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HIGHER ORDER CONTACT OF REAL CURVES IN A REAL HYPERQUADRIC PART II

Yuli Villarroel

Abstract. Let Φ be an Hermitian quadratic form, of maximal rank and index $(n,1)$, defined over a complex $(n+1)$ vector space V. Consider the real hyperquadric defined in the complex projective space P^nV by

$$Q = \{[\varsigma] \in P^nV,\ \Phi(\varsigma) = 0\}.$$

Let G be the subgroup of the special linear group which leaves Q invariant and D the $(2n)-$ distribution defined by the Cauchy Riemann structure induced over Q. We study the real regular curves of constant type in Q, tangent to D, finding a complete system of analytic invariants for two curves to be locally equivalent under transformations of G.

The real hypersurfaces of real codimension one, are the boundaries of domains in a complex manifold. Among the non-degenerate real hypersurfaces in C^{n+1} the simplest and most important are the real hyperquadrics. S. Chern and J. Moser show how the geometry of a general non-degenerate real hypersurface can be considered as a generalization of a real hyperquadric [4].

The concept of Frenet frame for holomorphic curves in complex projective spaces played an important role in the classical theory of the equidistribution of these curves [5]. In [16], we use contact theory to study the real curves in the hyperquadric,

$$Q = \{[\zeta] \in P^nC^n, \quad \Phi(\zeta) = 0\};$$

transversal to the distribution D, and find the Frenet frames for these curves.

The purpose of the present article is to study the real curves, of constant type in the hyperquadric Q, tangent to the distribution D, finding a complete system of analytic invariants for two curves to be locally equivalent under transformation

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of the group G. Moreover we find the Frenet frames for these curves. Using contact theory and the action of the group on each contact manifold of order k, a transversal section to the orbits of maximal dimension, can be naturally obtained.

It is interesting to observe that in the study of the Frenet frame of regular curves of constant type tangent to the distribution D, we obtain three visibly differents cases, namely: $n = 2$, $4 \leq n \leq 6$, y $n \geq 7$. For $n = 2$, we obtain a real invariant of order 5, in contrast with the curves transverse to the distribution, which have two real invariants of order 3 and 4 [16].

1. The higher order contact in the hyperquadric

Let Φ be the Hermitian quadratic form over a $(n+1)-$ vector space V

$$\Phi(\zeta) = \zeta^\alpha \overline{\zeta^\alpha} + i(\zeta^n \overline{\zeta^0} - \overline{\zeta^n}\zeta^0), \quad \zeta \in V,$$

and $G \subset SL(n+1, C)$ be the subgroup which leaves Φ invariant. The Lie algebra

$\mathcal{G}$ of G is given by

$$\mathcal{G} = \left\{ \ell \in T_eG : \quad \ell = \begin{pmatrix} \ell_0^0 & \ell_1^0 & \ldots & \ell_{n-1}^0 & \ell_n^0 \\ \ell_0^1 & & & & -i\overline{\ell_1^0} \\ \vdots & & (\ell_\beta^\alpha) & & \vdots \\ \ell_0^{n-1} & & & & -i\overline{\ell_{n-1}^0} \\ \ell_0^n & i\overline{\ell_O^1} & \ldots & i\overline{\ell_{n-1}^0} & -\overline{\ell_0^0} \end{pmatrix} ; \quad \begin{matrix} \ell_\beta^\alpha + \overline{\ell_\alpha^\beta} = 0, \\ \\ tr\ell = 0, \\ \\ \ell_n^0, \ell_0^n \in \Re \end{matrix} \right\}.$$

Denote by ω, the canonical form over G with components ω_γ^α respect to the usual basis . Let Q be the $(2n+1)$-dimensional real hyperquadric [5], defined in the complex projective space P^nV by the equation

$$Q = \{[\zeta] \in P^nV, \quad \Phi(\zeta) = 0\}.$$

The group G acts on P^nV by $g.[\zeta] = [g.\zeta]$, and the quadric Q is invariant by the action of G on P^nV, moreover, G acts transitively on Q. Then, given $p_0 \in Q$, the map

$$\psi^0 : g \in G \longmapsto g.p_0 \in Q,$$

defines an isomorphism: $G/G^o \simeq Q$. The isotropy group G^0 at $p_0 = [(1, 0, \cdots, 0)]$, is $G^0 = \{g \in G : \ g_0^\alpha = 0, 1 \leq \alpha \leq n\}$ and its Lie algebra $\mathcal{G}^0$ is given by

$$\mathcal{G}^0 = \left\{ \ell \in \mathcal{G} : \ell = \begin{pmatrix} \ell_0^0 & \ldots & \ell_n^0 \\ 0 & (\ell_\beta^\alpha) & \vdots \\ 0 & 0 & -\overline{\ell_0^0} \end{pmatrix} \right\}.$$

In the following we will agree that small Greek indices α, γ, run from 1 to $n-1$, the indices α_k run from k to $n-1$, unless otherwise specified, and we will use the summation convention.

The forms ω_0^α which vanish on T_eG^0, allow us to define a basis $\{\tilde{\omega}_0^\alpha\}$ of $T^*_{p_0}Q$ as follows. Given $\tilde{v} \in T_{p_0}Q$, let

$$\tilde{\omega}_0^\alpha(\tilde{v}) = \omega_{0_e}^\alpha(v_e), \quad where \quad v \in \mathcal{G}, \quad and \quad T_e\psi_e^0(v) = \tilde{v}. \tag{1.1}$$

Proposition 1.1. *The Lie algebra $\mathcal{G}^0$ acts on $T^*_{p_0}Q$ as follows:*

$$(\ell, \tilde{\omega}) \in \mathcal{G}^0 \times T^*_{p_0}Q \mapsto \ell.\tilde{\omega}, \quad \ell.\tilde{\omega}(\tilde{v}) = -d\omega|_e(\ell_e, v_e), \text{ with } v \in \mathcal{G}, \quad T_e\psi^0(v_e) = \tilde{v},$$

and its expression in coordinates is

$$\begin{array}{lclclcll} \ell.\tilde{\omega}_0^\alpha & = & \ell_\alpha^\gamma \tilde{\omega}_0^\gamma & + & (-\ell_0^0 + \ell_\alpha^\alpha)\tilde{\omega}_0^\alpha & - & i\overline{\ell_\alpha^0}\tilde{\omega}_0^n & 1 \leq \alpha,\ \gamma \leq n-1, \\ \ell.\tilde{\omega}_0^n & = & & & & - & 2Re\ell_n^0\tilde{\omega}_0^n & \gamma \neq \alpha\,. \end{array}$$

Proof. See [16]. □

We observe that the subspace $D_{p_0} \subset T_{p_0}Q$ defined by $\tilde{\omega}_0^n = 0$ is invariant by G^0, since $\mathcal{G}^0$ transforms ω_0^n in a multiple of itself, and G^0 is connected.

The transitivity of the action of G on Q, allow us to define a $(2n)$-dimensional distribution over Q, as follows

$$D : p \in Q \longmapsto (l_g)_*(D_{p_0}), \quad \text{where } g \in G \text{ and } l_g(p_0) = g.p_0 = p.$$

To study the real curves in Q, it is natural to consider two cases: the curves transversal to the distribution D at all its points, and the curves tangent to D at all points. In [16] we consider the first case, using contact theory. In this paper we will study the second case. We will use the following theorem about Lie groups.

