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LOWER-DIMENSIONAL DECOMPOSITIONS USING COMPLEX VARIABLES

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Abstract. The purpose of the present paper is to represent non-holomorphic functions depending on one or several complex variables by holomorphic and anti-holomorphic functions depending on only one complex variable. Similarly as in the case of functions of real variables, the obtained criteria can also be interpreted as conditions for the solvability of functional equations.

1. Statement of the problem

A function $h = h(x, y)$ depending on two real variables x and y can be represented in the form

$$h(x,y) = \sum_{k=1}^{n} f_k(x) g_k(y) \tag{1}$$

where the f_k and the g_k depend only on one real variable x and y resp. if and only if the determinant

$$D^n_{xy} h = \begin{vmatrix} h & \partial_x h & \partial_x^2 h & \dots & \partial_x^n h \\ \partial_y h & \partial_x \partial_y h & \partial_x^2 \partial_y h & \dots & \partial_x^n \partial_y h \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \partial_y^n h & \partial_x \partial_y^n h & \partial_x^2 \partial_y^n h & \dots & \partial_x^n \partial_y^n h \end{vmatrix}$$

vanishes identically. A complete and correct version of this statement was first given by F. Neuman in his papers [7, 8], while the survey article [10] written by F. Neuman and Th. M. Rassias contains historical comments and sketches present trends. The present state of the the theory of decomposition (including

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the decomposition of matrix valued functions and decompositions in more than two variables) is summarized in Th. M. Rassias's and J. Šimša's book [11].

Using the concept of partial complex derivatives, in the sequel we are going to discuss some contributions of Complex Analysis to the problem under consideration.

2. The basic lemmas

Suppose z and ζ are complex variables and the function $h = h(z, \zeta)$ is defined in the bicylinder $\Omega_z \times \Omega_\zeta$ where Ω_z and Ω_ζ are domains in the z- and in the ζ-plane resp. Suppose, further, that h depends holomorphically on both complex variables. Define the determinant $D^n_{z\zeta} h$ by replacing the differentiations with respect to x and y in $D^n_{xy} h$ by the ordinary complex differentiations with respect to z and ζ resp. Then the following statement is true:

Lemma 1. *In $\Omega_z \times \Omega_\zeta$ the function $h = h(z, \zeta)$ can be represented in the form*

$$h(z, \zeta) = \sum_{k=1}^{n} f_k(z) g_k(\zeta) \tag{2}$$

with linearly independent f_k and g_k (defined in Ω_z and Ω_ζ resp.) if and only if $D^n_{z\zeta} h \equiv 0$ and $D^{n-1}_{z\zeta} h \not\equiv 0$.

This statement can be proved by repeating the arguments of F. Neuman's proof of Theorem 1 in his paper [8]. One has to take into consideration, only, that one has to solve an ordinary differential equation in the complex domain of order n instead of an ordinary differential equation in one real variable. As in the real case, its solutions are linear combinations of n linearly independent solutions (see, for instance, H. Herold's book [5]) whose coefficients depend of the second variable.

Introduce complex differentiations ∂_z and $\partial_{\overline{z}}$ by

$$\begin{aligned} \partial_z &= \frac{1}{2}(\partial_x - i\partial_y) \\ \partial_{\overline{z}} &= \frac{1}{2}(\partial_x + i\partial_y) \end{aligned}$$

where $z = x + iy$. Suppose $h = h(x, y) = h(z)$ is a (real or complex valued) function depending n times continuously differentiable on x and y defined in the domain Ω of the $z = x + iy$-plane without necessarily being holomorphic in z. Applying F. Neuman's arguments to $D^n_{\overline{z}z} h$ instead to $D^n_{xy} h$, one gets the following statement:

Lemma 2. *Suppose $D^n_{\overline{z}z} h \equiv 0$ and $D^{n-1}_{\overline{z}z} h \not\equiv 0$ in the domain Ω of the complex plane. Then h satisfies an (ordinary) differential equation of order n with respect to the operator ∂_z in Ω whose coefficients are holomophic functions in Ω.*

Indeed, the derivatives with respect to $\overline{z}$ of the coefficients of the linear combinations mentioned above have to vanish identically, i.e., these coefficients are holomorphic.

Remark 1. Concerning all possible representations of form (1) the same statement can be made as in the case of real variables (see F. Neuman's papers [7, 8]).

