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FIXED POINT THEORY FOR COMPACT PERTURBATIONS OF PSEUDOCONTRACTIVE MAPS

Donal O'Regan

Abstract. Some new fixed point results are established for mappings of the form $F_1 + F_2$ with F_2 compact and F_1 pseudocontractive.

1. Introduction

This paper presents two new fixed point theorems for the sum of two operators (for example a pseudocontractive plus a compact operator) between Banach spaces. First however we will establish some general nonlinear alternatives of Leray–Schauder type. These can be established using the degree theory of Browder [2]. However it is of interest to provide elementary proofs. We do so by using the topological transversality of Granas [9] (see [6,9,11,12] for an elementary proof of this result). We remark here that our results were motivated by work of Browder [2], Deimling [5], Furi and Pera [7], Granas [9] and Kirk and Schöneberg [10].

We next gather together some definitions and some well known facts. Let E be a Banach space and Ω_E the family of all bounded subsets of E. The *Kuratowskii measure of noncompactness* is the map $\alpha : \Omega_E \to [0, \infty)$ defined by

$$\alpha(X) = \inf \{\epsilon > 0 : X \subseteq \cup_{i=1}^n X_i \ \text{ and } \ diam\,(X_i) \le \epsilon\}; \ \text{ here } \ X \in \Omega_E.$$

Of course if $S, T \in \Omega_E$ then

(i) $\alpha(S) = 0$ iff $\overline{S}$ is compact
(ii) $\alpha(\overline{S}) = \alpha(S)$
(iii) if $S \subseteq T$ then $\alpha(S) \le \alpha(T)$
(iv) $\alpha(co\,(S)) = \alpha(S)$
(v) $\alpha(T + S) \le \alpha(T) + \alpha(S)$.

Let B_1 and B_2 be two Banach spaces and let $F : Y \subseteq B_1 \to B_2$ be continuous and map bounded sets into bounded sets. We call F a α–Lipschitzian map if F is continuous, bounded and there is a constant $k \ge 0$ with $\alpha(F(X)) \le k\alpha(X)$

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for all bounded sets $X \subseteq Y$. We call F a condensing map if F is α–Lipschitzian with $k=1$ and $\alpha(F(X)) < \alpha(X)$ for all bounded sets $X \subseteq Y$ with $\alpha(X) \neq 0$.

Let B be a real Banach space and let $B^\star$ denote the dual of B. Notice from the Hahn–Banach theorem that

$$\{x^\star \in B^\star : x^\star(x) = \|x\|^2, \|x^\star\| = \|x\|\} \neq \emptyset$$

for every $x \in B$. The mapping $F : B \to 2^{B^\star}$ defined by

$$F(x) = \{x^\star \in B^\star : x^\star(x) = \|x\|^2 = \|x^\star\|^2\}$$

is called the *duality map* [2,4] of B. By means of F, the semi inner product $(.\,,.)_+ : B \times B \to R$, is defined by

$$(x,y)_+ = \sup\{y^\star(x) : y^\star \in F(y)\}.$$

Let $\Omega \subseteq B$. A mapping $T : \Omega \to B$ is said to be

(i) *strongly accretive* if for some $c > 0$,

$$(T(x) - T(y), x - y)_+ \geq c\|x-y\|^2 \quad \text{for all} \quad x, y \in \Omega \tag{1.1}$$

(ii) *accretive* if

$$(T(x) - T(y), x - y)_+ \geq 0 \quad \text{for all} \quad x, y \in \Omega$$

(iii) *pseudocontractive* if $I - T$ is accretive.

We next state some well known results.

Theorem 1.1. [4]. *Let E be a real Banach space and $T : E \to E$ a continuous and strongly accretive map (i.e. (1.1) holds for some $c > 0$). Then T is a homeomorphism from E onto E. Also $T^{-1} : E \to E$ is a Lipschitz map with Lipschitz constant $\frac{1}{c}$.*

Theorem 1.2. $[5,17]$. *(Deimling's invariance of domain).*
Let $U \subseteq E$ (E a Banach space) be open and $T : U \to E$ a continuous and strongly accretive map. Then $T(U)$ is open.

Theorem 1.3. $[16]$. *Let B be a uniformly convex Banach space, Q a bounded, closed, convex subset of B and Ω an open set containing Q with dist $(Q,\, B/\Omega) > 0$. Suppose $T : \overline{\Omega} \to B$ is a continuous pseudocontractive mapping which sends bounded sets into bounded sets. Then $I - T$ is demiclosed on Q.*

Remark. A mapping $T : \Gamma \subseteq B \to B$ is called demiclosed on Γ if for every sequence $\{x_n\} \in \Gamma$ with $x_n \rightharpoonup x$ and $T(x_n) \to y$ as $n \to \infty$ we have $x \in \Gamma$ and $T(x) = y$; here $\rightharpoonup$ denotes weak convergence.

Next we state the topological transversality theorem of Granas [6,9,11,14]. Let E be a Banach space, C a closed convex subset of E and U an open subset of C. We call $N : \overline{U} \times [0,1] \to C$ a condensing map if N is continuous, bounded (i.e. $N(\overline{U} \times [0,1])$ is a subset of a bounded set in C), $\alpha(N(W)) \leq \alpha(\pi W)$ for all bounded sets W of $\overline{U} \times [0,1]$ and $\alpha(N(\Omega)) < \alpha(\pi\,\Omega)$ for all bounded non precompact subsets Ω of $\overline{U}\times[0,1]$; here $\pi:\overline{U}\times[0,1]\to\overline{U}$ is the natural projection. $K_{\partial U}(\overline{U},C)$ denotes the set of all condensing maps $H:\overline{U}\to C$ with $H(\overline{U})$ a subset of a bounded set in C and with H fixed point free on ∂U. A mapping $F\in K_{\partial U}(\overline{U},C)$ is *essential* if for every $H\in K_{\partial U}(\overline{U},C)$ which agrees with F on ∂U we have that H has a fixed point in U.

Theorem 1.4. [6, 9, 11, 14]. *Let U, C and E be as above. Assume $N:\overline{U}\times[0,1]\to C$ is a condensing map with the following conditions satisfied:*

$$N(u,\lambda)\neq u \ \text{ for all } \ u\in\partial U \ \text{ and } \ \lambda\in[0,1] \tag{1.2}$$

and

$$N(\,.\,,0) \ \text{ is essential on } \ U. \tag{1.3}$$

Then for each $\lambda\in[0,1]$ there exists at least one fixed point in U for $N(\,.\,,\lambda)$.

