ARCHIVUM MATHEMATICUM (BRNO)
Tomus 34 (1998), 393 – 399

ON A CONJECTURE ABOUT AN INTEGRAL CRITERION FOR OSCILLATION

Uri Elias, Anton Škerlík

Abstract. We discuss an open question of Kiguradze and Chanturia about Property A and Property B for the equation $y^{(n)} + py = 0$. The proposed integral criterion is proved in a few cases.

1. Introduction.

The aim of this note is to give some partial answers to a conjecture about an integral criterion for the oscillation of the differential equation

$$y^{(n)} + p(t)y = 0, \qquad a \le t \le \infty, \tag{1}$$

with $p(t)$ of a fixed sign.

It is obvious that if $p(t) \le 0$ then each initial value problem $y^{(i)}(t_0) > 0$, $t_0 \ge a$, $i = 0, 1, \dots, n-1$ yields an increasing solution of (1) such that

$$y^{(i)} > 0, \qquad i = 0, 1, \dots, n, \qquad \text{on } [t_0, \infty). \tag{2}$$

It is also known that if $(-1)^n p(t) \le 0$ then (1) has at least one decreasing solution on the whole domain of definition such that

$$(-1)^i y^{(i)} > 0, \qquad i = 0, 1, \dots, n, \qquad \text{on } [a, \infty). \tag{3}$$

The existence of solutions of type (2),(3) depends on nothing else but the parity of n and the sign of $p(t)$. That is clearly not the case for the rest of the solutions. They may be oscillatory or nonoscillatory, depending on other circumstances, say the size of $p(t)$. This suggests, traditionally, the classification of (1) according to Property A and Property B. If $p(t) \ge 0$, (1) is said to have Property A if for even n each of its solutions is oscillatory and for odd n either is oscillatory or satisfies (3). If $p(t) \le 0$, we say that (1) has Property B if for odd n every solution either is oscillatory or satisfies (2) and for even n either is oscillatory or satisfies (2) or satisfies (3).

The following conjecture [K. p. 29, Problem 1.14] states sufficient conditions for Properties A and B:

1991 *Mathematics Subject Classification*: 34C10.
Key words and phrases: oscillation, Property A, Property B.
Received October 23, 1997.

Conjecture. *If $p(t) \geq 0$ and*

$$\int^{\infty} t^{n-1}\big[p(t) - \frac{M_{n*}}{t^n}\big]\,dt = +\infty \tag{4}$$

where $M_{n*} = \max_{[0,n-1]}\big(\lambda(\lambda-1)\dots(\lambda-n+1)\big)$, then (1) has Property A.
If $p(t) \leq 0$ and

$$\int^{\infty} t^{n-1}\big[|p(t)| - \frac{M_n^*}{t^n}\big]\,dt = +\infty \tag{5}$$

where $M_n^* = \max_{[0,n-1]}\big(-\lambda(\lambda-1)\dots(\lambda-n+1)\big)$, then (1) has Property B.

The conjecture is verified for $n = 3$ by Škerlík [S1,S2]. Our aim is to prove it in additional cases.

The conjecture is intimately connected to the concept of $(k, n-k)$-disfocality. Recall that (1) is called $(k, n-k)$-disfocal on an interval I if no nontrivial solution satisfies the boundary value conditions

$$\begin{aligned} y^{(i)}(\alpha) &= 0, \qquad i = 0, \dots, k-1, \\ y^{(j)}(\beta) &= 0, \qquad j = k, \dots, n-1 \end{aligned} \tag{6}$$

for any $\alpha, \beta \in I$, $\alpha < \beta$. It is known that (1) is $(k, n-k)$-disfocal on I if and only if it has a solution y such that

$$\begin{aligned} y^{(i)} &> 0, \qquad i = 0, \dots, k-1, \\ (-1)^{j-k} y^{(j)} &> 0, \qquad j = k, \dots, n-1 \end{aligned} \tag{7}$$

on I. We say that (1) is eventually $(k, n-k)$-disfocal if it is $(k, n-k)$-disfocal on some (c, ∞).

On the other hand, it is also known that every nonoscillatory solution of (1) on (a, ∞) satisfies inequalities (7) for some k and on some subinterval (c, ∞). Of course, the parity of k in (7) must be compatible with the sign of $p(t)$, namely

$$(-1)^{n-k} p(t) \leq 0. \tag{8}$$

Thus, (7) links nonoscillation with various types of disfocality.

