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v-PROJECTIVE SYMMETRIES OF FIBERED MANIFOLDS

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Abstract. We prove that the set of the v-projective symmetries is a Lie algebra.

Introduction

Let $\pi : E \to B$ be a fibered manifold and let us suppose that every leaf is endowed with a symmetric covariant derivative. In other words, this means that there is defined the morphism

$$\nabla : V(E) \times V(E) \to V(E) \qquad (U, V) \to \nabla(U, V)$$

which satisfies the following properties:

(a) it is biadditive
(b) $\nabla(f \cdot U, V) = f \cdot \nabla(U, V)$
(c) $\nabla(U, fV) = f \cdot \nabla(U, V) + (Uf) \cdot \mathrm{V}$
(d) $[U, V] = \nabla(U, V) - \nabla(V, U)\,, \qquad f \in F(E), \quad U, V \in V(E)$

where $V(E)$ represents the Lie algebra of the vertical vector fields.

A v-projective symmetry is a projectable vector field X with the property in which every diffeomorphism φ_t of its one – parametric group is a projective map between leaves. The purpose of this note is to investigate the set of the v-projective symmetries. We prove that the set of the v-projective symmetries is a Lie algebra.

In the second paragraph we prove some results concerning with v-symmetries with respect to the Levi-Civita connection induced by a vertical metric. In a further paper we shall study the v-projective symmetries in the context of Riemannian submersion and Riemannian foliations.

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1. The set of the v-symmetries

We use the notation:

– $V(E)$ is the Lie algebra of the vertical vector fields of π,
– $P(E)$ is the Lie algebra of the projectable vector fields of π. we recall that $V(E)$ is an ideal of $P(E)$ and that a projectable vector field has the property in which $\varphi_t(E_b) = E_{\varphi_t(b)}$, where $\{\varphi_t : E \to E\,,\ t \in R\}$ represents the flow of X.

Definition 1. The vector field $X \in P(E)$ is called v-symmetry of ∇ if and only if there exists a family $\omega : R \times V(E) \to R$ of 1-forms which satisfies the relation

$$\varphi_t \cdot \nabla_U V = \nabla_{U^t} V^t + \omega(t, U) \cdot V^t + \omega(t, V) \cdot U^t \tag{1}$$

where we denote $U^t = \varphi_t \cdot U$ and we also denote φ_t for the differential of φ_t.

If $\omega = 0$ we call X a v-affine symmetry.
The result of the paragraph is the following:

Theorem 1. *The set of v-projective symmetries of ∇ is a Lie algebra.*

In order to prove the theorem we need the following result:

Proposition 1. *The vector field $X \in V(E)$ is a v-projective symmetry if and only if there exists a form $\omega_0 : V(E) \to R$ such that the relation*

$$[X, \nabla_U V] + \omega_0(U) \cdot V + \omega_0(V) \cdot U = \nabla_{[X,U]} V + \nabla_U [X, V] \tag{2}$$

holds good, for every $U, V \in V(E)$.

Proof. Let us suppose that X is a v-projective symmetry: one obtains

$$\lim_{t\to 0}(1/t)(\nabla_U V - \varphi_t \cdot \nabla_{U^t} V^t) = \{\lim_{t\to 0}(1/t)\omega(t, U)\} \cdot V + \{\lim_{t\to 0}(1/t)\omega(t, V)\} \cdot U\,.$$

The first limit is equal with

$$-[X, \nabla_U V] + \nabla_{[X,U]} V + \nabla_U [X, V]\,.$$

Let us put

$$\lim_{t\to 0}(1/t)\omega(t, U) = \omega_0(U) \qquad \text{for every} \quad U \in V(E)\,. \tag{*}$$

Hence, one obtains (2).
Let us suppose that the relation (2) holds good. We define the family $\omega : R \times V(E) \to R$ of 1-forms as follows:

$$\omega(t, U) = \int_0^t \omega_0(\varphi_t \cdot U)\, dt\,, \quad U \in V(E)\,.$$

It is easy to check that $\omega(t,U)$ is the unique solution of the differential equation

$$(**)\qquad \frac{d\omega}{dt}(t,U)=\omega_0(\varphi_t\cdot U)=\omega_0(U^t)$$
$$\omega(0,U)=0\,.$$

for every $U\in V(E)$.

