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NEW EXAMPLES OF COMPACT COSYMPLECTIC SOLVMANIFOLDS

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ABSTRACT. In this paper we present new examples of $(2n+1)$-dimensional compact cosymplectic manifolds which are not topologically equivalent to the canonical examples, i.e., to the product of the $(2m+1)$-dimensional real torus and the r-dimensional complex projective space, with $m, r \geq 0$ and $m+r = n$. These new examples are compact solvmanifolds and they are constructed as suspensions with fibre the $2n$-dimensional real torus. In the particular case $n = 1$, using the examples obtained, we conclude that a 3-dimensional compact flat orientable Riemannian manifold with non-zero first Betti number admits a cosymplectic structure. Furthermore, if the first Betti number is equal to 1 then such a manifold is not topologically equivalent to the global product of a compact Kähler manifold with the circle S^1.

1. INTRODUCTION

It is well-known that the odd-dimensional counterpart of Kähler manifolds are cosymplectic manifolds. Let us recall that an almost contact metric structure (φ, ξ, η, g) on a manifold M is cosymplectic if it is integrable and the 1-form η and the fundamental 2-form of the structure are closed (see [1]).

The canonical example of compact cosymplectic manifold is given by the product of a compact Kähler manifold with the circle S^1 (see [2]). Thus, the natural examples of $(2n+1)$-dimensional compact cosymplectic manifolds are the products of the $(2m+1)$-dimensional real torus $\mathbb{T}^{2m+1}$ and the r-dimensional complex projective space $\mathbb{C}P^r$, with $m, r \geq 0$ and $m+r = n$. In fact, a compact cosymplectic manifold has topological properties similar to the product of a compact Kähler manifold with the circle S^1 (see [2] and [3]). In particular, in [5], the authors prove that a $(2n+1)$-dimensional compact cosymplectic manifold with positive constant φ-sectional curvature is diffeomorphic to the product manifold $\mathbb{C}P^n \times S^1$.

However, in [3] the authors give an example of 3-dimensional compact cosymplectic manifold which is not topologically equivalent to a global product of a compact Kähler manifold with the circle S^1. This fact yields a good motivation for studying the cosymplectic manifolds.

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The purpose of this paper is to show some examples of $(2n+1)$-dimensional compact cosymplectic manifolds which are not topologically equivalent to the natural examples $\mathbb{T}^{2m+1} \times \mathbb{C}P^r$, with $m, r \geq 0$ and $m + r = n$. These new examples are constructed as suspensions with fibre a compact Kähler manifold of representations defined by Hermitian isometries and, moreover, we have:

(1) All the examples are compact solvmanifolds, that is, they are compact homogeneous spaces of the form $\Gamma\backslash G$, where G is a connected simply connected solvable non-nilpotent Lie group and Γ is a discrete cocompact subgroup.
(2) Using the examples obtained we conclude that a 3-dimensional compact flat orientable Riemannian manifold with non-zero first Betti number admits a cosymplectic structure. Furthermore, if the first Betti number is equal to 1 then such a manifold is not topologically equivalent to the global product of a compact Kähler manifold with S^1. In fact, the example given in [3] is a 3-dimensional compact flat cosymplectic manifold with first Betti number equal to 1.

All the manifolds considered in this paper are assumed to be connected and of class C^∞.

2. Suspensions with fibre a compact Kähler manifold of representations defined by Hermitian isometries

Let (φ, ξ, η, g) be an *almost contact metric structure* on M. Then we have

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for $X, Y \in \mathfrak{X}(M)$, I being the identity transformation and $\mathfrak{X}(M)$ the Lie algebra of vector fields on M.

The fundamental 2*-form* Φ of M is defined by

$$\Phi(X, Y) = g(X, \varphi Y),$$

for $X, Y \in \mathfrak{X}(M)$. The almost contact metric structure (φ, ξ, η, g) is said to be [1]: *integrable* if $N_\varphi = 0$, N_φ being the Nijenhuis tensor of φ; *cosymplectic* if it is integrable and $d\eta = 0$, $d\Phi = 0$.