For convenience we rephrase theorem 1.4. Recall [6,9,11,14] two maps $F,G\in K_{\partial U}(\overline{U},C)$ are homotopic in $K_{\partial U}(\overline{U},C)$, written $F\cong G$ in $K_{\partial U}(\overline{U},C)$ if there is a condensing map $N:\overline{U}\times[0,1]\to C$ with $N_t(u)=N(u,t):\overline{U}\to C$ belonging to $K_{\partial U}(\overline{U},C)$ for each $t\in[0,1]$ and $N_0=F$, $N_1=G$.

Theorem 1.5. [6, 9, 11, 14]. *Let U,C and E be as above. Suppose F and G are two maps in $K_{\partial U}(\overline{U},C)$ such that $F\cong G$ in $K_{\partial U}(\overline{U},C)$. Then F is essential iff G is essential.*

Theorem 1.6. [6, 9, 11, 14]. *Let U,C and E be as above and let $u_0\in U$. Define $F:\overline{U}\to C$ by $F(u)=u_0$. Then the constant map $F\in K_{\partial U}(\overline{U},C)$ is essential.*

Theorem 1.4 is valid if the family of maps $N(\,.\,,\lambda)$, $\lambda\in[0,1]$ are defined on the same domain $\overline{U}$. However to prove our fixed point results in section 2 we need to have results for families of maps $N(\,.\,,\lambda)$, $\lambda\in[0,1]$ which may be defined on different domains. In fact it is easy to extend theorem 1.4 to this situation; this extension is due to Precup [16] if the maps are compact. However new arguments are needed if the mappings are condensing. We conclude the introduction by stating and proving such a result.

Let E be a Banach space and C a closed convex subset of E. Let $G\subseteq C\times[0,1]$ be open in $C\times[0,1]$. For any $\Omega\subseteq E\times[0,1]$ let $\Omega_\lambda=\{x\in E:(x,\lambda)\in\Omega\}$ denote the section of Ω at λ.

Theorem 1.7. *Let G, C and E be as above. Assume $N:\overline{G}\to C$ is a condensing map with*

$$N(x,\lambda)\neq x \ \text{ for all } \ (x,\lambda)\in\partial G. \tag{1.4}$$

In addition suppose there exists $p\in G_0$ with

$$(1-\mu)p+\mu N(x,0)\neq x \ \text{ for all } \ (x,0)\in\partial G,\ 0<\mu<1 \tag{1.5}$$

holding. Then for each $\lambda\in[0,1]$ there exists at least one fixed point in G_λ for $N(\,.\,,\lambda)$.

Proof. Let

$$N^\star : \overline{G} \times [0,1] \to C \times [0,1]$$

be given by

$$N^\star(x,\lambda,\mu) = (N(x,\lambda),\, \mu) \quad \text{for} \quad (x,\lambda) \in \overline{G} \quad \text{and} \quad \mu \in [0,1].$$

The idea is to apply theorem 1.4 with the Banach space $E \times R$ with norm $\|(x,t)\|_{E\times R} = \max\{\|x\|_E,\, |t|_R\}$, the convex set $C \times [0,1]$, the open set G, and the map $N^\star$. We <u>claim</u> that

$$N^\star : \overline{G} \times [0,1] \to C \times [0,1] \quad \text{is a condensing map} \tag{1.6}$$

that

$$N^\star(x,\lambda,\mu) \neq (x,\lambda) \quad \text{for all} \quad (x,\lambda) \in \partial G \quad \text{and} \quad \mu \in [0,1] \tag{1.7}$$

and that

$$N^\star(x,\lambda,0) = (N(x,\lambda),\, 0) \quad \text{is essential on} \quad G. \tag{1.8}$$

If (1.6), (1.7) and (1.8) are true then theorem 1.4 implies for each $\mu \in [0,1]$, there exists $(x,\lambda) \in G$ with

$$N^\star(x,\lambda,\mu) = (x,\lambda)$$

i.e. $N(x,\lambda) = x$ and $\mu = \lambda$. Thus $x \in G_\mu$ with $N(x,\mu) = x$ and we are finished.

It remains to prove (1.6), (1.7) and (1.8). We first show that $N^\star : \overline{G} \times [0,1] \to C \times [0,1]$ is a condensing map.

Remark. If $N : \overline{G} \to C$ is a compact map then clearly $N^\star : \overline{G}\times[0,1] \to C\times[0,1]$ is a compact map from Tychonoff's theorem and the fact that $N^\star(\overline{G} \times [0,1]) \subseteq N(\overline{G}) \times [0,1]$.

Fix $t \in [0,1]$. Let $N_t^\star : \overline{G} \to E \times \{t\}$ be given by $N_t^\star(x,\lambda) = (N(x,\lambda),\, t)$ for $(x,\lambda) \in \overline{G}$. We first show

$$N_t^\star : \overline{G} \to E \times \{t\} \quad \text{is a condensing map for each} \quad t \in [0,1]. \tag{1.9}$$

To see this fix $t \in [0,1]$ and let W be a bounded non precompact subset of $\overline{G}$. Then

$$\alpha(N_t^\star(W)) \leq \alpha(N(W) \times \{t\}) = \alpha(N(W)) < \alpha(W)$$

so (1.9) is true.

Remark. Note we used above the fact that $\alpha_E(\Omega) = \alpha_{E\times R}(\Omega \times \{t\})$ for any bounded set Ω in E; here $t \in [0,1]$ is fixed. To show this suppose $\alpha_E(\Omega) < \epsilon$; here $\epsilon > 0$. Then there exists subsets $\Omega_1, \ldots, \Omega_m$ of E with $\Omega \subseteq \cup_{i=1}^m \Omega_i$ and $diam\,(\Omega_i) \leq \epsilon$. Also

$$\Omega \times \{t\} \subseteq \cup_{i=1}^m \left(\Omega_i \times B_t\left(\frac{\epsilon}{2}\right)\right)$$

where $diam\,(\Omega_i \times B_t(\frac{\epsilon}{2})) \leq \epsilon$ (using the norm in $E \times R$); here $B_t(\frac{\epsilon}{2})$ is the ball with center t and radius $\frac{\epsilon}{2}$. Thus $\alpha_E(\Omega) < \epsilon$ implies $\alpha_{E\times R}(\Omega \times \{t\}) \leq \epsilon$ and so

$$\alpha_{E\times R}(\Omega \times \{t\}) \leq \alpha_E(\Omega) \tag{1.9a}$$

(there exists a sequence ϵ_n with $\epsilon_n \downarrow \alpha_E(\Omega)$ and since $\alpha_{E\times R}(\Omega \times \{t\}) \le \epsilon_n$ for all n we deduce (1.9a) immediately).