After these introductory notes we are ready to return to the conjecture. As mentioned above, every nonoscillatory solution eventually satisfies some inequality (7), which in turn is related to a certain type $(k, n-k)$-disfocality. Thus it is clear that (1) has Property A (Property B) if and only if it is not eventually $(k, n-k)$-disfocal for any k, $1 \leq k \leq n-1$, $(-1)^{n-k} p \leq 0$. Our approach is, consequently, to break up Properties A, B to their components of disfocality and treat each of them separately. We prefer to restate the conjecture in a more detailed form:

Conjecture. *If* $(-1)^{n-k}p(t) \le 0$ *and*

$$\int^{\infty} t^{n-1}\big[|p(t)| - \frac{M_{k,n}}{t^n}\big]\,dt = +\infty \tag{9}$$

where

$$M_{k,n} = \max_{[k-1,k]} |\lambda(\lambda-1)\dots(\lambda-n+1)|,$$

then (1) *cannot be* $(k, n-k)$*-disfocal on any* (c,∞).

If we maximize the left hand size of (9) for the values of k which satisfy the parity condition (8), we get either case (4) or (5) of the original conjecture, depending on the sign of p. Recall that if (1) is $(k, n-k)$-disfocal, it is also $(k-2, n-k+2)$-disfocal if $k \le (n+1)/2$ and it is $(k+2, n-k-2)$-disfocal if $k \ge (n-1)/2$ [E, Theorem 7.12].

(9) measures the distance from equation (1) to the Euler equation

$$y^{(n)} + \frac{\mu}{t^n} y = 0, \qquad 1 \le t < \infty. \tag{10}$$

Equation (10) with $(-1)^{n-k}\mu < 0$ is eventually $(k, n-k)$-disfocal if and only if

$$0 \le (-1)^{n-k-1}\mu \le M_{k,n}. \tag{11}$$

It is well known that (11) implies the existence of a solution $y = t^{\lambda}$ of type (7) and thus it is sufficient for the $(k, n-k)$-disfocality of (10) on $(0,\infty)$. The necessity of (11) for eventual $(k, n-k)$-disfocality is not trivial and it is proved in [E, Theorem 6.24]. Another related result is that (1) is eventually $(k, n-k)$-disfocal even if

$$0 \le (-1)^{n-k-1}p(t) \le \frac{M_{k,n}}{t^n} + \mathcal{O}\big(\frac{1}{t^{n+\varepsilon}}\big), \qquad \varepsilon > 0,$$

[E, Lemma 6.26].

Note that in the study of the conjecture we have to consider only the case $\int^{\infty} t^{n-1}|p|\,dt = \infty$, since otherwise, if $\int^{\infty} t^{n-1}|p|\,dt < \infty$, the solutions of (1) are asymptotic to those of $y^{(n)} = 0$, and are automatically nonoscillatory.

2. We shall adopt the method of [S1, S2] to prove additional cases of the conjecture.

Proposition. *The conjecture holds for* $n = 2$, $n = 3$, $p > 0$, $n = 3$, $p < 0$, $n = 4$, $p < 0$.

Proof. Suppose that (1) is $(k, n-k)$-disfocal, say for simplicity, on $[0,\infty)$, for some k, $1 \le k \le n-1$, which obeys (8). Then it has a solution y which satisfies inequalities (7) on $(0,\infty)$. Moreover, y may be taken so that it also satisfies the boundary conditions

$$y^{(i)}(0) = 0, \qquad i = 0, \dots, k-1. \tag{12}$$

We need the following facts:

(i) *If a function $y(t)$ satisfies $y^{(i)} > 0$, $i = 0, \dots, k$, $y^{(k+1)} < 0$, on $(0, \infty)$, then*

$$\frac{y}{t^k/k!} \geq \frac{y'}{t^{k-1}/(k-1)!}.$$

(ii) *If, in addition, $y(t)$ satisfies (12), then also*

$$\frac{y}{t^{k-1}/(k-1)!} \leq \frac{y'}{t^{k-2}/(k-2)!}.$$

Part (i) is the lemma of Kiguradze. (ii) is proved in [E, Corollary 6.15]. Thus, our solution y satisfies

$$k - 1 \leq t\,\frac{y'}{y} \leq k, \qquad a < t < \infty. \tag{13}$$

(a) Let us summarize Škerlík's proof for (1) with $n = 3$, $p \geq 0$, slightly modified. ([S1] discusses, in fact, the more general equation $y''' + py' + qy = 0$). Due to (13), let

$$z(t) = t\frac{y'}{y}.$$

Then

$$z' = \frac{y'}{y} + t\Big(\frac{y''}{y} - \frac{y'^2}{y^2}\Big) = t\,\frac{y''}{y} + \frac{1}{t}(z - z^2). \tag{14}$$

Next,

$$\begin{aligned}
(tz')' &= \frac{d}{dt}\Big(t^2\frac{y''}{y} + z - z^2\Big) \\
&= t^2\Big(\frac{y'''}{y} - \frac{y''y'}{y^2}\Big) + 2t\frac{y''}{y} + \frac{d}{dt}(z - z^2) \\
&= t^2\,\frac{y'''}{y} + \Big(t\frac{y''}{y}\Big)\Big(-t\,\frac{y'}{y} + 2\Big) + \frac{d}{dt}(z - z^2)
\end{aligned}$$

which is by (14)

$$= t^2\,\frac{y'''}{y} + \Big(\frac{dz}{dt} + \frac{z(z-1)}{t}\Big)(-z + 2) + \frac{d}{dt}(z - z^2)$$

and with $\frac{dz}{dt}(-z + 2) = \frac{d}{dt}(-\frac{1}{2}z^2 + 2z)$,

$$= t^2\,\frac{y'''}{y} - \frac{z(z-1)(z-2)}{t} + \frac{d}{dt}\Big(3z - \frac{3}{2}z^2\Big). \tag{15}$$

Suppose that $y''' + py = 0$, $p > 0$, is (2,1)-disfocal on $[0, \infty)$ and nevertheless (9) holds with $n = 3$, $k = 2$. Take the solution y of (1) which satisfies (7),(12) and the corresponding function $z(t)$. Then by (13), $1 \leq z \leq 2$,

$$0 \leq -z(z-1)(z-2) \leq \max_{[1,2]} |\lambda(\lambda - 1)(\lambda - 2)| = M_{2,3} = \frac{2}{3\sqrt{3}},$$

and, of course, $y'''/y = -p$. (15) becomes

$$(tz')' \le -t^2 p + \frac{M_{2,3}}{t} + \frac{d}{dt}(3z - \frac{3}{2}z^2),$$

$$tz'\Big|_1^t \le \int_1^t s^2\Big[-p(s) + \frac{M_{2,3}}{s^3}\Big]\,ds + (3z - \frac{3}{2}z^2)\Big|_1^t.$$

It follows from assumption (9) and the bounds $1 \le z \le 2$, that

$$tz' \le c_1 - \int_1^t s^2\Big[p(s) - \frac{M_{2,3}}{s^3}\Big]\,ds \to -\infty,$$

and $tz' \le c_2 < 0$ for sufficiently large values of t. Another integration of $z' \le c_2/t$ leads to $z(t) \le c_3 + c_2\ln(t) \to -\infty$, contradicting $z \ge 1$.

(b) This proof fits also to $n = 3$, $p \le 0$ with minor modifications. Suppose that $y''' + py = 0$, $p < 0$, is (1,2)-disfocal while (9) holds with $n = 3$, $k = 1$. Now $y, y' > 0, y'' < 0, y''' > 0$ and $0 \le z \le 1$,

$$0 \le z(z-1)(z-2) \le \max_{[0,1]}|\lambda(\lambda-1)(\lambda-2)| = M_{1,3} = \frac{2}{3\sqrt{3}},$$

and (15) becomes

$$(tz')' \ge t^2|p(t)| - \frac{M_{1,3}}{t} + \frac{d}{dt}(3z - \frac{3}{2}z^2).$$

As above,

$$tz' \ge c_1 + \int_1^t s^2\Big[|p(s)| - \frac{M_{2,3}}{s^3}\Big]\,ds \to +\infty,$$

and the next integration yields a contradiction with $z \le 1$.