Now, let N be a $2n$-dimensional compact *Kähler manifold* with *Hermitian structure* (J, h). Consider an *Hermitian isometry* $f : N \to N$, i.e., f is a diffeomorphism and

$$f_* \circ J = J \circ f_* \quad , \quad f^* h = h. \tag{2.1}$$

We define the action A of $\mathbb{Z}$ on the product manifold $N \times I\!R$ by

$$A(n, (x, z)) = (f^n(z), z - n), \tag{2.2}$$

for all $n \in \mathbb{Z}$ and $(x, z) \in N \times I\!R$. This action is free and properly discontinuous. Thus, the orbit space $(N \times I\!R)/A$ of the $\mathbb{Z}$-action is a $(2n+1)$-dimensional compact manifold and the canonical projection $p' : N \times I\!R \to M$ is a covering map. Moreover, we can define a fibration τ of M on $S^1 = I\!R/\mathbb{Z}$ by $\tau([(x,z)]) = [z]$, for all $(x,z) \in N \times I\!R$. It is clear that the fibers of τ are diffeomorphic to N.

Denote by $\varrho : \mathbb{Z} \to Diff(N)$ the representation of $\mathbb{Z}$ on the group of the diffeomorphisms of N, $Diff(N)$, given by $\varrho(k) = f^k$, for all $k \in \mathbb{Z}$. Then the manifold M is called the *suspension with fibre N of the repesentation ϱ* (see [4]).

Next, we shall obtain a cosymplectic structure on M (see [3]).

We consider on $N \times I\!R$ the cosymplectic structure $(\bar\varphi, \bar\xi, \bar\eta, \bar g)$ given by

$$\bar\varphi = J \circ (pr_1)_*, \quad \bar\xi = \frac{\partial}{\partial t}, \quad \bar\eta = pr_2^*(dt), \quad \bar g = pr_1^*(h) + pr_2^*(dt^2), \tag{2.3}$$

where $pr_1 : N \times I\!R \to N$ and $pr_2 : N \times I\!R \to I\!R$ are the canonical projections onto the first and second factor, respectively and t is the usual coordinate on $I\!R$.

Since f is an Hermitian isometry, using (2.1), (2.2) and (2.3), we deduce that the cosymplectic structure $(\bar\varphi, \bar\xi, \bar\eta, \bar g)$ is invariant under the action A of $\mathbb{Z}$ on $N \times I\!R$. Therefore, it induces a cosymplectic structure (φ, ξ, η, g) on M (see [3]).

Remark 2.1.

(1) If the Hermitian isometry f is the identity then $M = N \times S^1$ and (φ, ξ, η, g) is the usual cosymplectic structure on M.
(2) If the Riemannian manifold (N, h) is flat then it is clear that the Riemannian manifold (M, g) is also flat.

3. The examples

In this Section, using the construction of Section 2, we shall obtain some examples of $(2n+1)$-dimensional compact cosymplectic solvmanifolds. For this purpose, we shall consider two Kähler structures on the $2n$-dimensional real torus $\mathbb{T}^{2n}$.

CASE A: Let (J, h) be the natural Kähler structure on the $2n$-dimensional real torus $\mathbb{T}^{2n}$ given by

$$JX_i = -Y_i, \qquad JY_i = X_i, \qquad h = \sum_{j=1}^{n} (\alpha_j \otimes \alpha_j + \beta_j \otimes \beta_j),$$

for all $i \in \{1, \ldots, n\}$, where $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ is the canonical global basis of vector fields on $\mathbb{T}^{2n}$ and $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}$ is its dual basis of 1-forms.

Case A1: We consider the diffeomorphism $\widetilde{f_1} : I\!R^{2n} \to I\!R^{2n}$ defined by

$$\widetilde{f_1}(x_1, \ldots, x_n, y_1, \ldots, y_n) = (y_1, \ldots, y_n, -x_1, \ldots, -x_n),$$

for all $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in I\!R^{2n}$. It is easy to prove that $\widetilde{f_1}$ induces an Hermitian isometry $f_1 : (\mathbb{T}^{2n}, J, h) \to (\mathbb{T}^{2n}, J, h)$.