Theorem 1.1. *Let G be a Lie group that acts on a smooth manifold M and $\mathcal{G}$ its Lie algebra. Let $\chi(M)$ denote the smooth (C^∞) vector fields on M, and let F be the map:*

$$F : \mathcal{G} \longrightarrow \chi(M), \qquad \ell \longmapsto F_\ell, \quad with \quad F_\ell(x) = \frac{d}{dt}|_{t=o}(\exp(t\ell).x)\,,$$

then we have:

a) The integral curve $y(x)$ of the field F_ℓ, at the point $x \in M$, is contained in the orbit $G(x)$ of x.

b) The action of G on M is transitive if and only if for any $x \in M$, and for any $v \in T_xM$, there exists $\ell \in \mathcal{G}$ such that $F_\ell(x) = v$. □

1. The contact elements tangent to D

Let $C^s_q Q$ be the manifold of contact element of order s, and dimension 1 at $q \in Q$, and $C^s Q$ the manifold of all contact elements $C^s_q Q$, with $q \in Q$.

For $k \leq s$ we consider the canonical projection $\pi^s_k : C^s Q \to C^k Q$ given by $C^s_q \Gamma \mapsto C^k_q \Gamma$, with $\Gamma \subset Q$ a 1-dimensional submanifold [6, 12, 15, 17].

Denote by i^s, $i^{1,s}$ the canonical immersion $\quad i^s : q \in \Gamma \mapsto C^s\Gamma \in C^s Q.$

$$i^{1,s} : C^{s+1}_q \Gamma \in C^{s+1} Q \mapsto C^1_{C^s_q \Gamma} C^s \Gamma \in C^1 C^s Q,$$

The action $G \times C^s Q \to C^s Q \quad$ is given by $g.C^s_q \Gamma = C^s_{g.q} g.\Gamma$.

Let $\mathcal{H}^1$ be the fiber of the contact elements of order 1 tangent to D, which project onto p_0, i. e.,

$$\mathcal{H}^1 = \{X^1 \in C^1_{p_0} Q : \qquad \tilde{\omega}^n_0 | X = 0\},$$

where $\tilde{w}^\alpha_0 | X^1$ denotes the restriction of $\tilde{\omega}^\alpha_0$ to the 1-dimensional subspace defined by the contact element X^1.

Consider the open set $\mathcal{U}^\alpha$, $\mathcal{V}^\alpha \subset \mathcal{H}^1$, $1 \leq \alpha \leq n-1$, defined as follows,

$$\mathcal{U}^\alpha = \{X^1 \in \mathcal{H}^1 : \qquad (\tilde{\omega}^\alpha_0 + \overline{\tilde{\omega}^\alpha_0}) | X^1 \neq 0\},$$
$$\mathcal{V}^\alpha = \{X^1 \in \mathcal{H}^1 : \qquad (\tilde{\omega}^\alpha_0 - \overline{\tilde{\omega}^\alpha_0}) | X^1 \neq 0\},$$

the coordinates in $\mathcal{U}^\alpha$, (respectively $\mathcal{V}^\alpha$), are defined as in [13],

$$\tilde{v}^{\alpha_0}_0 | X^1 = b^\alpha_0 \tilde{u}^{\alpha_0}_0 | X^1,$$
$$\tilde{\omega}^\gamma_0 | X^1 = \lambda^\gamma_0 \tilde{u}^\alpha_0 | X^1,$$

where, $\gamma \neq \alpha_0$, and $\tilde{\omega}^\gamma_0 = \tilde{u}^\gamma_0 + \tilde{v}^\gamma_0$. (respectively $\tilde{u}^\alpha_0 | X^1 = b^\alpha \tilde{v}^\alpha_0 | X^1, \quad \tilde{\omega}^\gamma_0 | X^1 = \lambda^\gamma_0 \tilde{v}^\alpha_0 | X^1$.)

Proposition 2.1. *Given $\ell \in \mathcal{G}^0$, we have*

$$\begin{array}{rcl} \ell.\tilde{u}^{\alpha_0}_0 &=& Re(\ell^{\alpha_0}_{\alpha_0} - \ell^0_0) \quad \tilde{u}^{\alpha_0}_0 \; - Im(\ell^{\alpha_0}_{\alpha_0} - \ell^0_0) \quad \tilde{v}^{\alpha_0}_0 \; + \sum Re(\ell^\gamma_{\alpha_0} \omega^\gamma_0), \\ \ell.\tilde{v}^{\alpha_0}_0 &=& Im(\ell^{\alpha_0}_{\alpha_0} - \ell^0_0) \quad \tilde{u}^{\alpha_0}_0 \; + Re(\ell^{\alpha_0}_{\alpha_0} - \ell^0_0) \quad \tilde{v}^{\alpha_0}_0 \; + \sum Im(\ell^\gamma_{\alpha_0} \omega^\gamma_0), \\ \ell.\omega^\alpha_0 &=& (\ell^{\alpha_0}_{\alpha_0} - \ell^0_0) \quad \tilde{\omega}^{\alpha_0}_0 \; + \sum \ell^\gamma_\alpha \omega^\gamma_0. \end{array} \tag{2.1}$$

Proof. Applying Proposition 1.1 we have

$$\begin{array}{rcl} \ell.(\frac{\omega^{\alpha_0}_0 + \overline{\omega^{\alpha_0}_0}}{2}) &=& Re(\ell^{\alpha_0}_{\alpha_0} - \ell^0_0) \quad \tilde{u}^{\alpha_0}_0 \; - \; Im(\ell^{\alpha_0}_{\alpha_0} - \ell^0_0) \quad \tilde{v}^{\alpha_0}_0 \; + \sum Re(\ell^\gamma_{\alpha_0} \omega^\gamma_0), \\ \ell.(\frac{\omega^{\alpha_0}_0 - \overline{\omega^{\alpha_0}_0}}{2i}) &=& iIm(\ell^{\alpha_0}_{\alpha_0} - \ell^0_0) \quad \tilde{u}^{\alpha_0}_0 \; + \; Re(\ell^{\alpha_0}_{\alpha_0} - \ell^0_0) \quad \tilde{v}^{\alpha_0}_0 \; + \sum Im(\ell^\gamma_{\alpha_0} \omega^\gamma_0), \end{array}$$

simplify we obtain the result. □

Denote by $\tilde{C}^1 Q$ all the contact elements of order 1, tangent to D, and express $X \in \mathcal{U}^\alpha$ in coordinates as

$$X = (\lambda^1_0, \cdots, \lambda^{\alpha-1}_0, b^\alpha_0, \lambda^{\alpha+1}_0, \cdots, \lambda^{n-1}_0),$$

Proposition 2.2. *Let $F^0 : \mathcal{G}^0 \to \chi(\mathcal{U}^\alpha)$, $\quad F_\ell(X) = \frac{d}{dt}|_{t=o}(\exp(t\ell).X)$, then we have:*
For $n > 2$,

$$F^0_\ell(X) = \sum_{\gamma\neq\alpha}^{n-1}(\ell^\alpha_\gamma + g(b^\alpha_0, \lambda^\gamma_0))\frac{\partial}{\lambda^\gamma_0}|_X + (\ell^0_0 - Im\ell^\alpha_\alpha + f(b^\alpha_0, \lambda^\gamma_0))\frac{\partial}{b^\alpha_0}|_X .$$

where $f(0,0) = g(0,0) = 0$.