On the other hand suppose $\alpha_{E\times R}(\Omega \times \{t\}) < \epsilon$. Then there exist subsets $V_1, ..., V_m$ of E with $\Omega \times \{t\} \subseteq \cup_{i=1}^m V_i$ and $diam\,(V_i) \le \epsilon$. Thus

$$\Omega \subseteq \cup_{i=1}^m \pi\, V_i \quad \text{with} \quad diam\,(\pi\, V_i) \le \epsilon,$$

and so $\alpha_{E\times R}(\Omega \times \{t\}) < \epsilon$ implies $\alpha_E(\Omega) \le \epsilon$. Consequently

$$\alpha_E(\Omega) \le \alpha_{E\times R}(\Omega \times \{t\}). \tag{1.9b}$$

We now prove (1.6). Let W be a bounded non precompact subset of $\overline{G} \times [0,1]$. Now let $\epsilon(t) > 0$ be such that

$$\alpha(N_t^\star(\pi\, W)) < \alpha(\pi\, W) - 2\epsilon(t) \tag{1.10}$$

and let $V(t)$ be a neighborhood of t such that

(1.11)

$$\|N_t^\star(x,\lambda) - N_s^\star(x,\lambda)\| = \|(0, t-s)\| = |t-s| \le \epsilon(t) \quad \text{for all} \quad s \in V(t) \quad \text{and} \quad (x,\lambda) \in \pi\, W.$$

Remark. In (1.10) we used the fact that if W is a non precompact subset of $\overline{G} \times [0,1]$ then $\pi\, W$ is a non precompact subset of $\overline{G}$.

Also if $s, s_1 \in V(t)$ and $(u,\lambda), (u_1,\lambda_1) \in \pi\, W$ we have

$$\begin{aligned} N^\star(u,\lambda,s) - N^\star(u_1,\lambda_1,s_1) \quad &= [N^\star(u,\lambda,s) - N^\star(u,\lambda,t)] + [N^\star(u_1,\lambda_1,t) \\ &\quad -N^\star(u_1,\lambda_1,s_1)] + [N_t^\star(u,\lambda) - N_t^\star(u_1,\lambda_1)] \end{aligned}$$

and so (1.10) and (1.11) imply

$$\alpha(N^\star(\pi\, W \times V(t))) < \alpha(\pi\, W). \tag{1.12}$$

Now $\{V(t) : t \in [0,1]\}$ is an open cover of $[0,1]$ and since $[0,1]$ is compact we suppose

$$\{V(t_i),\, i = 1, ..., n\} \quad \text{is a finite covering of} \quad [0,1].$$

Now (1.12) together with properties of α imply

$$\begin{aligned} \alpha(N^\star(W)) &\le \alpha(N^\star(\pi\, W \times [0,1])) \\ &\le \max\,\{\alpha(N^\star(\pi\, W \times V(t_i))),\; i = 1,..,n\} < \alpha(\pi\, W) \end{aligned}$$

so (1.6) is true.

Remark. Another way of proving (1.6) is to first show that $\alpha_E(\pi\,\Omega) = \alpha_{E\times R}(\Omega)$ for any bounded subset Ω of $E \times [0,1]$; this follows from the second last remark and the fact that one can show $\alpha_{E\times R}(\Omega) = \alpha_E(\pi\,\Omega \times \{0\})$ (notice $\Omega \subseteq \pi\,\Omega \times \{0\} + \{0\} \times [0,1]$ so $\alpha_{E\times R}(\Omega) \le \alpha(\pi\, W \times \{0\})$ and the reverse inequality is also easy). Thus if W is a bounded non precompact subset of $\overline{G} \times [0,1]$, then

$$\alpha(N^\star(W)) \le \alpha(N(\pi\, W) \times [0,1]) = \alpha(N(\pi\, W)) < \alpha(\pi\, W).$$

Next we show (1.7) is satisfied. Suppose not i.e. suppose there exists $(x_1, \lambda_1) \in \partial G$ and $\mu_1 \in [0,1]$ with

$$(x_1, \lambda_1) = N^\star(x_1, \lambda_1, \mu_1) = (N(x_1, \lambda_1),\, \mu_1).$$

Then $\mu_1 = \lambda_1$ and $N(x_1, \lambda_1) = x_1$ with $(x_1, \lambda_1) \in \partial G$. This contradicts (1.4). Consequenty (1.7) is true. It remains to show (1.8).

The idea is to apply theorem's 1.5 and 1.6. Let the homotopy $H : \overline{G} \times [0,1] \to C \times [0,1]$ be given by

$$H(x, \lambda, \mu) = ((1-\mu)p + \mu N(x, \lambda),\, 0) \quad \text{for} \quad (x, \lambda) \in \overline{G} \text{ and } 0 \le \mu \le 1.$$

First notice the map $H(x, \lambda, 0) = (p, 0)$ is essential on G by theorem 1.6 (note $(p, 0) \in G$ since $p \in G_0$). Next we show $H : \overline{G} \times [0,1] \to C \times [0,1]$ is a condensing map. To see this let W be a bounded non precompact subset of $\overline{G} \times [0,1]$. Then

$$\begin{aligned} \alpha(H(W)) &\le \alpha(\, co\,(N(\pi\, W) \cup \{p\}) \times \{0\}) \\ &= \alpha(\, co\,(N(\pi\, W) \cup \{p\})) = \alpha(N(\pi\, W)) < \alpha(\pi\, W). \end{aligned}$$

Before we apply theorem 1.5 we need to show that $H_\mu : \overline{G} \to C \times [0,1]$ belongs to $K_{\partial G}(\overline{G}, C \times [0,1])$ for each $\mu \in [0,1]$. Suppose not i.e. suppose there exists $(x, \lambda) \in \partial G$ and $\mu \in [0,1]$ with $H_\mu(x, \lambda) = (x, \lambda)$. Then $(1-\mu)p + \mu N(x, \lambda) = x$ and $\lambda = 0$ i.e. $(1-\mu)p + \mu N(x, 0) = x$. Now if $0 < \mu < 1$ we have a contradiction since (1.5) holds. If $\mu = 1$ then $\lambda = 0$ and $N(x, \lambda) = N(x, 0) = x$, which is a contradiction since (1.4) holds. If $\mu = 0$ then $\lambda = 0$ and $(p, 0) = (x, \lambda) \in \partial G$ which is a contradiction since $p \in G_0$ (i.e. $(p, 0) \in G$). Thus $H_\mu \in K_{\partial G}(\overline{G}, C \times [0,1])$ for each $\mu \in [0,1]$. Theorem 1.5 now implies that $H_1(x, \lambda) = (N(x, \lambda),\, 0)$ is essential so (1.8) follows. □

2. Fixed point theory

We begin this section by presenting some nonlinear alternatives of Leray–Schauder type. Our first result is motivated by work of Browder [2].