Suppose that $M_1(n)$ is the suspension with fibre $\mathbb{T}^{2n}$ of the representation $\varrho_1 : \mathbb{Z} \to Diff(\mathbb{T}^{2n})$ given by $\varrho_1(k) = (f_1)^k$, for all $k \in \mathbb{Z}$. A direct computation shows that the fundamental group of $M_1(n)$, $\pi_1(M_1(n))$, is the semidirect product

$$\pi_1(M_1(n)) = \mathbb{Z}^{2n} \times_{\psi_1} \mathbb{Z}, \tag{3.1}$$

where $\psi_1 : \mathbb{Z} \to Aut(\mathbb{Z}^{2n})$ is the homomorphism of $\mathbb{Z}$ on the automorphism group of $\mathbb{Z}^{2n}$, $Aut(\mathbb{Z}^{2n})$, defined by $\psi_1(k) = ((\widetilde{f}_1)_{|Z^{2n}})^{-k}$ for all $k \in \mathbb{Z}$.

From (3.1), we deduce that the commutator subgroup $[\pi_1(M_1(n)), \pi_1(M_1(n))]$ of $\pi_1(M_1(n))$ is

$$\begin{aligned}&[\pi_1(M_1(n)), \pi_1(M_1(n))] = \\ &\qquad = \{(p_1, \ldots, p_n, q_1, \ldots, q_n, 0) \in \mathbb{Z}^{2n+1} \ / \ (p_1 + q_1, \ldots, p_n + q_n) \in (2\mathbb{Z})^n\}.\end{aligned}$$

This implies that the first integral homology group $H_1(M_1(n), \mathbb{Z})$ is $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \oplus \overset{n)}{\cdots} \oplus \mathbb{Z}_2$.

Next, we shall describe the manifold $M_1(n)$ as a compact solvmanifold.

For this purpose, we consider the vector field $\widetilde{\zeta}$ on $\mathbb{R}^{2n}$ defined by

$$\widetilde{\zeta} = \sum_{i=1}^{n} \frac{3\pi}{2} \left(y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i} \right). \tag{3.2}$$

The vector field $\widetilde{\zeta}$ is an infinitesimal automorphism of the usual Kähler structure $(\widetilde{J}, \widetilde{h})$ of $\mathbb{R}^{2n}$, i.e., $\mathfrak{L}_{\widetilde{\zeta}}\widetilde{J} = 0$ and $\mathfrak{L}_{\widetilde{\zeta}}\widetilde{h} = 0$, $\mathfrak{L}$ being the Lie derivative operator on $\mathbb{R}^{2n}$. In fact, if $\widetilde{\psi} : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the flow of $\widetilde{\zeta}$, we have that

$$\begin{aligned}&\widetilde{\psi}(z, (x_1, \ldots, x_n, y_1, \ldots, y_n)) = \tag{3.3}\\ &= \left(x_1 \cos\left(\frac{3\pi}{2} z\right) + y_1 \sin\left(\frac{3\pi}{2} z\right), \ldots, x_n \cos\left(\frac{3\pi}{2} z\right) + y_n \sin\left(\frac{3\pi}{2} z\right), \right. \\ &\qquad \left. y_1 \cos\left(\frac{3\pi}{2} z\right) - x_1 \sin\left(\frac{3\pi}{2} z\right), \ldots, y_n \cos\left(\frac{3\pi}{2} z\right) - x_n \sin\left(\frac{3\pi}{2} z\right) \right).\end{aligned}$$

Thus, the diffeomorphism

$$\widetilde{\psi}(1) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

given by $\widetilde{\psi}(1)(x_1, \ldots, x_n, y_1, \ldots, y_n) = \widetilde{\psi}(1, (x_1, \ldots, x_n, y_1, \ldots, y_n))$ is just the map $\widetilde{f}_1^{-1}$. Furthermore, if on $\mathbb{R}^{2n}$ we consider the structure of additive Lie group then, for all $z \in \mathbb{R}$, the diffeomorphism $\widetilde{\psi}(z) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is an automorphism of $\mathbb{R}^{2n}$. Consequently, the map $\widetilde{\psi}$ induces a Lie group homomorphism of $\mathbb{R}$ into the automorphism group of $\mathbb{R}^{2n}$, $Aut(\mathbb{R}^{2n})$, which we also denote by $\widetilde{\psi}$.