For $n = 2$, $\quad F^0_\ell(X) = (3Im\ell^0_0(1 + b^1_0)^2)\frac{\partial}{\partial b^1_0}|_X$.

Proof. For simplicity, consider the case $\alpha_0 = 1$.

Given $\ell \in \mathcal{G}^0$, and $X = (b^1_0, \lambda^2_0, \cdots, \lambda^{n-1}_0) \in \mathcal{U}^1$, let $r(t) = \exp(t\ell).X$, which is expressed in coordinates as

$$r(t) = (b^1_0(t), \lambda^2_0(t), \cdots, \lambda^{n-1}_0(t)),$$

where

$$\tilde{v}^1_0|r(t) = b^1_0\tilde{u}^1_0|r(t), \quad \tilde{\omega}^\alpha_0|r(t) = \lambda^\alpha_0\tilde{u}^1_0|r(t)\,,$$

deriving with respect to t and evaluating at $t = 0$, we have

$$\frac{db^1_0}{dt}|_{t=0} = \ell.\tilde{v}^1_0|X - b^1_0\ell.\tilde{u}^1_0|X\,,$$
$$\frac{d\lambda^\alpha_0}{dt}|_{t=0} = \ell.\tilde{\omega}^\alpha_0|X - \lambda^\alpha_0\ell.\tilde{u}^1_0|X\,.$$

Applying Proposition 2.1 and substituting coordinates we have the result. □

Proposition 2.3. *The group G acts transitively on $\tilde{C}^1Q$.*

Proof. Since the action of G on Q is transitive, it is sufficient to prove that the action of G^0 on $\mathcal{U}^\alpha$ is transitive.

Now given $X \in \mathcal{U}^\alpha$, by Proposition 2.2 we can choose $\ell_1, \cdots, \ell_{n-1} \in \mathcal{G}^0$, such that $F^0_{\ell_1}(X)$, $\cdots$, $F^0_{\ell_{n-1}}(X)$ generate $T_X\mathcal{U}^\alpha$ Then by theorem 1.1. we have that G^0 acts transitively on $\mathcal{U}^\alpha$. □

Proposition 2.4. *Let $X^1_0 \in C^{1,1}_{p_0}Q$ be given by $X^1_0 = (0, \cdots, 0)$, $\quad$ i.e.,*

$$\tilde{v}^1_0|X^1 = \tilde{\omega}^\alpha_0|X^1 = 0, \quad \tilde{u}^1_0|X^1 \neq 0\,,$$

then

i) the Lie algebra $\mathcal{G}^1$ of the isotropy group $G^1 \subset G^0$ of X^1_0 is given by

$$\mathcal{G}^1 = \{\ell \in \mathcal{G}^0 : Im(\ell^1_1 - \ell^0_0) = \ell^1_2 = \cdots = \ell^1_{n-1} = 0\}, \quad i.e.,$$

$$\mathcal{G}^1 = \left\{ \ell = \begin{pmatrix} \ell^0_0 & \ell^0_1 & \ell^0_2 & \cdots & \ell^0_{n-1} & \ell^0_n \\ 0 & iIm\ell^0_0 & 0 & \cdots & 0 & i\overline{\ell^0_1} \\ 0 & 0 & & (\ell^\alpha_\beta) & & . \\ 0 & 0 & 0 & \cdots & 0 & -\overline{\ell^0_0} \end{pmatrix} ; \quad \ell^\alpha_\beta + \overline{\ell^\beta_\alpha} = 0, \quad tr\ell = 0 \right\},$$

ii) $dim\mathcal{G}^1 = dim\mathcal{G}^0 - (2n-3)$

iii) For $n = 2$,

$$\mathcal{G}^1 = \left\{ \ell = \begin{pmatrix} Re\ell_0^0 & \ell_1^0 & \ell_2^0 \\ 0 & 0 & i\overline{\ell_1^0} \\ 0 & 0 & -Re\ell_0^0 \end{pmatrix} \right\}.$$

Proof. By Proposition 2.2., we have

$$F_\ell^0(X_0^1) = 0 \quad \Leftrightarrow \quad Im(\ell_1^1 - \ell_0^0) = \ell_2^1 = \cdots = \ell_{n-1}^1 = 0\,. \qquad \square$$

Let $\mathcal{O}^1 = G.X_0^1$ be the orbit of the action of G on $\tilde{C}^1Q$ which contains X_0^1. Then, the map

$$\psi^1 : G \to \mathcal{O}^1, \quad \text{given by} \quad \psi^1(g) = g.X_0^1$$

induces a diffeomorphism $\mathcal{O}^1 \simeq G/G^1$. The forms

$$u_0^1, \omega_0^\alpha, \omega_1^1 - Im\omega_0^0, \omega_{\alpha_2}^1,$$

vanishing on G^1, define linearly independent real forms which can be projected onto $T_{X_0^1}\mathcal{O}^1$, using a similar argument as in (1.1). The projected forms, denoted by

$$\tilde{u}_0^1, \tilde{\omega}_0^\alpha, \tilde{\omega}_1^1 - Im\tilde{\omega}_0^0, \tilde{\omega}_{\alpha_2}^1,$$

define a basis of $T^*_{X_0^1}\mathcal{O}^1$.

Let $\mathcal{H}^2$ be the fiber of the contact elements of order 2 which project onto X_0^1. Denote by $i : C^2Q \to C^1(C^1Q)$ the canonical immersion and

$$\pi_0^1 : C^1Q \to Q, \quad \pi_0^{1,1} : C^1(C^1Q) \to C^1Q,$$

the canonical projections. Then [17]

$$i(\mathcal{H}^2) = \{X^2 \in C^2_{p_0}Q : \quad \tilde{\omega}_0^n|X^2 = \tilde{v}_0^1|X^2 = \tilde{\omega}_0^{\alpha_2}|X^2 = 0, \quad \tilde{u}_0^1|X^2 \neq 0\},$$

Consider coordinates in $\mathcal{H}^2$ defined as

$$X^2 = (b_1^1, \lambda_2^1, \cdots, \lambda_{n-1}^1),$$

where

$$(\tilde{\omega}_1^1 - Im\tilde{\omega}_0^0)|X^2 = ib_1^1\tilde{u}_0^1|X^2; \quad \tilde{\omega}_\alpha^1|X^2 = \lambda_\alpha^1\tilde{u}_0^1|X^2.$$

Let $\tilde{C}^2Q$ be the contact elements of order 2, tangent to D, which project onto $\tilde{C}^1Q$.