Theorem 2.1. *Let U be an open subset of a real Banach space E and $\Omega \supseteq \overline{U}$ a subset of E. Assume $p \in U$, and $F : \overline{U} \to E$ is given by $F = F_1 + F_2$. Here $I - F_1 : \Omega \to E$ is continuous and strongly accretive (single valued) with $F_1(\overline{U})$ bounded and $F_2 : \overline{U} \to E$ is a continuous, compact map. Then either*

(A1) *F has a fixed point in $\overline{U}$; or*
(A2) *there exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(u) + (1-\lambda)p$.*

Proof. Now there exists $c > 0$ with

$$((I - F_1)(x) - (I - F_1)(y),\, x - y)_+ \ge c\, \|x - y\|^2 \quad \text{for all} \quad x, y \in \Omega. \tag{2.1}$$

Clearly $I - F_1$ is one to one and $(I - F_1)^{-1} : (I - F_1)(\Omega) \to E$ is Lipschitz with Lipschitz constant $\frac{1}{c}$ since for $z_1, z_2 \in (I - F_1)(\Omega)$ we have

$$\begin{aligned} c\,\|(I - F_1)^{-1}(z_1) - (I - F_1)^{-1}(z_2)\|^2 \\ \le \big(z_1 - z_2,\, (I - F_1)^{-1}(z_1) - (I - F_1)^{-1}(z_2)\big)_+ \\ \le \|z_1 - z_2\|\, \|(I - F_1)^{-1}(z_1) - (I - F_1)^{-1}(z_2)\|. \end{aligned}$$

Let

$$G = \{(x, \lambda) :\, x \in E,\, \lambda \in [0, 1] \text{ and } x \in (I - \lambda F_1)(U)\,\} \tag{2.2}$$

and for each $\lambda \in [0, 1]$ let G_λ be the section of G at level λ i.e.

$$G_\lambda = (I - \lambda F_1)(U) = \{\, u \in E :\, (u, \lambda) \in G\,\}.$$

Let $J : G_0 \to E$ be given by $J(x) = p$ and $N_1 : G_1 \to E$ be given by $N_1(u) = F_2(I - F_1)^{-1}(u)$.

Remark. Fix $0 \le \lambda \le 1$. Then $I - \lambda F_1 : \Omega \to E$ is strongly accretive. This is immediate since for $x, y \in \Omega$,

$$\begin{aligned} ((I - \lambda F_1)(x) - (I - \lambda F_1)(y),\, x - y)_+ \\ = (\lambda[(I - \lambda F_1)(x) - (I - \lambda F_1)(y)] + (1 - \lambda)(x - y),\, x - y)_+ \\ = \lambda\,((I - F_1)(x) - (I - F_1)(y),\, x - y)_+ + (1 - \lambda)\|x - y\|^2 \\ \ge (\lambda c + (1 - \lambda))\,\|x - y\|^2 \end{aligned}$$

since $(z_1 + \alpha z_2,\, z_2)_+ = (z_1, z_2)_+ + \alpha\|z_2\|^2$ (here $z_1, z_2 \in E$ and α is a scaler). Also $(I - \lambda F_1)^{-1} : (I - \lambda F_1)(\Omega) \to E$ is a Lipschitz map with Lipschitz constant $\frac{1}{c_\lambda}$; here $c_\lambda = \lambda c + (1 - \lambda)$ and notice $\frac{1}{c_\lambda} \le \frac{1}{\min\{1, c\}}$.

Consider the homotopy $N : \overline{G} \to E$ joining J and N_1 given by

$$N(u, \lambda) = \lambda F_2(I - \lambda F_1)^{-1}(u) + (1 - \lambda)p. \tag{2.3}$$

Fix $\lambda \in [0, 1]$. Define $h_\lambda : \overline{U} \to E$ by $h_\lambda(u) = (I - \lambda F_1)(u)$. Now Deimling's invariance of domain theorem (theorem 1.2) implies that $G_\lambda = h_\lambda(U)$ is open. Next we claim that $h_\lambda(\overline{U})$ is closed and $h_\lambda(\overline{U}) = \overline{h_\lambda(U)} = \overline{G_\lambda}$. To see that $h_\lambda(\overline{U})$ is closed let $w \in \overline{h_\lambda(\overline{U})}$. Then there exists $u_n \in \overline{U}$ with $h_\lambda(u_n) \to w$. Now since

$$(\lambda c + (1 - \lambda))\|u_n - u_m\| \le \|(I - \lambda F_1)(u_n) - (I - \lambda F_1)(u_n)\|$$

we have that $\{u_n\}$ is a Cauchy sequence in $\overline{U}$. Thus there exists $u \in \overline{U}$ with $u_n \to u$. Since h_λ is continuous we have that $h_\lambda(u_n) \to h_\lambda(u)$ so $w = h_\lambda(u)$. Thus $h_\lambda(\overline{U})$ is closed. In addition since h_λ is continuous we have that $h_\lambda(\overline{U}) \subseteq \overline{h_\lambda(U)}$. On the other hand $\overline{h_\lambda(U)} \subseteq \overline{h_\lambda(\overline{U})} = h_\lambda(\overline{U})$ since $h_\lambda(\overline{U})$ is closed. Consequently $h_\lambda(\overline{U}) = \overline{h_\lambda(U)} = \overline{G_\lambda}$. Next since $F_1(\overline{U})$ is bounded there exists a constant M with $\|F_1(u)\| \le M$ for all $u \in \overline{U}$. Thus if $t, \lambda \in [0, 1]$ and $u \in \overline{U}$ we have

$$\|h_\lambda(u) - h_t(u)\| = \|(\lambda - t)F_1(u)\| \le M|\lambda - t|. \tag{2.4}$$

The above together with a result of F. E. Browder [2, Prop. 12.2,p. 189] implies that G given in (2.2) is an open subset of $E \times [0,1]$ and

$$\partial G = \{(x,\lambda) : x \in E,\, \lambda \in [0,1] \text{ and } x \in (I - \lambda F_1)(\partial U)\}. \tag{2.5}$$