Remark 2. In view of Hartogs's Continuity Theorem (cf., for instance, L. Hörmander's book [6]) a function $h = h(z, \zeta)$ is a holomorphic function in (z, ζ) if only h is partially holomorphic in z and in ζ as well. In view of the complex version of the Weyl Lemma (see, for instance, I. N. Vekua's book [17] or the booklet [14]) for the holomorphy in one complex variable it is sufficient that the function is integrable with respect to the variable under consideration and that the Cauchy-Riemann system is satisfied in the distributional sense.

3. Preliminaries from Complex Analysis

In order to solve the differential equation of Lemma 2, one needs a certain amount of Complex Analysis going beyond solving ordinary differential equations in the complex domain. Note, first, that the differential operator ∂_z appearing in the differential equation of Lemma 2 is defined for non-holomorphic functions, too. Consequently, the differential equation under consideration has non-holomorphic solutions, in general[1].

Solutions $w = \lambda(z)$ of the complex differential equation

$$\partial_{\overline{z}}^n w = 0$$

(where $n \geq 2$) are called poly-analytic functions[2]. In a given domain Ω they can be represented in the form

$$\lambda(z) = \sum_{j=0}^{n-1} \alpha_j(z)\overline{z}^j \tag{3}$$

where the $\alpha_j(z)$ are holomorphic in z. In view of Weyl's Lemma this represention formula holds if only λ and its $n-1$ first derivatives $\partial_{\overline{z}}\lambda, \ldots, \partial_{\overline{z}}^{n-1}\lambda$ are integrable.

Differentiating (3) $(n-1)$ times with respect to $\overline{z}$, one gets the representation formula

$$\alpha_{n-1} = \frac{1}{(n-1)!}\partial_{\overline{z}}^{n-1}\lambda$$

for the leading coefficient α_{n-1}. The remaining coefficients can be calculated by the recursion formula

$$\alpha_k = \frac{1}{k!}\partial_{\overline{z}}^k\left(\lambda - \alpha_{n-1}\overline{z}^{n-1} \cdots - \alpha_{k+1}\overline{z}^{k+1}\right).$$

[1]Non-holomorphic solutions of ordinary differential equations in the complex domain have been discussed already in the paper [15].

[2]Cf. M. B. Balk's book [1].

Replacing $\overline{z}$ by an independent variable ζ, one gets an extension

$$\Lambda(z,\zeta)=\sum_{j=0}^{n-1}\alpha_j(z)\zeta^j$$

which is holomorphic in z and ζ in the fundamental domain[3] $\Omega\times \mathbb{C}$. Differentiating k times with respect to ζ and substituting $\zeta=0$, one gets another representation

$$\alpha_k(z)=\frac{1}{k!}\partial_\zeta^k\Lambda(z,0)$$

of the holomorphic coefficients of a poly-analytic function. The given $\lambda(z)$ is connected with $\Lambda(z,\zeta)$ by the relation

$$\lambda(z)=\Lambda(z,\overline{z}).$$

Let Ω_1 and Ω_2 be domains in the z_1- and z_2-plane resp. Then a (complex valued) function $h=h(z_1,z_2)$ defined in $\Omega_1\times\Omega_2$ is called poly-analytic if it is a solution of a system of differential equations of type

$$\partial_{\overline{z}_1}^{n_1}h=0,\quad \partial_{\overline{z}_2}^{n_2}h=0 \tag{4}$$

Such a function can be represented in the form

$$h(z_1,z_2)=\sum_{\nu_1=0}^{n_1-1}\sum_{\nu_2=0}^{n_2-1}h_{\nu_1\nu_2}(z_1,z_2)\overline{z}_1^{\nu_1}\overline{z}_2^{\nu_2}$$

where the $h_{\nu_1\nu_2}$ are holomorphic functions in z_1 and z_2. Again, the coefficients $h_{\nu_1\nu_2}$ can be represented by the derivatives of h. Similarly, the $h_{\nu_1\nu_2}$ can also be represented by the derivatives of the holomorphic extension

$$H(z_1,z_2,\zeta_1,\zeta_2)=\sum_{\nu_1=0}^{n_1-1}\sum_{\nu_2=0}^{n_2-1}h_{\nu_1\nu_2}(z_1,z_2)\zeta_1^{\nu_1}\zeta_2^{\nu_2}.$$

4. Decompositions by holomorphic factors in the complex plane

Since a holomorphic function of a complex variable z can be interpreted as a function depending on one variable only, it makes sense to ask under which conditions a (real- or complex-valued) non-holomorphic function $h=h(z)$ can be represented by finitely many holomorphic functions. For instance, the real valued function h defined by

$$h(x,y)=x^2+y^2+\sum_{k=0}^{\infty}\left(2x-x^2-y^2\right)^k$$

in the unit disk (i.e., $x^2+y^2<1$) can be represented in the form

$$h=f_1\overline{f_1}+f_2\overline{f_2}$$

[3] Concerning the concept of a fundamental domain see, for instance, I. N. Vekua's book [16].

where

$$f_1(z) = z \quad \text{and} \quad f_2(z) = \frac{1}{1-z}.$$

Using the results of the previous section, we are now in a position to solve the differential equation mentioned in Lemma 2.