Proposition 2.5. *Let $F^1 : \mathcal{G}^1 \to \chi(\mathcal{H}^2)$ be defined as*

$$F_\ell^1(X^2) = \frac{d}{dt}|_{t=o}(\exp{(t\ell)}.X^2)\,,$$

then, for $n=2$,

$$F^1_\ell(X^2)=(-Im\ell^0_1+bRe\ell^0_0)\frac{\partial}{\partial b}|X^2\,,\,where\ \tilde{v}^0_0=b\tilde{u}^1_0\,,\tilde{v}^1_0=0\,.$$

For $n>2$,

$$\begin{aligned}F^1_\ell(X^2)&=(2Im\ell^0_1+b^1_1Re\ell^0_0)\frac{\partial}{\partial b^1_1}|X^2+\\&+(-\ell^0_2+(\ell^1_1-\ell^2_2+Re\ell^0_0)\lambda^1_2+\sum_{\gamma=3}^{n-1}\ell^\gamma_2\lambda^1_\gamma)\frac{\partial}{\partial\lambda^1_2}|X^2_0+\\&+\cdots+(-\ell^0_{n-1}+(\ell^1_1-\ell^{n-1}_{n-1}+Re\ell^0_0)\lambda^1_{n-1}+\textstyle\sum_{\gamma=2}^{n-2}\ell^\gamma_{n-1}\lambda^1_\gamma)\frac{\partial}{\partial\lambda^1_{n-1}}|X^2_0\,.\end{aligned}$$

Proof. For $n=2$. Given $\ell\in\mathcal{G}^1$, we have

$$\ell.\tilde{v}^0_0=\ell.\frac{\tilde{\omega}^0_0-\overline{\tilde{\omega}^0_0}}{2i}=\lambda_1\tilde{\omega}^1_0+\lambda_2\overline{\tilde{\omega}^1_0}+\lambda_3\tilde{\omega}^2_0+\lambda_4\tilde{v}^0_0\,,$$

Consider in $T_eG^0_{p_0}$ the basis,

$$E^1_0=\begin{pmatrix}0&0&0\\1&0&0\\0&\mathrm{i}&0\end{pmatrix}\,,\quad E^2_0=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}\,,$$

$$F^1_0=\begin{pmatrix}0&0&0\\\mathrm{i}&0&0\\0&1&0\end{pmatrix}\,,\quad F^1_1=\begin{pmatrix}\mathrm{i}&0&0\\0&-2\mathrm{i}&0\\0&0&\mathrm{i}\end{pmatrix}\,,$$

then

$$\ell.\tilde{v}^0_0(E^1_0)=\lambda_1+\lambda_2=\frac{1}{2i}d(\tilde{\omega}^0_0-\overline{\tilde{\omega}^0_0})(X,\ell)=Im\ell^0_1,\qquad \ell.\tilde{v}^0_0(F^1_1)=0\,,$$
$$\ell.\tilde{v}^0_0(F^1_0)=i(\lambda_1-\lambda_2)=Re\ell^0_1,\qquad \ell.\tilde{v}^0_0(E^2_0)=\lambda_3=Im\ell^0_2\,,$$

and we have

$$\ell.\tilde{v}^0_0=Re\ell^0_1\tilde{v}^1_0+Im\ell^0_1\tilde{u}^1_0+Im\ell^0_2\tilde{\omega}^2_0\,.$$

Since $\pi X^2=X^1_0$, we obtain $\tilde{v}^1_0|X^2=\tilde{\omega}^2_0|X^2=0$. Hence
If $\ell\in\mathcal{G}^1$, and $r(t)=\exp t\ell.X^2$, we have

$$\frac{db}{dt}|_{t=0}=-Im\ell^0_1-bRe\ell^0_0\,,$$

For $n>2$ with a similar procedure we obtain the result. □

Corollary 1. *The group G acts transitively on $\tilde{C}^2Q$.*

Proof. Using the map $F^1:\mathcal{G}^1\to\chi(\mathcal{H}^2)$, the proposition 2.5 , and Theorem 1.1. □

Corollary 2. *The Lie algebra of the element $\tilde{X}^2_0$ is given as follows:*

In the case $n = 2$,

$$\tilde{\mathcal{G}}^2 = \left\{ \ell = \begin{pmatrix} Re\ell_0^0 & Re\ell_1^0 & \ell_2^0 \\ 0 & 0 & -iRe\ell_1^0 \\ 0 & 0 & -Re\ell_0^0 \end{pmatrix} \right\}. \qquad \square$$

In the case $n > 2$,

$$\mathcal{G}^2 = \left\{ \ell = \begin{pmatrix} \ell_0^0 & Re\ell_1^0 & 0 & \cdots & 0 & \ell_n^0 \\ 0 & iIm\ell_0^0 & 0 & \cdots & 0 & iRe\ell_1^0 \\ 0 & 0 & & \ell_\beta^\alpha & & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\overline{\ell_0^0} \end{pmatrix}, \ell_\beta^\alpha + \overline{\ell_\alpha^\beta} = 0, \quad tr\ell = 0 \right\};$$

For $n > 2$, using a similar argument as in (1.1), the forms

$$\omega_0^\alpha, \omega_1^0 - Im\omega_0^0, \omega_{\alpha_2}^1, Im\omega_1^0, \omega_{\alpha_2}^0,$$

can be projected onto $T_{\tilde{X}_0^2}\tilde{\mathcal{O}}^2$ and define a basis of $T^*_{\tilde{X}_0^2}\tilde{\mathcal{O}}^2$.

Let $\mathcal{H}^3 = \{X^3 \in C^3Q : \pi_2^3(X^3) = \tilde{X}_0^2\}$. We identify $\mathcal{H}^3$ with its image by $i : C^3Q \to C^1(C^2Q)$.
We consider coordinates in $\mathcal{H}^3$ defined as

$$Im\tilde{\omega}_1^0|X^3 = b_1^0\tilde{u}_0^1|X^3, \quad \tilde{\omega}_\alpha^0|X^3 = \lambda_\alpha^0\tilde{u}_0^1|X^3,$$

and write $X^3 = (b_1^0, \lambda_2^0, \cdots, \lambda_{n-1}^0)$.

Proposition 2.6. *Let $\mathcal{F}^3 : \mathcal{G}^2 \to \mathcal{X}(\mathcal{H}^3)$ be defined as in Theorem 1.1 then:*
For $n = 2$,

$$F_\ell^3(b_1^0, \lambda_1^0) = -b_0^1 Re\ell_0^0 \frac{\partial}{\partial b_1^0}|X^3$$

For $n > 2$,

$$F_\ell^3(X^3) = -b_1^0 Re\ell_0^0 \frac{\partial}{\partial b_1^0}|X^3 + (Im(\ell_0^0 - \ell_2^2)\lambda_2^0 - \sum_{\gamma=3}^{n-1} \ell_2^\gamma\lambda_\gamma^0)\frac{\partial}{\partial\lambda_2^0}|X^3 +$$
$$+ \cdots + (Im(\ell_0^0 - \ell_{n-1}^{n-1})\lambda_{n-1}^0 - \sum_{\gamma=2}^{n-2} \ell_{n-1}^\gamma\lambda_\gamma^0)\frac{\partial}{\partial\lambda_{n-1}^0}|X^3.$$