Now, let $I\!R^{2n} \times_{\widetilde{\psi}} I\!R$ be the semidirect product defined by the homomorphism $\widetilde{\psi} : I\!R \to Aut(I\!R^{2n})$. From (3.3), we deduce that a basis for the left invariant vector fields on $I\!R^{2n} \times_{\widetilde{\psi}} I\!R$ is given by

$$X_i = \cos\left(\frac{3\pi}{2}z\right)\frac{\partial}{\partial x_i} - \sin\left(\frac{3\pi}{2}z\right)\frac{\partial}{\partial y_i}, \quad Y_i = \sin\left(\frac{3\pi}{2}z\right)\frac{\partial}{\partial x_i} + \cos\left(\frac{3\pi}{2}z\right)\frac{\partial}{\partial y_i},$$

$$Z = \frac{\partial}{\partial z},$$

for all $i \in \{1, \ldots, n\}$. Then, for all $i \in \{1, \ldots, n\}$,

$$[X_i, Z] = \frac{3\pi}{2}Y_i, \qquad [Y_i, Z] = -\frac{3\pi}{2}X_i, \tag{3.4}$$

and the other brackets being zero. Using (3.4), we conclude that $I\!R^{2n} \times_{\widetilde{\psi}} I\!R$ is a $(2n+1)$-dimensional simply connected solvable non-nilpotent Lie group.

On the other hand, since $\widetilde{\psi}(k)_{|Z^{2n}} = \psi_1(k)$ for all $k \in \mathbb{Z}$, we obtain that the fundamental group $\pi_1(M_1(n))$ of $M_1(n)$ is a discrete subgroup of $I\!R^{2n} \times_{\widetilde{\psi}} I\!R$.

Finally, it is easy to prove that the compact solvmanifold $\pi_1(M_1(n))\backslash(I\!R^{2n} \times_{\widetilde{\psi}} I\!R)$ is diffeomorphic to the suspension $M_1(n)$.

Case A2: Now, we consider the diffeomorphism $\widetilde{f}_2 : I\!R^{2n} \to I\!R^{2n}$ defined by

$$\widetilde{f}_2(x_1, \ldots, x_n, y_1, \ldots, y_n) = (-x_1, \ldots, -x_n, -y_1, \ldots, -y_n),$$

for all $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in I\!R^{2n}$. Then, the diffeomorphim $\widetilde{f}_2$ induces an Hermitian isometry $f_2 : (\mathbb{T}^{2n}, J, h) \to (\mathbb{T}^{2n}, J, h)$.

We denote by $M_2(n)$ the suspension with fibre $\mathbb{T}^{2n}$ of the representation $\varrho_2 : \mathbb{Z} \to Diff(\mathbb{T}^{2n})$ given by $\varrho_2(k) = (f_2)^k$, for all $k \in \mathbb{Z}$. The fundamental group of $M_2(n)$, $\pi_1(M_2(n))$, is the semidirect product

$$\pi_1(M_2(n)) = \mathbb{Z}^{2n} \times_{\psi_2} \mathbb{Z}, \tag{3.5}$$

where $\psi_2 : \mathbb{Z} \to Aut(\mathbb{Z}^{2n})$ is the homomorphism defined by $\psi_2(k) = ((\widetilde{f}_2)_{|Z^{2n}})^{-k}$ for all $k \in \mathbb{Z}$.

From (3.5), we deduce that the commutator subgroup of $\pi_1(M_2(n))$ is

$$[\pi_1(M_2(n)), \pi_1(M_2(n))] = (2\mathbb{Z})^{2n} \times \{0\}$$

and the first integral homology group $H_1(M_2(n), \mathbb{Z})$ is $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \overset{2n)}{\cdots} \oplus \mathbb{Z}_2$.

Using the vector field $2\widetilde{\zeta}$ (see (3.2)) and the fact that $f_2 = (f_1)^2$, by a similar device to used for the manifold $M_1(n)$, we obtain that $M_2(n)$ is also a compact solvmanifold.