We now return to the homotopy $N : \overline{G} \to E$ joining J and N_1 given in (2.3). Either $N(x,\mu) \neq x$ for all $(x,\mu) \in \partial G$ or not. Suppose not i.e. suppose there exists $(y,\lambda) \in \partial G$ with $N(y,\lambda) = y$. Then there exists $u \in \partial U$ (by (2.5)) with $N(y,\lambda) = y = (I - \lambda F_1)(u)$. Now $\lambda \neq 0$ since if $\lambda = 0$ then $p = N(y,0) = y = I\,u = u \in \partial U$, a contradiction. Thus $0 < \lambda \leq 1$. Also $N(y,\lambda) = y$ means $\lambda F_2(I - \lambda F_1)^{-1}(y) + (1-\lambda)p = y$ and so

$$\lambda F_2(u) = \lambda F_2(I - \lambda F_1)^{-1}(y) = y - (1-\lambda)p = (I - \lambda F_1)(u) - (1-\lambda)p.$$

That is

$$\lambda F(u) + (1-\lambda)p = u,\ 0 < \lambda \leq 1 \text{ and } u \in \partial U.$$

Hence $(A2)$ occurs if $0 < \lambda < 1$ and $(A1)$ occurs if $\lambda = 1$ and we are finished. So for the remainder of the proof we assume $N(x,\mu) \neq x$ for all $(x,\mu) \in \partial G$.

Next we claim that $N : \overline{G} \to E$ is a continuous, compact map. To see the continuity let $(y_n,\lambda_n),\, (y,\lambda) \in \overline{G}$ with $(y_n,\lambda_n) \to (y,\lambda)$. We first show

$$h_{\lambda_n}^{-1}(y_n) \to h_\lambda^{-1}(y). \tag{2.6}$$

To see this recall (2.4) implies that given $\epsilon > 0$ there exists a positive integer k such that for $n > k$ we have

$$\|h_{\lambda_n}(x) - h_\lambda(x)\| \leq \epsilon \text{ for all } x \in \overline{U}.$$

Let $x_n = h_{\lambda_n}^{-1}(y_n)$. Thus for $n > k$ we have

$$\|y_n - h_\lambda(x_n)\| = \|h_{\lambda_n}(x_n) - h_\lambda(x_n)\| \leq \epsilon.$$

Also since $y_n \to y$ then there exists an integer $n_0 \geq k$ such that

$$\|h_\lambda(x_n) - y\| \leq 2\epsilon \text{ for } n > n_0.$$

Thus as $n \to \infty$ we have $h_\lambda(x_n) \to y$ in E. Consequently

$$h_\lambda^{-1}(y_n) = h_\lambda^{-1}(h_\lambda(x_n)) \to h_\lambda^{-1}(y)$$

since h_λ^{-1} is continuous on $\overline{h_\lambda(U)} = h_\lambda(\overline{U})$. Next notice

$$\begin{aligned}\|N(y_n,\lambda_n) - N(y,\lambda)\| &\leq \|\lambda_n F_2 h_{\lambda_n}^{-1}(y_n) - \lambda F_2 h_\lambda^{-1}(y)\| + |\lambda_n - \lambda|\,\|p\| \\ &\leq \|\lambda_n F_2 h_{\lambda_n}^{-1}(y_n) - \lambda_n F_2 h_\lambda^{-1}(y)\| \\ &\quad + \|\lambda_n F_2 h_\lambda^{-1}(y) - \lambda F_2 h_\lambda^{-1}(y)\| + |\lambda_n - \lambda|\,\|p\| \\ &= |\lambda_n|\,\|F_2 h_{\lambda_n}^{-1}(y_n) - F_2 h_\lambda^{-1}(y)\| \\ &\quad + |\lambda_n - \lambda|\,\|F_2 h_\lambda^{-1}(y)\| + |\lambda_n - \lambda|\,\|p\|.\end{aligned}$$

Now $F_2 : \overline{U} \to E$ being continuous together with (2.6) and $F_2(\overline{U})$ bounded implies that $N : \overline{G} \to E$ is continuous. To see that N is a compact map let

$(y,\lambda) \in \overline{G}$. Then $y = (I-\lambda F_1)(\overline{U})$, i.e. $y = (I-F_1)(u)$ for some $u \in \overline{U}$, and $N(y,\lambda) = \lambda F_2(I-\lambda F_1)^{-1}(y) + (1-\lambda)p = \lambda F_2(u) + (1-\lambda)p \subseteq co\,(F_2(\overline{U}) \cup \{p\})$. Consequently

$$N(\overline{G}) \subseteq co\,(F_2(\overline{U}) \cup \{p\})$$

and so

$$\alpha(N(\overline{G})) \leq \alpha(co\,(F_2(\overline{U}) \cup \{p\})) = \alpha(F_2(\overline{U}) \cup \{p\}) = 0.$$

Consequently $N : \overline{G} \to E$ is a compact map.

Remark. Alternatively one can deduce that N is a compact map if one notices

$$F_2(\overline{U}) \subseteq K,\ K \text{ compact};\ N(\overline{G}) \subseteq \overline{co}\,(K \cup \{p\})$$

and that $\overline{co}\,(K \cup \{p\})$ is compact by Mazur's theorem.

We are also assuming $N(x,\lambda) \neq x$ for all $(x,\lambda) \in \partial G$. Also since $N(x,0) = p$ we have $(1-\mu)p + \mu N(x,0) \neq x$ for all $(x,0) \in \partial G$ and $0 < \mu < 1$ since if $p = (1-\mu)p + \mu N(x,0) = x$ for some $(x,0) \in \partial G$ and $0 < \mu < 1$ then $(p,0) \in \partial G$ which is a contradiction since $p \notin \partial U = I(\partial U)$. Now theorem 1.7 implies that there exists $y \in G_1 = (I-F_1)(U)$ with $N(y,1) = y$. So there exists $u \in U$ with $N(y,1) = y = (I-F_1)(u)$. Now $N(y,1) = y$ means $F_2(I-F_1)^{-1}(y) = y$ so

$$F_2(u) = F_2(I-F_1)^{-1}(y) = y = (I-F_1)(u).$$

That is $F(u) = u$ with $u \in U$ so $(A1)$ occurs. □

Remark. The assumption that $h_1 = I - F_1 : \Omega \to E$ is continuous and strongly accretive in theorem 2.1 could be replaced by the more general condition

$$\text{(2.7)} \qquad \begin{cases} h_1 : \Omega \to E \text{ is continuous with } h_1^{-1} : h(\Omega) \to E \text{ continuous} \\ \text{(assuming the inverse } h_1^{-1} \text{ exists)},\ h_1(U) \text{ open},\ h_1(\overline{U}) = \overline{h_1(U)} \\ \text{and (2.4) holds for some } M > 0 \text{ (independent of } u \in \overline{U}). \end{cases}$$