Proof. Similarly to Proposition 2.5. $\square$

Corollary 1. *For $n = 2$ we have*
i. The group G^2 acts transitively on

$$\mathcal{H}_0^3 = \{X^3 \in \mathcal{H}^3 : Im\omega_1^0(X^3) \neq 0\},$$

and

$$\mathcal{G}^3 = \left\{ \ell = \begin{pmatrix} 0 & Re\ell_1^0 & \ell_2^0 \\ 0 & 0 & -iRe\ell_1^0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Corollary 2. *If $b_1^0(X^3) = 0$ then $\mathcal{G}^3 = \mathcal{G}^2$ and $G.X_0^3$ is a singular orbit in $\tilde{C}_0^3Q$.*

Corollary 3. *For $n > 2$ we have*

$$\mathcal{H}_0^3, \hat{\mathcal{H}}_0^3, \mathcal{H}_1^3, \mathcal{H}_2^3 \subset \mathcal{H}_0^3$$

defined as

$$\mathcal{H}_0^3 = \{X^3 \in \mathcal{H}^3 : b_1^0(X^3) \neq 0, \quad \lambda_{\alpha_2}^0 \neq 0, \text{ for any } \alpha_2\},$$

$$\hat{\mathcal{H}}_0^3 = \{X^3 \in \mathcal{H}^3 : b_1^0(X^3) = \lambda_{\alpha_2}^0 = 0\},$$

$$\mathcal{H}_1^3 = \{X^3 \in \mathcal{H}^3 : b_1^0(X^3) = 0, \quad \lambda_{\alpha_2}^0 \neq 0 \text{ for any } \alpha_2\},$$

$$\mathcal{H}_2^3 = \{X^3 \in \mathcal{H}^3 : b_1^0(X^3) \neq 0, \quad \lambda_{\alpha_2}^0 = 0\},$$

then $\mathcal{H}_0^3$ contain the orbits of maximal dimension and $\hat{\mathcal{H}}_0^3$ contain the orbits of minimal dimension.

The following Proposition gives, for $n > 2$, a real differential invariant of order 3 [9, 7, 10], defined by a transversal section to the orbits of maximal dimension on $\tilde{C}^3Q$.

For the case $n = 2$, the orbits of the action of G^2 are lines given by the axes b_1^0. A transversal section of these orbits is given by

$$I^3 = \{X^3 \in \mathcal{H}_0^3 : \quad Im\lambda_1^0 = 1\}\,.$$

Proposition 2.7. *If $X_0^3 \in \tilde{\mathcal{H}}_0^3$, then the orbit $\tilde{\mathcal{O}}^3 = G.X_0^3$ is given by*

$$\tilde{\mathcal{O}}^3 = \{X^3 \in \tilde{\mathcal{H}}^3 : b_1^0(X^3) = b_1^0(X_0^3);\ \sum |\lambda_{\alpha_2}^0(X^3)|^2 = \rho^2,\ \rho^2 = \sum |\lambda_{\alpha_2}^0(X_0^3)|^2\}.$$

Proof. We will prove the statement for $n = 4$, and similarly for $n > 4$. Denote I_β^α the matrix defined by

$$I_\beta^\alpha = (j_\beta^\alpha) = \delta_\alpha^1 \delta_\beta^1\,,$$

consider a basis of fields defined as:

$$E_{\alpha_2}^{\alpha_2} = iI_0^0 + iI_1^1 - 3iI_{\alpha_2}^{\alpha_2} + iI_n^n\,,$$

$$E_\beta^\alpha = iI_\beta^\alpha + iI_\alpha^\beta, \qquad J_\beta^\alpha = I_\beta^\alpha + I_\alpha^\beta,$$

$$J_0^0 = I_0^0 - I_n^n\,, \qquad J_\alpha^0 = I_\alpha^0 + iI_n^\alpha.$$

By Proposition 2.6, $\quad F_{J_0^0}^3 = -b_1^0 \frac{\partial}{\partial b_1^0}|X^3$. Then, to study the distribution D^3 on H^3, generated by the fields

$$\{F_\ell^3(X^3), \quad \ell \in \mathcal{G}^3\}\,,$$

it is enough to consider the components in the coordinates

$$\frac{\partial}{\partial \lambda_{\alpha_2}^0}|_{X^3}\,, \qquad 2 \leq \alpha_2 \leq n-1\,.$$

We identify the tangent to the fiber with the corresponding real dimensional space, i.e.

$$\lambda^{\alpha}_{\beta} = a^{\alpha}_{\beta} + ib^{\alpha}_{\beta} \to (a^{\alpha}_{\beta}, b^{\alpha}_{\beta}),$$

and we have

$$F^3_{E^2_2}(X^3) = (0, 4i\lambda^0_2, i\lambda^0_3) \leftrightarrow (0, 4b^0_2, -b^0_3, 4a^0_2, a^0_3),$$

$$F^3_{E^3_3}(X^3) = (0, i\lambda^0_2, 4i\lambda^0_3) \leftrightarrow (0, -b^0_2, -4b^0_3, a^0_2, 4a^0_3),$$

$$F^3_{J^2_3}(X^3) = (0, \lambda^0_3, -\lambda^0_2) \leftrightarrow (0, a^0_3, -a^0_2, b^3_2, -b^0_2),$$

$$F^3_{E^3_2}(X^3) = (0, i\lambda^0_3, i\lambda^0_2) \leftrightarrow (0, -b^0_3, -b^0_2, a^0_3, a^0_2),$$

we observe that the vectors defined by

$$F^3_{E^2_2}, F^3_{E^3_3}, F^3_{J^2_3}, F^3_{E^3_2}, \tag{2.2}$$

are perpendicular to the vector with components given by X^3. Moreover, for $a^0_\alpha \neq 0$ (respectively $b^0_\alpha \neq 0$), it is possible to find a 4×4 matrix inside the matrix defined by (2.2). Then, the integral submanifold of the distribution D^3 generated by $F^3_{\ell_j}$, is given by

$$\mathcal{O}^3 = \{X^3 \in \mathcal{H}^3_0 : \quad \sum_2^{n-1} |\lambda^0_\alpha(X^3)|^2 = \rho^2, \quad b^0_1 = b^0_1(X^3_0), \quad \rho^2 = \sum_2^{n-1} |\lambda^1_\alpha(X^3_0)|^2 . \qquad \square$$

A transversal section to the orbits of the action of G^3 on $\mathcal{H}^3_0$ is given by

$$I^3_0 = \left\{ X^3 \in \mathcal{H}^3 : \begin{array}{ll} Re\lambda^0_2(X^3_0) > 0, & Im\lambda^0_2(X^3) = 0 \\ Im\lambda^0_1(X^3) = 1, & \lambda^0_{\alpha_3}(X^3) = 0, \end{array} \right\}.$$

We will consider coordinate $\rho_3 > 0$ in I^3_0 given by

$$\tilde{\omega}^0_2 | X^3 = \quad \rho_3 \tilde{u}^1_0 | X^3 .$$

This real invariant of C^3Q, correspond with the point where the orbit of X^3 meets I^3.