If we put $U=V$ in the relation (2), one obtains

$$(3)\qquad [X,\nabla_U U]+2\omega_0(U)=\nabla_{[X,U]}U+\nabla_U[X,U]\quad\text{for every}\quad U\in V(E)\,.$$

We prove that, if (3) holds, then the flow has the property in which

$$\varphi_t\cdot\nabla_U U=\nabla_{u^t}U^t+2\omega_0(t,U)\cdot U^t\,,$$

hence the map φ_t is a projective map between leaves.

Let us put $V(s)=\nabla_U U_{,e}-(\varphi_{-s})_{,\varphi_s(e)}\cdot\nabla_{U^s}U^s\in T_eE_b$, where $e\in E_b$, $\pi(e)=b$. Then one obtains

$$\begin{aligned}\frac{dV}{ds}(s)&=\lim_{s\to 0}(1/s)\cdot\left(\nabla_U U_{,e}-(\varphi_{-s})_{,\varphi_s(e)}\cdot\nabla_{U^s}U^s\right)=\\&=\varphi_{-s}\cdot\left\{-[X,\nabla_{U^s}U^s]+\nabla_{U^s}[X,U^s]+\nabla_{[X,U^s]}U^s\right\}=\\&=\varphi_{-s}\cdot\{2\omega_0(U^s)\cdot U^s\}=2\omega_0(U^s)\cdot U_{,e}\end{aligned}$$

where we put $U^s=\varphi_s\cdot U$. So the vector field $V(s)$ is the solution of the equation

$$\frac{dV}{ds}(s)=2\omega_0(U^s)\cdot U_{,e}$$
$$V(0)=0$$

Taking into account by (**), one deduces that $V(s)=2\omega(s,U)\cdot U_{,e}$. Consequently, the diffeomorphism φ_t is a projective map between leaves. This concludes the proof. □

We notice that the relation (3) is a necessary and sufficient condition for a projective vector field X to be v-projective symmetry. Indeed, if we put in (3) the sum $U+V$ and if we take into account by the fact that the torsion of ∇ is zero, we obtain the relation (2). But, in general, the relations (2) and (3) are not equivalent.

Proof of the Theorem 1. Let X and Y be two v-projective symmetries and let ω and η be their 1-forms respectively. We verify the relation (3) holds good for the bracket $[X,Y]$. Let us denote ω_0 and η_0 the derivatives as in the relation (*). One obtains

$$\begin{aligned}&L_{[X,Y]}\nabla_U U=L_XL_Y\nabla_U U-L_YL_X\nabla_U U=\\&=L_X(\nabla_{[Y,U]}U+\nabla_U[Y,U]-2\eta_0(U)\cdot)-L_Y(\nabla_{[X,U]}U+\nabla_U[X,U]-2\omega_0(U)\cdot U)\,.\end{aligned}$$

Taking into account by (2) we obtain the relation

$$L_{[X,Y]}\nabla_U U=\nabla_U\,[[X,Y],U]+\nabla_{[[X,Y],U]}U-2(L_X\eta_0-L_Y\omega_0)\,(U)\cdot U\,.$$

Because the relation (3) is fulfiled, we conclude the proof. □

2. The v-Weyl projective tensor field

In the theory of the projective transformations of connections the Weyl projective tensor plays an important role. In our case, the correspondent tensor will be defined on the vertical vectors only. So, the v-Weyl projective tensor field is the morphism $P : V(E) \times V(E) \times V(E) \to V(E)$ defined by the relation

$$\begin{aligned}(4)\quad P(U,V)W = R(U,V)W - S(V,W)\cdot U \\ + S(U,W)\cdot V + [S(U,V) - S(V,U)]\cdot W\end{aligned}$$

where

$$S(U,V) = \frac{1}{n^2-1}\left[\mathrm{Ric}\,(U,V) + n\mathrm{Ric}\,(V,U)\right] \tag{5}$$

where n represents the common dimension of the leaves (see [1] for details).

Proposition 2. *The vector field $X \in P(E)$ is a v-projective symmetry if and only if the relation*

$$L_X P = 0 \tag{6}$$

holds good.