CASE B: Let $\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ be the canonical global basis of vector fields on $\mathbb{T}^{2n}$ and $\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n\}$ its dual basis of 1-forms. Denote by α_i' and β_i' the 1-forms on $\mathbb{T}^{2n}$ given by

$$\alpha_i' = \alpha_i + \cos\frac{\pi}{3}\beta_i, \qquad\qquad \beta_i' = -\sin\frac{\pi}{3}\beta_i,$$

for all $i \in \{1, \dots, n\}$. If $\{X_1', \dots, X_n', Y_1', \dots, Y_n'\}$ is the dual basis of vector fields of the basis of 1-forms $\{\alpha_1', \dots, \alpha_n', \beta_1', \dots, \beta_n'\}$ then we consider on $\mathbb{T}^{2n}$ the Kähler structure (J', h') defined by

$$J'X_i' = -Y_i', \qquad J'Y_i' = X_i', \qquad h' = \sum_{j=1}^{n}(\alpha_j' \otimes \alpha_j' + \beta_j' \otimes \beta_j').$$

Case B1: The diffeomorphism $\widetilde{f_1'} : I\!R^{2n} \to I\!R^{2n}$ defined by

$$\widetilde{f_1'}(x_1, \dots, x_n, y_1, \dots, y_n) = (-y_1, \dots, -y_n, x_1 + y_1, \dots, x_n + y_n),$$

for all $(x_1, \dots, x_n, y_1, \dots, y_n) \in I\!R^{2n}$, induces an Hermitian isometry $f_1' : (\mathbb{T}^{2n}, J', h') \to (\mathbb{T}^{2n}, J', h')$.

Denote by $M_1'(n)$ the suspension with fibre $\mathbb{T}^{2n}$ of the representation $\varrho_1' : \mathbb{Z} \to Diff(\mathbb{T}^{2n})$ given by $\varrho_1'(k) = (f_1')^k$, for all $k \in \mathbb{Z}$. We have that the fundamental group of $M_1'(n)$, $\pi_1(M_1'(n))$, is the semidirect product

$$\pi_1(M_1'(n)) = \mathbb{Z}^{2n} \times_{\psi_1'} \mathbb{Z}, \tag{3.6}$$

where $\psi_1' : \mathbb{Z} \to Aut(\mathbb{Z}^{2n})$ is the homomorphism defined by $\psi_1'(k) = ((\widetilde{f_1'})_{|Z^{2n}})^{-k}$ for all $k \in \mathbb{Z}$.

From (3.6), we deduce that the commutator subgroup of $\pi_1(M_1'(n))$ is

$$[\pi_1(M_1'(n)), \pi_1(M_1'(n))] = \mathbb{Z}^{2n} \times \{0\}.$$

This implies that the first integral homology group $H_1(M_1'(n), \mathbb{Z})$ is $\mathbb{Z}$.

Next, we shall describe the manifold $M_1'(n)$ as a compact solvmanifold.

Let $(\widetilde{J'}, \widetilde{h'})$ be the induced Kähler structure on $I\!R^{2n}$ by the Kähler structure (J', h') of $\mathbb{T}^{2n} = I\!R^{2n}/\mathbb{Z}^{2n}$. Then, the vector field $\widetilde{\zeta'}$ on $I\!R^{2n}$ given by

$$\widetilde{\zeta'} = \sum_{i=1}^{n} \frac{\pi}{3}\sin\frac{\pi}{3}\left(\left(\frac{4}{3}y_i + \frac{2}{3}x_i\right)\frac{\partial}{\partial x_i} - \left(\frac{4}{3}x_i + \frac{2}{3}y_i\right)\frac{\partial}{\partial y_i}\right) \tag{3.7}$$

in an infinitesimal automorphism of the structure $(\widetilde{J'}, \widetilde{h'})$. In fact, if $\tilde{\psi}' : I\!R \times I\!R^{2n} \to I\!R^{2n}$ is the flow of $\widetilde{\zeta'}$ we have that

$$(3.8)\quad \widetilde{\psi}'(z,(x_1,\ldots,x_n,y_1,\ldots,y_n)) = \\ = (x_1\theta(z+1)+y_1\theta(z),\ldots,x_n\theta(z+1)+y_n\theta(z), \\ -x_1\theta(z)-y_1\theta(z-1),\ldots,-x_n\theta(z)-y_n\theta(z-1)),$$