An important observation is that the isotropy algebra of $X^3_{\rho_3}$ does not depend on ρ_3. Then a curve $\Gamma \subset Q$, tangent to D, with $C^3\Gamma \subset \tilde{C}^3Q$, is a curve of constant type [7]. The isotropy group of $X^3_{\rho_3} \in I_3$ is given by

$$\mathcal{G}^2 = \left\{ \ell = \begin{pmatrix} iIm\ell^0_0 & Re\ell^0_1 & 0 & \cdots & 0 & \ell^0_n \\ 0 & iIm\ell^0_0 & 0 & \cdots & 0 & -iRe\ell^0_1 \\ 0 & 0 & & (iIm\ell^\alpha_\beta) & & 0 \\ 0 & 0 & 0 & \cdots & 0 & iIm\ell^0_0 \end{pmatrix} \right\}, \text{ in the case } n > 2;$$

$$\tilde{\mathcal{G}}^2 = \left\{ \ell = \begin{pmatrix} 0 & Re\ell^0_1 & \ell^0_2 \\ 0 & 0 & -iRe\ell^0_1 \\ 0 & 0 & 0 \end{pmatrix}, \ell^0_2 \in \Re \right\}, \quad \text{in the case n=2}. \qquad \square$$

3. Action of G on the k, $(k \geq 4)$, order contact elements

Let $X_0^3 = X_{\rho_3}^3$, and $\mathcal{O}^3 = G.X_0^3$. The map $\psi^3 : g \in G \mapsto g.X_0^3 \in \mathcal{O}^3$, allows us to project the forms which vanish on G^3,on $T_{X_0^3}\mathcal{O}^3$. Moreover the real form $d\rho_3$, given by the function ρ_3, define a basis of $T^*_{X_0^3}I^3$.

If $\tilde{C}^3Q$ denotes the contact elements of order 3 tangent to D, which project onto $\tilde{\mathcal{O}}^2$, then we have

$$T_{X_0^3}\tilde{C}^3Q = T_{X_0^3}\tilde{\mathcal{O}}^3 \oplus T_{X_0^3}I^3 \,,$$

and we can define a basis of $T^*_{X_0^3}\tilde{C}^3Q$ extending the forms, as follows

$$\omega^\alpha_\beta | T_{X_0^3}I^3 = 0\,, \qquad d\rho_3 | T_{X_0^3}\tilde{\mathcal{O}}^3 = 0\,.$$

Let $\mathcal{H}^4 = \{X^4 \in C^4Q : \quad \pi^4_2(X^4) = X_0^3\}$, be defined in coordinates as follows

$$(\omega^0_0 - \omega^2_2)|X^4 = (a^0_0 + ib^2_2)\tilde{u}^1_0\,, \quad \tilde{\omega}^2_{\alpha_3} = \lambda^2_{\alpha_3}\tilde{u}^1_0\,.$$

If $\mathcal{G}^3$ denotes the Lie algebra of the isotropy group of $X^3_{\rho_3}$ then, using the fact that the isotropy algebra $\mathcal{G}^3$ is independent of ρ_3, we prove $\ell.d\rho_3 = 0$.

Similarly to paragraph 2, and using induction we obtain the transversal section I^k of the orbits of maximal dimension, defined as:

In case $n = 2$, we have:

i) G^3 acts transitively on H^4.

$$\mathcal{G}^4 = \left(\begin{array}{ccc} 0 & 0 & \ell^0_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right)\,, \quad \ell^0_2 \in \Re\,,$$

ii) G^4 acts transitively on H^5 and $\mathcal{G}^5 = (0)$.

For $n > 2$,

I^4 :

$$\begin{array}{rcl} \tilde{\omega}^0_0 - \tilde{\omega}^2_2 & = & i\tau_4\tilde{u}^1_0\,, \quad d\rho_3 \quad = \quad \rho^1_3\tilde{u}^1_0\,, \\ \tilde{\omega}^2_3 & = & \rho_4\tilde{u}^1_0\,, \quad \rho_4 > 0\,. \end{array}$$

I^5 :

$$\begin{array}{rclcrclcrcl} \tilde{\omega}^0_0 - \tilde{\omega}^3_3 & = & i\tau_5\tilde{u}^1_0\,, & & d\tau_4 & = & \tau^1_4\tilde{u}^1_0\,, & & d\rho^1_3 & = & \rho^2_3\tilde{u}^1_0\,, \\ \tilde{\omega}^3_4 & = & \rho_5\tilde{u}^1_0\,, & & d\rho_4 & = & \rho^1_4\tilde{u}^1_0\,. \end{array}$$

I^6 :

$$\begin{array}{rclrclrclrcl} \tilde{\omega}^0_n & = & \mu_6\tilde{u}^1_0\,, & d\tau_5 & = & \tau^1_5\tilde{u}^1_0\,, & d\tau^1_4 & = & \tau^2_4\tilde{u}^1_0\,, & d\rho^2_3 & = & \rho^3_3\tilde{u}^1_0\,, \\ \tilde{\omega}^3_3 - \tilde{\omega}^4_4 & = & i\tau_6\tilde{u}^1_0\,, & d\rho_5 & = & \rho^1_5\tilde{u}^1_0\,, & d\rho^1_4 & = & \rho^2_4\tilde{u}^1_0\,, \\ \tilde{\omega}^4_5 & = & \rho_6\tilde{u}^1_0\,. \end{array}$$

I^7 : For $4 \leq n \leq 6$,

$$\begin{array}{lllll} Im\tilde{\omega}_0^0 = \nu_7 \tilde{u}_0^1\,, & d\mu_6^1 = \mu_6^2 \tilde{u}_0^1\,, & d\tau_5^2 = \tau_5^3 \tilde{u}_0^1\,, & d\tau_4^2 = \tau_4^3 \tilde{u}_0^1\,, & d\rho_3^3 = \rho_3^4 \tilde{u}_0^1\,, \\ \tilde{\omega}_4^4 - \tilde{\omega}_5^5 = i\tau_7 \tilde{u}_0^1\,, & d\tau_6 = \tau_6^1 \tilde{u}_0^1\,, & d\rho_5^1 = \rho_5^2 \tilde{u}_0^1\,, & d\rho_4^2 = \rho_4^3 \tilde{u}_0^1, & \\ Re\tilde{\omega}_6^5 = \rho_7 \tilde{u}_0^1\,. & & & & \end{array}$$

I^7 : For $n \geq 7$, I^7 :,

$$\begin{array}{lll} \tilde{\omega}_{k-2}^{k-2} = i\tau_k \tilde{u}_0^1\,, & d\tau_{k-l}^l = \tau_{k-1}^{l+1} \tilde{u}_0^1\,, & d\mu_{k-1}^l = \mu_{k-1}^{l+1} \tilde{u}_0^1\,, \\ Re\tilde{\omega}_{k-1}^{k-2} = \rho_k \tilde{u}_0^1\,, & d\rho_{k-l}^l = \rho_{k-l}^{l+1} \tilde{u}_0^1\,. & \end{array}$$

Given $X^k \in I^k$, the Lie algebra of the isotropy group of order k is given by:

In case $n = 2$:

i) G^3 acts transitively on H^4.

$$\mathcal{G}^4 = \begin{pmatrix} 0 & 0 & \ell_2^0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \ell_2^0 \in \Re\,,$$

ii) G^4 acts transitively on H^5 and $\mathcal{G}^5 = (0)$.