Theorem 2.2. *Let U be an open set in a a real Banach space E and $\Omega \supseteq \overline{U}$ a subset of E. Assume $0 \in U$ and $F : \overline{U} \to E$ is given by $F = F_1 + F_2$. Here $I - F_1 : \Omega \to E$ is continuous and accretive (i.e $F_1 : \Omega \to E$ is pseudocontractive) with $F_1(\overline{U})$ bounded and $F_2 : \overline{U} \to E$ is a continuous, compact map. Also assume $(I-F)(\overline{U})$ is closed. Then either*

(A1) *F has a fixed point in $\overline{U}$; or*
(A2) *there exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

Proof. Assume $(A2)$ does not hold. Consider for each $n \in \{2,3,...\}$ the mapping

$$\text{(2.8)} \qquad S_n = \left(1 - \frac{1}{n}\right) F : \overline{U} \to E.$$

Notice $\left(1-\frac{1}{n}\right)F_2:\overline{U}\to E$ is compact and $I-\left(1-\frac{1}{n}\right)F_1:\Omega\to E$ is strongly accretive since for $x,y\in\Omega$ we have

$$
\begin{aligned}
&\left(\left(I-\left(1-\frac{1}{n}\right)\right)F_1(x)-\left(I-\left(1-\frac{1}{n}\right)\right)F_1(y),\,x-y\right)_+\\
&=\left(\left(1-\frac{1}{n}\right)[(I-F_1)(x)-(I-F_1)(y)]+\frac{1}{n}(x-y),\,x-y\right)_+\\
&\geq\frac{1}{n}\|x-y\|^2.
\end{aligned}
$$

Remark. $(z_1+\alpha z_2,z_2)_+=(z_1,z_2)_++\alpha\|z_2\|^2$; here $z_1,z_2\in E$ and α is a scaler.

Apply theorem 2.1 to S_n. If there exists $\lambda\in(0,1)$ and $u\in\partial U$ with $u=\lambda S_n(u)$ then

$$u=\lambda\left(1-\frac{1}{n}\right)F(u)=\eta F(u)\quad\text{where}\quad 0<\eta=\lambda\left(1-\frac{1}{n}\right)<1,$$

which is a contradiction since $(A2)$ was assumed not to hold. Consequently for each $n\in\{2,3,...\}$ we have that S_n has a fixed point $u_n\in\overline{U}$. Notice also since $u_n=\left(1-\frac{1}{n}\right)F(u_n)$ we have that $u_n-F(u_n)=-\frac{1}{n}F(u_n)$ and so $u_n-F(u_n)\to 0$ as $n\to\infty$ (since $F(\overline{U})$ is bounded). Consequently $0\in(I-F)(\overline{U})$ since $(I-F)(\overline{U})$ is closed. Thus there exists $u\in\overline{U}$ with $0=(I-F)(u)$. □

Theorem 2.3. *Let U be a bounded, open, convex subset of a uniformly convex Banach space E. Suppose Ω is an open set containing $\overline{U}$ with dist $(\overline{U},E/\Omega)>0$. Assume $0\in U$ and $F:\overline{U}\to E$ is given by $F=F_1+F_2$. Here $I-F_1:\Omega\to E$ is a continuous accretive mapping which sends bounded sets into bounded sets and $F_2:\overline{U}\to E$ is a continuous, compact map. In addition suppose $F_2:\overline{U}\to E$ is strongly continuous. Then either*

(A1) *F has a fixed point in $\overline{U}$; or*
(A2) *there exists $u\in\partial U$ and $\lambda\in(0,1)$ with $u=\lambda F(u)$.*

Remark. $F_2:\overline{U}\to E$ is said to be strongly continuous [18] if $x_x\rightharpoonup x$ implies $F_2(x_n)\to F_2(x)$; here $x_n,x\in\overline{U}$.

Proof. Assume $(A2)$ does not hold. Consider for each $n\in\{2,3,...\}$ the mapping S_n given by (2.8). Essentially the same reasoning as in theorem 2.2 implies that S_n has a fixed point $u_n\in\overline{U}$.

A standard result in functional analysis (if E is a reflexive Banach space then any norm bounded sequence in E has a weakly convergent subsequence) implies (since $\overline{U}$ is bounded) that there exists a subsequence S of integers and a $u\in\overline{U}$ (notice $\overline{U}$ is strongly closed and convex so weakly closed) with

$$u_n\rightharpoonup u\quad\text{as}\quad n\to\infty\quad\text{in } S.$$

Also since $u_n = \left(1-\frac{1}{n}\right)F_1(u_n) + \left(1-\frac{1}{n}\right)F_2(u_n)$ we have

$$\begin{aligned}\|(I-F_1)(u_n)-F_2(u)\| &= \|-\frac{1}{n}F_1(u_n)+\left(1-\frac{1}{n}\right)F_2(u_n)-F_2(u)\| \\ &\leq \frac{1}{n}\|F(u_n)\|+\|F_2(u_n)-F_2(u)\|\end{aligned}$$

so since F_2 is strongly continuous and $F(\overline{U})$ is bounded we have $(I-F_1)(u_n) \to F_2(u)$.

Theorem 1.3 (i.e. $I-F_1$ is demiclosed on $\overline{U}$) implies $(I-F_1)(u) = F_2(u)$. □

Remark. Of course one can prove theorem 2.3 directly from theorem 2.2 by showing that $(I-F)(\overline{U})$ is closed. To see this let $y \in \overline{(I-F)(\overline{U})}$ so there exists $u_n \in \overline{U}$ with $(I-F)(u_n) \to y$. Since $u_n \in \overline{U}$ there exists a subsequence S of integers and a $u \in \overline{U}$ with $u_n \rightharpoonup u$ as $n \to \infty$ in S. Consequently $(I-F)(u_n) \to (I-F)(u)$ i.e. $y = (I-F)(u)$.

Next we present two new fixed point results.

Theorem 2.4. *Let Q be a closed, convex subset of a a real Banach space E with $0 \in Q$. Also let $\Omega \supseteq Q$ be a subset of E with $U_i = \{x \in E : d(x,Q) < \frac{1}{i}\} \subseteq \Omega$ for i sufficiently large; here d denotes the metric induced by the norm. Now $F: Q \to E$ is given by $F = F_1 + F_2$ where $I - F_1 : \Omega \to E$ is continuous, strongly accretive (i.e. (2.1) is satisfied) with $F_1(\overline{U_1})$ bounded and $F_2 : Q \to E$ is a bounded continuous,compact map. In addition suppose $F_2(Q) \subseteq (I-F_1)(\Omega)$ with $(I-F_1)(\Omega)$ closed and also that*

$$(2.9)\qquad \begin{cases} \text{if } \{(x_j,\lambda_j)\}_{j=1}^{\infty} \text{ is a sequence in } \partial Q \times [0,1] \text{ converging} \\ \text{to } (x,\lambda) \text{ with } x = \lambda F(x) \text{ and } 0 \leq \lambda < 1, \text{ and if } \{z_j\} \\ \text{is a sequence in } U_m\ (m \text{ sufficiently large) with} \\ z_j \in \partial U_j \text{ for } j = m+1, m+2, \ldots \text{ and } z_j \to x, \text{ then} \\ \lambda_j\,[F_1(z_j) + F_2(x_j)] \in Q \text{ for } j \text{ sufficiently large} \end{cases}$$

holds. Then F has a fixed point in Q.