where $\theta : I\!R \to I\!R$ is the map defined by $\theta(z) = \frac{4}{3}\sin\frac{\pi}{3}\sin(\frac{\pi}{3}z)$, for all $z \in I\!R$. Thus, the diffeomorphism

$$\widetilde{\psi}'(1) : I\!R^{2n} \to I\!R^{2n}$$

given by $\widetilde{\psi}'(1)(x_1,\ldots,x_n,y_1,\ldots,y_n) = \widetilde{\psi}'(1,(x_1,\ldots,x_n,y_1,\ldots,y_n))$ is just the map $(\widetilde{f}_1')^{-1}$. Moreover, for all $z \in I\!R$, the diffeomorphism $\widetilde{\psi}'(z) : I\!R^{2n} \to I\!R^{2n}$ is an automorphism of $I\!R^{2n}$. Consequently, the map $\widetilde{\psi}'$ induces a Lie group homomorphism of $I\!R$ into the automorphism group of $I\!R^{2n}, Aut(I\!R^{2n})$, which we also denote by $\widetilde{\psi}'$.

Now, let $I\!R^{2n} \times_{\widetilde{\psi}'} I\!R$ be the semidirect product defined by the homomorphism $\widetilde{\psi}' : I\!R \to Aut(I\!R^{2n})$. From (3.8), we deduce that a basis for the left invariant vector fields is given by

$$X_i' = \theta(z+1)\frac{\partial}{\partial x_i} - \theta(z)\frac{\partial}{\partial y_i},\quad Y_i' = \theta(z)\frac{\partial}{\partial x_i} - \theta(z-1)\frac{\partial}{\partial y_i},\quad Z' = \frac{\partial}{\partial z},$$

for all $i \in \{1,\ldots,n\}$. Then, for all $i \in \{1,\ldots,n\}$,

$$(3.9)\quad [X_i',Z'] = \frac{\pi}{3}\sin\frac{\pi}{3}(\frac{4}{3}Y_i' - \frac{2}{3}X_i'),\qquad [Y_i',Z'] = \frac{\pi}{3}\sin\frac{\pi}{3}(\frac{2}{3}Y_i' - \frac{4}{3}X_i'),$$

and the other brackets being zero. Using (3.9), we conclude that $I\!R^{2n} \times_{\widetilde{\psi}'} I\!R$ is a $(2n+1)$-dimensional simply connected solvable non-nilpotent Lie group.

On the other hand, since $\widetilde{\psi}'(k)_{|Z^{2n}} = \psi_1'(k)$ for all $k \in \mathbb{Z}$, we obtain that the fundamental group $\pi_1(M_1'(n))$ of $M_1'(n)$ is a discrete subgroup of $I\!R^{2n} \times_{\widetilde{\psi}'} I\!R$.

Finally, it is easy to prove that the compact solvmanifold $\pi_1(M_1'(n))\backslash(I\!R^{2n}\times_{\widetilde{\psi}'} I\!R)$ is diffeomorphic to the suspension $M_1'(n)$.

Case B2: We consider the diffeomorphism $\widetilde{f}_2' : I\!R^{2n} \to I\!R^{2n}$ defined by

$$\widetilde{f}_2'(x_1,\ldots,x_n,y_1,\ldots,y_n) = (-x_1-y_1,\ldots,-x_n-y_n,x_1,\ldots,x_n),$$

for all $(x_1,\ldots,x_n,y_1,\ldots,y_n)\in I\!R^{2n}$. Then, the diffeomorphism $\widetilde{f}_2'$ induces an Hermitian isometry $f_2' : (\mathbb{T}^{2n},J',h') \to (\mathbb{T}^{2n},J',h')$.

We denote by $M_2'(n)$ the suspension with fibre $\mathbb{T}^{2n}$ of the representation $\varrho_2' : \mathbb{Z} \to Diff(\mathbb{T}^{2n})$ given by $\varrho_2'(k) = (f_2')^k$, for all $k \in \mathbb{Z}$. The fundamental group of $M_2'(n)$, $\pi_1(M_2'(n))$, is the semidirect product

$$(3.10)\qquad \pi_1(M_2'(n)) = \mathbb{Z}^{2n} \times_{\psi_2'} \mathbb{Z},$$

where $\psi_2' : \mathbb{Z} \to Aut(\mathbb{Z}^{2n})$ is the homomorphism defined by $\psi_2'(k) = ((\widetilde{f_2'})_{|Z^{2n}})^{-k}$ for all $k \in \mathbb{Z}$.