For $n > 2$,

$$\mathcal{G}^4 = \begin{pmatrix} iA_0^0 Id_{4\times 4} & 0 & \ell_0^0 \\ 0 & (\ell_\beta^\alpha) & 0 \\ 0 & 0 & iA_0^0 \end{pmatrix}, \quad \mathcal{G}^5 = \begin{pmatrix} iA_0^0 Id_{5\times 5} & 0 & 0 \\ 0 & (\ell_\beta^\alpha) & 0 \\ 0 & 0 & iA_0^0 \end{pmatrix},$$

For $k > 5$:

$$\mathcal{G}^k = \begin{pmatrix} 0 Id_{k\times k} & 0 & 0 \\ 0 & (\ell_\beta^\alpha) & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \text{for } n = 6, \quad \mathcal{G}^6 = (0), \quad \text{for } n \geq 7, \quad \mathcal{G}^{n-1} = (0)\,.$$

It is important to distinguish the case $n = 2$, where we have the following:

4. Equivalence of regular curves in Q

In this paragraph we give the structure equations for a regular curve in Q, and necessary and sufficient conditions for the equivalence of two regular curves.

Let Γ be a curve in Q tangent to the distribution D at all its points, viewed as a connected 1-dimensional submanifold. Given $p \in \Gamma$, denote by $C_p^k\Gamma$ the contact element of order k of Γ at p, and $G^0_{C_p^k\Gamma}$ the isotropy group by the induced action of G on C^kQ. We say that p is a *regular point* if:

For $2 \leq n \leq 5$, $\dim G^0_{C_p^5}\Gamma = 0$, for $n = 6$ $\dim G^0_{C_p^6}\Gamma = 0$.

For $n \geq 7$, $\dim G^0_{C_p^{n-1}}\Gamma = 0$, If p is regular and k is the smallest order of contact with $G^0_{C^k_p\Gamma} = 0$, we say that p is a *regular point* of order k.

Now if p is a regular point of order k then all the points in a neighborhood of p are regular of order k. We can see this, using the following results about Lie Groups [1].

Proposition 3.1. *Let G be a Lie group acting on a smooth manifold M. For $p \in M$, let G_p be the isotropy group of p and $d(p)$ the dimension of G_p. Then, given $p_0 \in M$, there exists a neighborhood V of p_0 in M, such that $d(p) \leq d(p_0)$ for all $p \in V$.*

We will say that a curve $\Gamma \subset Q$ is *regular* if all its points are regular.

Let I^k be the transversal section defined in paragraph 3., and $A = G.I^k$. Then we have the following,

Theorem 3.1. *Let $\Gamma \subset Q$ be a curve, and $p \in \Gamma$. Then Γ is regular of order k in p if and only if $C^k_p\Gamma \in A$.*

Proof. See [16] . □

Theorem 3.2. *Let $h : I^k \times G \to A$ be defined as $h(X, g) = g.X$. Then h has maximal rank.*

Proof. The manifold I^k is transversal to the orbits of the action of G on A. Then, if $X \in I^k$ we have,

$$T_X A = T_X I^k \oplus T_X \mathcal{O}_X \ , \text{ where } \mathcal{O}_X = G.X \, .$$

Now given $g \in G$

$$Tl_g(T_X A) = Tl_g(T_X I^k) \oplus Tl_g(T_X \mathcal{O}_X) = T_{g.X}\, g I^k \oplus T_{g.X} \mathcal{O}_X \ ,$$

moreover

$$T_{g.X} I^k \cap T_{g.X} \mathcal{O}_X = Tl_g(T_X I^k \cap T_X \mathcal{O}_X) = 0 \, .$$

Then, $dim(Tl_g(T_X A)) = dim A$ and h has maximal rank. □

Corollary 3.1. *Given $X \in A$ there exists $X_0 \in I^k$, $g_0 \in G$ and neighborhoods $\mathcal{U} \subset A$, $U \subset I^k$, $\mathcal{B} \subset G$ of X, X_0 and g_0 respectively such that*

$$h|_{U\times\mathcal{B}} : U \times \mathcal{B} \longrightarrow \mathcal{U} \text{ is a diffeomorphism} .$$

Then, there exist smooth sections $\eta : \mathcal{U} \to U \subset I^k$, $\sigma : \mathcal{U} \to \mathcal{B} \subset G$, such that

$$(h|_{U\times\mathcal{B}})^{-1} = (\eta, \sigma) \, . \tag{3.1}$$

Remarks.

1. If $\Gamma : J \in \Re \to Q$ is a regular curve of order k, then $C^k\Gamma \subset A$. Also given $X = C^k_p\Gamma$ the section σ allows us to define an immersion $\tilde{\sigma} = \sigma \circ C^k\Gamma$, of the curve

Γ in G. Then we can transport the invariant forms ω^α_β defined on G to the curve Γ, as follows

$$\tilde{\omega}^\alpha_\beta = \tilde{\sigma}^* \omega^\alpha_\beta \,.$$

2. Given $X \in I^k$ the forms ω^α_β, which vanish on G^k. Were projected onto $T_{X^k}\mathcal{O}^k$ using the map $\phi^k : g \in G \longmapsto g.X^k \in \mathcal{O}^k$.

For $v \in T_{X^k}\mathcal{O}_{X^k}$ we have defined $\tilde{\omega}^\alpha_\beta(v) = \omega^\alpha_\beta(V)$, where $V \in T_e G$ and $T\psi^j(V) = v$. The forms $\tilde{\omega}^\alpha_\beta$ can be extended, to the space $T_{X^k}(G.I^k)$.

3. Let $\Gamma \in Q$ be a regular curve of order k, and $p \in \Gamma$ a regular point of order k. Then, for $q \in \Gamma$ there are $g \in G$ and $X^k \in I^k$, such that $C^k_q\Gamma = g.X^k$. The left invariants forms $\tilde{\omega}^\alpha_\beta$ which vanish on G^k (with the exception of $\frac{\omega^0_0 - \overline{\omega^0_0}}{2i}$), were defined on $\Gamma'(0) \in T_pQ$ as

$$\tilde{\omega}^\alpha_\beta(\Gamma'(t)) = \tilde{\omega}^\alpha_\beta(\frac{d}{dt}|_{t=0}C^{k-1}\Gamma(t)) \,.$$

We can transport the form $Im\omega^\alpha_\beta$ to the curve Γ, using the section $\tilde{\sigma}$. Then we define

$$\sigma^* Im\omega^0_0(\Gamma'(t)) = i\tau_n(t)\tilde{\sigma}^*\omega^n_0(\Gamma'(t)), \quad \tau_n(t) \in \Re \,.$$

4. The Structure equations

Let

$$e_\alpha : G \to C^{n+1}, \qquad \text{given by} \qquad e_\alpha(g_0, \cdots, g_n) = g_\alpha \,,$$

where $g = (g^\alpha_\beta) = (g_0, \cdots, g_n)$. Then, we have $de_\alpha = \sum \omega^\alpha_\beta e_\beta$.
If $\Gamma : J \to Q$ is a regular curve tangent to D, we say that Γ is *parametrized by arc length* if $u^1_0(\Gamma'(t)) = 1$, for $t \in J$. A parametrization of Γ by arc length is given by

$$s(t) = \int_{t_0}^t u^1_0(\Gamma'(t))dt \,.$$

Given a regular curve parametrized by arc length , we have,
for $n = 2$:

$$\begin{array}{rcllll}
de_0 & = & & & dse_1 & \\
de_1 & = & dse_0 & & & + \quad idse_2 \\
de_2 & = & \nu_5 de_0 & + & idse_1 & \quad ,
\end{array}$$

where $\nu_5 = \frac{\sigma^* Im\tilde{\omega}^2_0}{\sigma^*\tilde{u}^1_0}$.