Since ∂_z is an elliptic operator and since the differential equation under consideration has holomorphic coefficients, its solutions have (local) power series representations in x and y (cf., for instance, L. Schwartz [12]). Without any loss of generality consider a neighbourhood of $z = 0$, and rewrite the desired solution in the form

$$\sum_{j=0}^{\infty} \alpha_j(z)\overline{z}^j$$

where the $\alpha_j(z)$ are power series in z, i.e., they are holomorphic in z. Note that $\partial_z \overline{z}^j = 0$, i.e., the $\overline{z}^j$ are anti-holomorphic. Moreover, the $\overline{z}^j$ are linearly independent. Consequently, all of the coefficients $\alpha_j(z)$ have to satisfy the given ordinary complex differential equation. Since the $\alpha_j(z)$ are holomorphic solutions, they are linear combinations

$$\sum_{k=1}^{n} c_k^{(j)} g_k(z)$$

of n linearly independent (holomorphic) solutions $g_k(z)$. Again rearranging the power series, one gets, finally,

$$\begin{aligned} h(z) &= \sum_{j=0}^{\infty} \left(\sum_{k=1}^{n} c_k^{(j)} g_k(z) \right) \overline{z}^j \\ &= \sum_{k=1}^{n} g_k(z)\overline{f_k(z)} \end{aligned}$$

where

$$f_k(z) = \sum_{j=0}^{\infty} \overline{\left(c_k^{(j)}\right)} z^j.$$

Hence the following decomposition theorem has been proved:

Theorem 1. *Suppose $D^n_{\overline{z}z} h \equiv 0$, while $D^{n-1}_{\overline{z}z} h \neq 0$ in the domain Ω of the z-plane. Then h can be represented in the form*

$$h(z) = \sum_{k=1}^{n} g_k(z)\overline{f_k(z)} \tag{5}$$

where the g_k and the f_k are holomorphic functions in Ω.

Remark 1. Since the solution of an ordinary differential equation exists globally in case the differential equation is linear, the function h can be decomposed globally, too.

Remark 2. If h is of form (5), then $D^{n}_{\overline{z}z} h \equiv 0$ necessarily. Indeed, (5) implies that h and its derivatives with respect to z (up to the order n) are linear combinations of the n rows

$$(\overline{g}_k \quad \partial_{\overline{z}}\overline{g}_k \quad \cdots \quad \partial^{n}_{\overline{z}}\overline{g}_k)\,,$$

$k = 1, ..., n$. Therefore, $D^{n}_{\overline{z}z} h$ has (as determinant with $n+1$ rows) to be equal to zero at each point z.

5. A decomposition theorem for functions depending on four real variables

Let h be a (real or complex valued) function depending on four real variables x_1, y_1, x_2, y_2. Introduce two complex variables $z_j = x_j + iy_j$, $j = 1, 2$. Suppose $h = h(z_1, z_2)$ is a solution of the system (4), i.e., h is a poly-analytic function in two complex variables z_1, z_2. Then the natural holomorphic extension $H(z_1, z_2, \zeta_1, \zeta_2)$ is defined in the corresponding fundamental domain in $\mathbb{C}^4$. Moreover, h is a polynomial

$$h(z_1, z_2) = \sum_{\nu_1=0}^{n_1-1} \sum_{\nu_2=0}^{n_2-1} \frac{1}{\nu_1!\nu_2!} \partial^{\nu_1}_{\zeta_1} \partial^{\nu_2}_{\zeta_2} H(z_1, z_2, 0, 0) \overline{z}_1^{\nu_1} \overline{z}_2^{\nu_2}$$

in $\overline{z}_1$ and $\overline{z}_2$ whose coefficients are holomorphic in z_1 and z_2. Applying Lemma 1 to these coefficients, the following statement has been proved:

Theorem 2. *Suppose h is a solution of the system*

$$\partial^{n_1}_{\overline{z}_1} h = 0, \quad \partial^{n_2}_{\overline{z}_2} h = 0.$$