From (3.10), we deduce that the commutator subgroup of $\pi_1(M_2'(n))$ is

$$[\pi_1(M_2'(n)), \pi_1(M_2'(n))] = \\ = \{(p_1, \ldots, p_n, q_1, \ldots, q_n, 0) \in \mathbb{Z}^{2n+1} / (p_1 - q_1, \ldots, p_n - q_n) \in (3\mathbb{Z})^n\}.$$

Thus, the first integral homology group $H_1(M_2'(n), \mathbb{Z})$ is $\mathbb{Z} \oplus \mathbb{Z}_3 \oplus \overset{n)}{\cdots} \oplus \mathbb{Z}_3$.

Similarly to the case B1, since $f_2' = (f_1')^2$, if we consider the vector field $2\widetilde{\zeta}'$ (see (3.7)) on $\mathbb{R}^{2n}$, we obtain that $M_2'(n)$ is also a compact solvmanifold.

From (3.1), (3.5), (3.6) and (3.10), we deduce that the fundamental group of the manifolds $M_i(n), M_i'(n)$ $(i = 1, 2)$ is not abelian. Therefore, using Remark 2.1 and the results obtained in this Section, we conclude that

Theorem 3.1. *The manifolds $M_1(n), M_2(n), M_1'(n)$ and $M_2'(n)$ are $(2n + 1)$-dimensional compact flat cosymplectic solvmanifolds which are not topologically equivalent to the compact cosymplectic manifolds $\mathbb{T}^{2m+1} \times \mathbb{C}P^r$, with $m, r \geq 0$ and $m + r = n$.*

Moreover, in the particular case $n = 1$, we have

Theorem 3.2. *Let M be a 3-dimensional compact flat orientable Riemannian manifold and $b_1(M)$ its first Betti number.*

i) *If $b_1(M) \neq 0$ then M admits a cosymplectic structure.*
ii) *If $b_1(M) = 1$ then M is not topologically equivalent to the global product of a compact Kähler manifold with S^1.*

Proof. Two 3-dimensional compact flat orientable Riemannian manifolds are diffeomorphic if their first integral homology groups are isomorphic (see [6]). Moreover, if M is a 3-dimensional compact flat orientable Riemannian manifold and $b_1(M) \neq 0$ then the first integral homology group of M is $\mathbb{Z} \oplus \mathbb{Z}_2$, $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}_3$ or $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ (see [6]). Consequently, M is either diffeomorphic to $M_1(1)$, or to $M_2(1)$, or to $M_1'(1)$, or to $M_2'(1)$, or to the 3-dimensional real torus $\mathbb{T}^3$. This implies that M admits a cosymplectic structure (see Theorem 3.1), which proves *i*).

Now, if $b_1(M) = 1$ then M is either diffeomorphic to $M_1(1)$, or to $M_2(1)$, or to $M_1'(1)$, or to $M_2'(1)$. Thus, if M is homeomorphic to the global product of a 2-dimensional compact Kähler manifold N with S^1 then the first Betti number of N is zero. Hence N would be diffeomorphic to the 2-dimensional unit sphere S^2 which yields a contradiction with the fact that the fundamental group of M is not abelian (see (3.1), (3.5), (3.6) and (3.10)). □

Remark 3.1.

(1) If M is a compact cosymplectic manifold and $b_p(M)$ is the pth Betti number of M, $0 \leq p \leq dim M$, then $b_p(M) \neq 0$ (see [2] and [3]). In particular, $b_1(M) \neq 0$.

(2) In [3], the authors obtain an example of 3-dimensional compact cosymplectic manifold which is not topologically equivalent to the global product of a compact Kähler manifold with S^1. Such a manifold is just the 3-dimensional compact flat cosymplectic solvmanifold $M_1(1)$.

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