Remarks. (i) If $\Omega = E$ then $(I-F_1)(\Omega) = E$. Notice theorem 1.1 implies that $I-F_1$ is a homeomorphism from E onto E.
(ii) In the statement of theorem 2.4, $F_1(\overline{U_1})$ bounded may be replaced by $F_1(\overline{U_m})$ bounded for some $m \in \{1, 2, \ldots\}$.
(iii) Theorem 2.4 was proved by Furi and Pera [7], by a different method, when $F_1 = 0$ and F_2 is a compact map.

Proof. Let $r : E \to Q$ be a continuous retraction [13] with $r(z) \in \partial Q$ for $z \in E \backslash Q$. Consider

$$B = \left\{x \in (I-F_1)(\Omega) : x = F_2\, r\,(I-F_1)^{-1}(x)\right\}.$$

We claim $B \neq \emptyset$. To see this we look at $r(I-F_1)^{-1}F_2 : Q \to Q$ (notice this is a well defined map since $F_2(Q) \subseteq (I-F_1)(\Omega)$). Now $r(I-F_1)^{-1}F_2 : Q \to Q$ is a compact map since $F_2 : Q \to E$ is a compact map and r, $(I-F_1)^{-1}$ are continuous maps. Schauder's fixed point theorem implies that there exists $y \in Q$ with $y = r(I - F_1)^{-1} F_2(y)$. Let $z = F_2(y)$. Then

$$F_2\, r\, (I - F_1)^{-1}(z) = F_2\, r\, (I - F_1)^{-1} F_2(y) = F_2(y) = z$$

so $z \in B$ (notice $y \in Q$ and $F_2(Q) \subseteq (I - F_1)(\Omega)$) and $B \neq \emptyset$. In addition the continuity of $F_2\, r\, (I - F_1)^{-1}$ together with $(I - F_1)(\Omega)$ closed implies that B is closed. Also

$$B \subseteq F_2(Q)$$

together with $F_2 : Q \to E$ being a compact map implies that B is compact. Let

$$\Phi = (I - F_1)^{-1}(B).$$

Notice Φ is a compact set. We claim $\Phi \cap Q \neq \emptyset$.

To do this we argue by contradiction. Suppose $\Phi \cap Q = \emptyset$. Then since Φ is compact and Q is closed there exists $\delta > 0$ with $dist\,(\Phi, Q) > \delta$. Define

$$U_i = \left\{ x \in E : \, d(x, Q) < \frac{1}{i} \right\} \quad \text{for} \quad i \in \{N, N+1, ...\}.$$

Here $N \in \{1, 2, ...\}$ is chosen so that $1 < \delta N$ and $\overline{U_i} \subseteq \Omega$ for $i \geq N$. Fix $i \in \{N, N+1, ...\}$. Notice U_i is open and since $dist\,(\Phi, Q) > \delta$ then $\Phi \cap \overline{U_i} = \emptyset$. Also $F_2\, r : \overline{U_i} \to E$ is a compact map. Now theorem 2.1 (with $F_1 + F_2\, r$) implies that there exists $(y_i, \lambda_i) \in \partial U_i \times (0, 1)$ with $y_i = \lambda_i\, [F_1(y_i) + F_2\, r(y_i)]$.

Remark. Notice there cannot exist a $y \in \overline{U_i}$ with $y = F_1(y) + F_2\, r(y)$ since $\Phi \cap \overline{U_i} = \emptyset$. To see this suppose there exists $y \in \overline{U_i}$ with $y = F_1(y) + F_2\, r(y)$. We claim $y \in \Phi$ (which will yield a contradiction). Let $x = (I - F_1)(y)$. Then $x \in B$ since

$$F_2\, r\, (I - F_1)^{-1}(x) = F_2\, r(y) = (I - F_1)(y) = x$$

and so $y \in \Phi$.

Consequently for each $j \in \{N, N+1, ...\}$ there exists $(y_j, \lambda_j) \in \partial U_j \times (0, 1)$ with $y_j = \lambda_j\, [F_1(y_j) + F_2\, r(y_j)]$. Notice in particular since $y_j \in \partial U_j$ that

$$\lambda_j\, [F_1(y_j) + F_2\, r(y_j)] \notin Q \quad \text{for} \quad j \in \{N, N+1, ...\}. \tag{2.10}$$

Now let

$$G = \{(x, \lambda) : \, x \in E, \, \lambda \in [0, 1] \text{ and } x \in (I - \lambda F_1)(U_N)\}.$$

As, in theorem 2.1,

$$\overline{G} = \left\{(x, \lambda) : \, x \in E, \, \lambda \in [0, 1] \text{ and } x \in (I - \lambda F_1)(\overline{U_N})\right\}.$$

Next let

$$D = \left\{ x \in E : \, x \in (I - \lambda F_1)(\overline{U_N}) \text{ for some } \lambda \text{ and } N_0(x, \lambda) = x \right\}$$

where $N_0 : \overline{G} \to E$ is given by

$$N_0(u, \lambda) = \lambda\, F_2\, r\, (I - \lambda F_1)^{-1}(u).$$

Also, as in theorem 2.1 since $F_2\, r : \overline{U_N} \to E$ is a compact map, we have that $N_0 : \overline{G} \to E$ is a continuous compact map. Notice $x_i \in D$, $i \in \{N, N+1, ...\}$ where $x_i = (I - \lambda_i F_1)(y_i)$. To see this notice $x_i \in (I - \lambda_i F_1)^{-1}(\partial U_i) \subseteq (I - \lambda_i F_1)^{-1}(\overline{U_N})$ and

$$\lambda_i F_2\, r\,(I - \lambda_i F_1)^{-1}(x_i) = \lambda_i F_2\, r\,(y_i) = (I - \lambda_i F_1)(y_i) = x_i.$$

Also D is closed. To see this let $x \in \overline{D}$. Then there exists $z_n \in D$ with $z_n \to x$. Also there exists $\mu_n \in [0,1]$ with $z_n \in (I - \mu_n F_1)(\overline{U_N})$. Without loss of generality assume $\mu_n \to \mu$. Then (z_n, μ_n), $(x, \mu) \in \overline{G}$ together with $N_0 : \overline{G} \to E$ continuous implies $N_0(x, \mu) = x$. Hence $x \in D$ and D is closed. Also since $D \subseteq N_0(\overline{G})$ we have that D is compact (so sequentially compact).