Suppose, further, that for $0 \leq \nu_1 \leq n_1$ and $0 \leq \nu_2 \leq n_2$ there exist integers $m_{\nu_1\nu_2}$ such that

$$D^{m_{\nu_1\nu_2}}_{z_1 z_2} \partial^{\nu_1}_{\zeta_1} \partial^{\nu_2}_{\zeta_2} H(z_1, z_2, 0, 0) \equiv 0$$

where $H(z_1, z_2, \zeta_1, \zeta_2)$ is the natural holomorphic extension of $h(z_1, z_2)$. Then h possesses a finite decomposition

$$h(z_1, z_2) = \sum_{\nu_1, \nu_2, k} f_k^{(\nu_1,\, \nu_2)}(z_1) g_k^{(\nu_1,\, \nu_2)}(z_2) \overline{z}_1^{\nu_1} \overline{z}_2^{\nu_2}$$

where the $f_k^{(\nu_1,\, \nu_2)}$ and the $g_k^{(\nu_1,\, \nu_2)}$, $1 \leq k \leq m_{\nu_1\nu_2}$, are holomorphic in z_1 and z_2 respectively.

6. Concluding remarks

Remark 1. The obtained theorems lead to conditions for the solvability of functional equations, too. For instance, in view of Theorem 1 the condition $D^n_{\overline{z}z}h \equiv 0$ in Ω implies that to a given h defined in Ω there exist holomorphic solutions f_k and g_k, $k = 1, ..., n$ of the functional equation (5).

Remark 2. The corollary to the proof of Theorem 2.1 in H. Gauchman's and L. A. Rubel's paper [4] states that in a decomposition (1) the functions f_k and g_k have to be power series in x and y resp. provided the function h is supposed to be a power series in x, y (at least locally). Of course, a real-analytic function (in x and y) is a power series in z and $\overline{z}$, too, but not every power series in z and $\overline{z}$ can be represented in the form (5). This shows that the above Theorem 1 goes further than H. Gauchman's and L. A. Rubel's result on real analyticity does (cf. also Theorem 2.1.4 in Th. M. Rassias's and J. Šimša's book [11]). - Replacing the real variables x and y in H. Gauchman's and L. A. Rubel's corollary under consideration by two independent complex variables, one gets, finally, a factorization theorem in four real variables.

Remark 3. As already shown in J. Šimša' paper [13], factorizations of matrix-valued functions can be reduced not only to factorizations of their entries, but also they can be described by matrix operations. Consider, for instance, a matrix valued function $h = h(z)$ defined in Ω whose entries are supposed to be real-analytic functions, i.e., the entries of $h(z)$ are locally representable as power series in $(z - z_0)$ and $(\overline{z} - \overline{z}_0)$. We are looking for factorizations

$$h(z) = f(z)g(\overline{z}) \tag{6}$$

where the entries of the matrices f and g are (locally) power series in their variables. Let $H = H(z, \zeta)$ be the natural holomorphic extension of $h(z)$ defined in $\Omega \times \overline{\Omega}$, i.e., $h(z) = H(z, \overline{z})$ in Ω. Then (6) holds if and only if

$$H(z, \zeta) = f(z)g(\zeta) \tag{7}$$

in $\Omega \times \overline{\Omega}$. Repeating the arguments in the proof of Theorem 1 in J. Šimša's paper [13], one can prove that (7) is true if and only if

$$H(z, \zeta) = H(z, \zeta_0)H^{-1}(z_0, \zeta_0)H(z_0, \zeta)$$

for each $z \in \Omega$ and $\zeta \in \overline{\Omega}$ where $z_0 \in \Omega$ and $\zeta_0 \in \overline{\Omega}$ are arbitrarily chosen.

Remark 4. Similar generalizations of the above approach are possible also for decompositions with two multi-dimensional (complex) variables and with several single (complex) variables (cf. M. Čadek's and J. Šimša's papers [3] and [2] resp.; see also F. Neuman's paper [9]).

Remark 5. The considerations on minimal decompositions (Section 2.3 in Th. M. Rassias's and J. Šimša's book [11]) are, of course, also applicable to decompositions of functions in complex variables.

Remark 6. Notice that in the case of more than two real variables some sufficient and necessary conditions for the existence of decompositions are given by a system of partial differential equations in real variables (see, e.g., the Theorems 3.4.4 and 3.4.6 in the book book [11] of Th. M. Rassias and J. Šimša). The conditions formulated in Theorem 2 of the present paper are a system of partial complex differential equations.

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