Proof. It is well known the property of the Weyl projective tensor to be invariant with respect to projective transformations (see [2] for details). Furthermore, this property is a sufficient condition for a diffeomorphism to be projective. In our case, this leads to the relation $\varphi_t \cdot P(U,V)W = P(U^t,V^t)W^t$ which means that $(L_X P)(U,V)W = 0$ (see Corollary 3.7, p. 33, [5]). This concludes the proof. □

Let us suppose from now on that the fibered manifold π is endowed with a vertical metric $g : V(E) \times V(E) \to R$ and let us suppose that ∇ represents the Levi-Civita connection induced by the metric. Then the v-Weyl projective tensor has the form

$$P(U,V)W = R(U,V)W - \frac{1}{n-1}\,B(U,V)W \tag{7}$$

where

$$B(U,V)W = \mathrm{Ric}\,(V,W)\cdot U - \mathrm{Ric}\,(U,W)\cdot V \tag{8}$$

(see [1], p. 94, (1.25)).

Proposition 3. *Let us suppose that π is endowed with a vertical metric and let us suppose that X is a v-projective symmetry. Then we have:*

(a) *the family $\omega : R \times V(E) \to R$ is closed, i.e. $d\omega(t,U,V) = 0$ for every $t \in R$*
(b) *the relation*

$$(\nabla_U \omega_0)(V) = (L_X Ric)(U,V) \tag{9}$$

holds good, for every $U, V \in V(E)$.

Proof. Before to start we recall that the family ω and ω_0 are related by the relations (*) and (**). We use the relations (0.5), (0.6) and (0.7) from [3]. So one obtains the formula

$$(***)\quad \begin{aligned} R(U,V)W - \varphi_{-t}\cdot R(U^t,V^t)W^t &= (\nabla\omega)\,(t,U,W)\cdot V - (\nabla\omega)\,(t,V,W)\cdot U \\ &+((\nabla\omega)\,(t,U,V) - (\nabla\omega)\,(t,V,U))\cdot W \end{aligned}$$

where we put $(\nabla\omega)\,(t,U,V) = (\nabla_U\omega)\,(t,V) - \omega(t,U)\omega(t,V)$. Because of

$$\lim_{t\to 0}(1/t)\omega(t,U)\omega(t,V) = \omega_0(U)\omega(0,V) = 0$$

one obtains

$$\lim_{t\to 0}(1/t)\,(\nabla\omega\,(t,U,V)) = (\nabla_U\omega_0)\,(V)\,.$$

Passing through the limit in (***) we find

$$(10)\quad \begin{aligned} (L_X R)\,(U,V)W = (\nabla_V\omega_0)\,(W)\cdot U - (\nabla_U\omega_0)\,V - \\ ((\nabla_U\omega_0)\,(V) - (\nabla_V\omega_0)\,(U))\cdot W\,. \end{aligned}$$

But the relation (6) implies the equality

$$(****)\quad (L_X R)\,(U,V)W = \frac{1}{n-1}\,(L_X B)\,(U,V)W\,.$$

A straightforward computation gives

$$(L_X B)\,(U,V)W = (L_X \mathrm{Ric}\,)\,(V,W)\cdot U - (L_X\;\mathrm{Ric}\,)\,(U,W)\,,\cdot V\,.$$

Comparing the two members of the equality (****), one obtains the following relations:

$$(\nabla_U\omega_0)\,(V) = \frac{1}{n-1}(L_X\;\mathrm{Ric}\,)\,(U,V)\,,\quad \text{i.e. the relation (9)}$$

and

$$(\nabla_U\omega_0)\,(V) = (\nabla_V\omega_0)\,(U)\,.$$

From the second relation we deduce $d\omega_0(U,V) = 0$ and so we conclude the point (a). We notice that the point (a) reconfirm the proposition 1.7 and the corollary 1.8 from [3]. As a consequence of this result, it is easy to prove

Corollary 1. *If the fibered manifold π endowed with the vertical metric g, allows a v-affine symmetry, then the relations*

$$L_X R = 0\,, \qquad L_X\, Ric = 0 \tag{11}$$

hold good.

We would like to present, in the end, some very simple examples of v-projective symmetries. Let us consider the Euclidean space E^{N+1} and let $\pi_n : E^{N+1} \to R$, $n \in \{1, \ldots, N\}$ be the submersions defined by the relations

$$\pi_n(x_1, \ldots, x_{N+1}) = \sqrt{(x_1)^2 + \cdots + (x_n)^2}$$

which generate the concentric hyperspheres arround the origin in the case $n = N$ and the generalized cilinders in the case $n < N$. It is easy to see that the position vector field $\bar{r}$ is a v-affine symmetry of the fibered manifold π_N. We shall prove, in a further paper, that the above examples, together with the parallel hyperplanes, are the only fibered manifolds defined on the Euclidean space which allow v-projective symmetries.

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