This together with $|\lambda_j| \leq 1$ (for $j \in \{N, N+1, ...\}$) implies that we may assume without loss of generality that $\lambda_j \to \lambda^\star$ and $x_j \to x^\star$. Now (x_j, λ_j), $(x^\star, \lambda^\star) \in \overline{G}$, $x_j = N_0(x_j, \lambda_j)$ together with $N_0 : \overline{G} \to E$ continuous implies $N_0(x,^\star, \lambda^\star) = x^\star$. Also as in theorem 2.1 (see (2.6)) we have immediately that

$$y_j = (I - \lambda_i F_1)^{-1}(x_i) \to (I - \lambda^\star F_1)^{-1}(x^\star).$$

Let $y^\star = (I - \lambda^\star F_1)^{-1}(x^\star)$. Then $y_j \to y^\star$ and $y^\star \in \partial Q$ since $y_j \in \partial U_j$ so $d(y_j, Q) = \frac{1}{j}$. Also

$$\lambda^\star\, F_2\,(y^\star) = \lambda^\star\, F_2\, r\,(y^\star) = \lambda^\star\, F_2\, r\,(I - \lambda^\star F_1)^{-1}(x^\star) = x^\star = (I - \lambda^\star F_1)(y^\star)$$

so $y^\star = \lambda^\star\, F(y^\star)$. If $\lambda^\star = 1$ then $y^\star = F(y^\star)$, $y^\star \in \partial Q$ and $x^\star = (I - F_1)(y^\star) \in B$ since

$$F_2\, r\,(I - F_1)^{-1}(x^\star) = F_2\, r\,(y^\star) = F_2(y^\star) = (I - F_1)(y^\star) = x^\star.$$

Hence $y^\star \in \Phi$ which contradicts $\Phi \cap Q = \emptyset$. Hence we may assume $0 \leq \lambda^\star < 1$. But in this case (2.9) with $x_j = r(y_j) \in \partial Q$, $x = y^\star = r(y^\star)$ and $z_j = y_j$, implies $\lambda_j\,[F_1(y_j) + F_2\, r(y_j)] \in Q$ for j sufficiently large. This contradicts (2.10). Thus $\Phi \cap Q \neq \emptyset$ so there exists $x \in \Phi \cap Q$. Let $z = (I - F_1)(x)$. Then $z \in B$ since $x \in \Phi$ so $F_2\, r\,(I - F_1)^{-1}(z) = z$. Consequently, since $x \in Q$,

$$F_2(x) = F_2\, r(x) = F_2\, r\,(I - F_1)^{-1}(z) = z = (I - F_1)(x).$$

That is $x = F(x)$. □

Remarks. (i) Notice we only need the assumptions $F_2(Q) \subseteq (I - F_1)(\Omega)$ and $(I - F_1)(\Omega)$ closed to show $B \neq \emptyset$ and closed.
(ii) Of course if we know that λF, $0 \leq \lambda < 1$ has no fixed points on ∂Q then (2.9) is trivially satisfied.
(iii) In theorem 2.4 if $0 \in int\,(Q)$ then the proof would be a lot simpler (simply show condition $(A2)$ in theorem 2.1 is not satisfied). In this situation $0 \leq \lambda < 1$ can be replaced by $0 < \lambda < 1$ in (2.9).

Theorem 2.5. *Let Q be a closed, convex subset of a real Banach space E with $0 \in Q$. Also let $\Omega \supseteq Q$ be a subset of E with $U_i = \{x \in E : d(x, Q) < \frac{1}{i}\} \subseteq \Omega$ for i sufficiently large. Now $F : Q \to E$ is given by $F = F_1 + F_2$ where $I - F_1 : \Omega \to E$ is continuous, accretive (i.e. $F_1 : \Omega \to E$ is pseudocontractive)* *with* $F_1(\overline{U_1})$ *bounded and* $F_2 : Q \to E$ *is a continuous, compact map. In addition suppose* $F_2(Q) \subseteq (I - F_1)(\Omega)$ *with* $(I - F_1)(\Omega)$ *closed and that* (2.9) *holds. Also assume* $(I - F)(Q)$ *is closed. Then* F *has a fixed point in* Q.

Proof. Consider for each $n \in \{2, 3, ...\}$ the mapping

$$S_n = \left(1 - \frac{1}{n}\right) F : Q \to E.$$

As in theorem 2.2, $\left(1 - \frac{1}{n}\right) F_2 : Q \to E$ is compact and $I - \left(1 - \frac{1}{n}\right) F_1 : \Omega \to E$ is strongly accretive. We will apply theorem 2.4. Let $\{(x_j, \lambda_j)\}_{j=1}^{\infty}$ be a sequence in $\partial Q \times [0, 1]$ converging to (x, λ) with $x = \lambda S_n(x)$ and $0 < \lambda < 1$. Also let $\{z_j\}$ be a sequence in U_m (m sufficiently large) with $z_j \in \partial U_j$ for $j = m+1, m+2, ...$ and $z_j \to x$. Then

$$\lambda_j \left(1 - \frac{1}{n}\right) F_1(z_j) + \lambda_j \left(1 - \frac{1}{n}\right) F_2(x_j) = \mu_j F_1(z_j) + \mu_j F_2(x_j) \in Q\,,$$

for j sufficiently large, since (2.9) is satisfied (note $\mu_j = \lambda_j \left(1 - \frac{1}{n}\right)$ is a sequence in $[0, 1]$ with $\mu_j \to \lambda \left(1 - \frac{1}{n}\right) = \mu$, $0 < \mu < 1$ and $x = \lambda S_n(x) = \lambda \left(1 - \frac{1}{n}\right) F(x) = \mu F(x)$). Apply theorem 2.4 to S_n to deduce that S_n has a fixed point $u_n \in Q$. Now since $u_n - F(u_n) = -\frac{1}{n} F(u_n)$ we have $0 \in (I - F)(Q)$ since $(I - F)(Q)$ is closed. Thus there exists $u \in Q$ with $0 = (I - F)(u)$. □

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Department of Mathematics, National University of Ireland
Galway, IRELAND
E-mail: DONAL.OREGAN@NUIGALWAY.IE