{(\alpha)}, \tag{5.1$'$}$$

and moreover that the equations (5.1) are equivalent to the following OED

$$ff'' - f'^2 + 1 = 0. \tag{5.2}$$

Thus we have the following.

Proposition 5.1. *Let $(M, \phi_{(\alpha)}, \xi_{(\alpha)}, \eta_{(\alpha)}), \alpha = 1, 2, 3$ be a 3-Sasakian manifold. An almost quaternionic Hermitian structure constructed on $M \times \mathbb{I}$ such as Proposition 4.1 is quaternionic Kähler if and only if the function f satisfies the ODE*

$$ff'' - f'^2 + 1 = 0$$

with the conditions $f > 0$ and $f' > 0$ on I.

Remark 1. After long calculations, it is shown that if the almost quaternionic Hermitian structure mentioned above satisfies the condition (IV) in Alekseevsky-Marchiafava [1, p.157], then the function f has to satisfy the ODE (5.2).

We shall write down the solutions of ODE (5.2) for later use. Usually, putting $p = f'$, we have

$$f'' = p\frac{dp}{df},$$

from which (5.2) reduces to

$$\frac{p}{p^2-1}dp = \frac{df}{f}. \tag{5.3}$$

Integrating the both sides of (5.3), we have

$$p^2 - 1 = kf^2, \tag{5.4}$$

where k is constant. Recall that $p = f'(t)$, and that $f(t) > 0$ and $f'(t) > 0$. We may, up to a motion of parameter t, have that a solution $f(t)$ of the ODE (5.4) is of the form:

Case 1. $k = 0$.

$$f(t) = t, \quad 0 < t < \infty.$$

Case 2. $k < 0$.

$$f(t) = \frac{1}{\sqrt{-k}} \sinh(\sqrt{-k}t), \quad 0 < t < \infty.$$

Case 3. $k > 0$.

$$f(t) = \frac{1}{\sqrt{k}} \sin(\sqrt{k}t), \quad 0 < t < \frac{\pi}{\sqrt{k}}.$$

Thus we now have a generalization of Theorem B in Boyer-Galicki-Mann [5], since $f' = 1$ in the case $k = 0$.

Theorem 5.2. *Let $(M, \phi_{(\alpha)}, \xi_{(\alpha)}, \eta_{(\alpha)}, g), \alpha = 1, 2, 3$ be a 3-Sasakian manifold. Let f be a real valued function, satisfying the ODE (5.2).*

(1) *(Boyer-Galicki-Mann) If $f(r) = r$, then the product manifold $M \times \mathbb{R}^+$ with the cone metric in (5.1)' is hyperkähler.*
(2) *If $f(r) = \frac{1}{\sqrt{-k}} \sin h(\sqrt{-k}r)$, then the product manifold $M \times \mathbb{R}^+$ with the metric in (5.1)' is quaternionic Kähler, where k is a negative constant.*
(3) *If $f(r) = \frac{1}{\sqrt{k}} \sin(\sqrt{k}r)$, then the product manifold $M \times (0, \frac{\pi}{\sqrt{k}})$ with the metric in (5.1)' is quaternionic Kähler, where k is a positive constant.*

Remark 2. In the above theorem, if $M = S^{4m+3}$ with the canonical metric, then the manifolds are abstract rotational manifolds in the sense of Hsiang [7], and the one constructed in (1) (resp. (2), (3)) is a geodesic coordinate neighbourhood of the quaternionic Euclidean n-space $\mathbb{H}^n$ with the canonical metric (resp. the quaternionic hyperbolic n-space $\mathbb{H}H^n$ with the canonical metric, the quaternionic projective n-space $\mathbb{H}P^n$ with the canonical metric), where $n = 4(m+1)$ (see [6] and [15] for the complex case).

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