For $3 \leq n \leq 6$ the structure equations are given by the matrix,

$$\begin{pmatrix}
i\nu_5 ds & ds & & & & & +ids & ids \\
ids & i\delta_4 ds & & & & & & \\
\rho_3 ds & & i\delta_5 ds & -\rho_4 ds & & & & \\
& & \rho_4 ds & & & & & \\
& & & i\delta_6 ds & -\delta_5 ds & & & \\
& & & & & & & +i\nu_6 ds \\
& & & \rho_5 ds & +i\rho_7 ds & -\rho_6 ds & & \\
\mu_5 ds & ds & -i\rho_3 ds & & & & &
\end{pmatrix}$$

where, ν_5, τ_α, ρ_α are the invariants of the orbits defined in paragraph and,

$$\nu_6 = \frac{\sigma^* Im\tilde{\omega}_0^0}{\sigma^*\tilde{u}_0^1}, \qquad \delta_4 = \nu_5 - \tau_4, \qquad \delta_\alpha = \delta_{\alpha-1} - \tau_\alpha.$$

For $n \geq 7$ the structure equations are given by the matrix,

$$\begin{pmatrix}
i\nu_7 ds & ds & & & & & & & ids \\
ids & i\nu_7 ds & & & & & & & \\
\rho_3 ds & & i\delta_4 ds & -\rho ds & & & & & \\
& & \rho_4 ds & \cdot & \cdot & & & & \\
& & & \cdot & \cdot & \cdot & & & \\
& & & & \cdot & \cdot & -\rho_{n-1} ds & & \\
& & & & & \rho_{n-1} ds & i\rho_{n-1} ds & -\nu_{n-1} ds & \\
& & & & & & \nu_{n-1} ds & i\epsilon ds & \\
ds & & & & & & & & i\nu_7 ds
\end{pmatrix}$$

where

$$\tau_k = \frac{\sigma^* Im\tilde{\omega}_0^{k-2}}{\sigma^*\tilde{u}_0^1} \qquad \nu_{n-1} = \frac{\sigma^*\tilde{\omega}_0^{n-2}}{\sigma^*\tilde{u}_0^1} \qquad \tau_n = \frac{\sigma^* Im\tilde{\omega}_0^0}{\sigma^*\tilde{u}_0^1}.$$

$$\delta_4 = \nu_7 - \tau_4 \qquad \delta_{\alpha_5} = \delta_{\alpha_5 - 1} - \tau_{\alpha_5} \qquad \epsilon = -3\delta_7 - \sum \delta_{\alpha_4}$$

Remark. In D_{p_0} we have a complex structure defined by the operator

$$I : v \in D_{p_0} \mapsto iv \in D_{p_0}\,,$$

and the Levi form defined by $(\alpha(v,v))^2 = \sum \overline{\omega_0^\gamma}(v)\omega_0^\gamma(v)$.
For regular curves tangent to D we have

$$\omega_0^\gamma = 0, \qquad \omega_0^1 = u_0^1, \qquad 2 \leq \gamma \leq n-1\,.$$

The following Theorem about Lie groups, and the structure equations defined above, allow us to prove the following Theorem of equivalence of curves.

Theorem 3.3. *Let G be a n-dimensional Lie group and $\omega^1, \cdots, \omega^n$ a basis of right invariants forms of G. Let $S_1, S_2 \in G$ be two m-dimensional connected submanifolds, with $n < m$. Then, there exists an element $g \in G$ such that $R_g(S_1) = S_2$ if and only if there exists a diffeomorphism $\psi : S_1 \to S_2$, which preserves the forms ω^j, $\psi^*(\omega^j|s_1) = \omega^j|S_2$.*

Proof. See [7].

Theorem 3.4. *Let $\Gamma^1, \Gamma^2 \subset Q$ be two regular curves of order k, transverses to D, parametrized by length arc. Let $p_1 \in \Gamma^1$, $p_2 \in \Gamma^2$. Then there exists an element $g \in G$ such that locally, $l_g(\Gamma^1) = \Gamma^2$, $g.p_1 = p_2$ if and only if there exists a local diffeomorphism*

$$\psi : \Gamma_1 \to \Gamma_2, \qquad \text{such that} \quad \begin{array}{ll} \tau^1_\alpha \circ \psi = \tau^2_\alpha, & \mu^1_\alpha \circ \psi = \mu^2_\alpha, \\ \rho^1_\alpha \circ \psi = \rho^2_\alpha, & \nu^1_\alpha \circ \psi = \nu^2_\alpha, \end{array}$$

where $\tau^j_\alpha, \mu^j_\alpha, \rho^j_\alpha, \nu^j_\alpha, \quad j = 1, 2$, are the invariants associated to Γ^j.

Proof. Let $\tilde{\sigma}^1, \tilde{\sigma}^2$ be the local sections of Γ^1, Γ^2 respectively in G, defined by the local diffeomorphism given in (3.1). These sections define two connected submanifolds $\tilde{\Gamma}^1, \tilde{\Gamma}^2 \in G$, and Γ^1, Γ^2 are locally equivalent if and only if $\tilde{\Gamma}^1, \tilde{\Gamma}^2$ are locally equivalent, by the following commutative diagram

$$\begin{array}{ccccccc} J^1 & \xrightarrow{\gamma^1} & \Gamma^1 \subset Q & \xrightarrow{i^k} & C^k Q & \xrightarrow{\sigma^1} & G \\ & & \Big\downarrow l_g & & & & \Big\downarrow l_g. \\ J^2 & \xrightarrow{\gamma^2} & \Gamma^2 \subset Q & \xrightarrow{i^k} & C^k Q & \xrightarrow{\sigma^2} & G \end{array}$$

Using Theorem 3.3, we have that $\tilde{\Gamma}^1 = \sigma^1 \circ i^k \circ \gamma^1$, and $\tilde{\Gamma}^2 = \sigma^2 \circ i^k \circ \gamma^2$, are equivalents if and only if there exists a local diffeomorphism ψ which preserves the forms ω^α. Since the curves are parametrized by arc length, we have using the structure equations that the forms are preserved if and only if the invariants $\tau^j_\alpha, \mu^j_\alpha, \rho^j_\alpha, \nu^j_\alpha, \quad j = 1, 2$, are preserved. $\square$

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Departamento de Matemática, Facultad de Ciencias
Universidad Central de Venezuela
Caracas, VENEZUELA
E-mail: yvillarr@euler.ciens.ucv.ve