(c) The same argument easily settles the conjecture for $n = 2, p > 0$. If $y'' + py = 0$, $p > 0$, is (1,1)-disfocal then $y, y' > 0$, $y'' < 0$ and $0 \le z \le 1$. (14) becomes $z' \le -tp(t) + M_{1,2}/t$, with $M_{1,2} = \max_{[0,1]} z(1-z) = 1/4$. If (9) holds with $n = 2$, $k = 1$, then

$$z(t) - z(1) \le -\int_1^t s\Big[p(s) - \frac{M_{1,2}}{s^2}\Big]\,ds \to -\infty,$$

contradicting the boundedness of $z(t)$.

(d) Now we turn to $n = 4$. Differentiating (15)

$$\begin{aligned}(t(tz')')' &= \frac{d}{dt}\Big(t^3\frac{y'''}{y} - z(z-1)(z-2) + t\frac{d}{dt}(3z - \frac{3}{2}z^2)\Big)\\ &= t^3\big(\frac{y^{iv}}{y} - \frac{y'''y'}{y^2}\big) + 3t^2\frac{y'''}{y} + \frac{d}{dt}z(z-1)(z-2) + \frac{d}{dt}\big(t\frac{d}{dt}(3z - \frac{3}{2}z^2)\big)\\ &= t^3\,\frac{y^{iv}}{y} + \big(3 - t\,\frac{y'}{y}\big)\big(t^2\frac{y'''}{y}\big) + \frac{d}{dt}P_1(z) + \frac{d}{dt}\big(t\frac{d}{dt}P_2(z)\big)\end{aligned}$$

Again by (15)

$$= t^3\,\frac{y^{iv}}{y} + (3-z)\Big((tz')' + \frac{z(z-1)(z-2)}{t} - \frac{d}{dt}(3z - \frac{3}{2}z^2)\Big) + \frac{d}{dt}P_1(z) + \frac{d}{dt}\big(t\frac{d}{dt}P_2(z)\big)$$

where $P_1(z), P_2(z)$ are polynomials. But

$$(3-z)\frac{d}{dt}(3z - \frac{3}{2}z^2) = \frac{d}{dt}P_3(z),$$

$$(3-z)(tz')' = \frac{d}{dt}\big(t(3-z)\frac{dz}{dt}\big) + tz'^2 = \frac{d}{dt}\big(t\frac{d}{dt}(3z - \frac{3}{2}z^2)\big) + tz'^2,$$

so

$$(t(tz')')' = t^3\,\frac{y^{iv}}{y} - \frac{z(z-1)(z-2)(z-3)}{t} + \frac{d}{dt}P_4(z) + \frac{d}{dt}\big(t\frac{d}{dt}P_5(z)\big) + tz'^2. \quad (16)$$

Suppose that the fourth order equation $y^{iv} + py = 0$, $p < 0$ is (2,2)-disfocal. Then $1 \le z \le 2$,

$$0 \le z(z-1)(z-2)(z-3) \le \max_{[1,2]} |\lambda(\lambda-1)(\lambda-2)(\lambda-3)| = M_{2,4} = \frac{9}{16},$$

and (16) becomes

$$(t(tz')')' \ge -t^3 p(t) - \frac{M_{2,4}}{t} + \frac{d}{dt}P_4(z) + \frac{d}{dt}\big(t\frac{d}{dt}P_5(z)\big).$$

After an integration it follows from the boundedness of z that

$$t(tz')' \ge c + t\frac{d}{dt}P_5(z) + \int_1^t s^3\big[|p(s)| - \frac{M_{2,4}}{s^4}\big]\,ds.$$

If the last integral diverges to $+\infty$ then for arbitrary $K_1 > 0$,

$$(tz')' \ge \frac{K_1}{t} + \frac{d}{dt}P_5(z), \qquad t \ge t_0.$$

Two more integrations lead to the conclusion that $z(t) \to +\infty$ as $t \to \infty$, contradicting $z \le 2$.

Analogous proofs are available if $(tz')'$, $(t(tz')')'$ are replaced by $(tz)''$, $(t^2z)'''$, respectively. □

The conjecture could have been proved for $n = 4$, $p > 0$, i.e., $k = 1$ or $k = 3$, if we could show that $\int^\infty tz'^2\,dt < \infty$. Once this is done, all one needs is to estimate (16) from above and reverse the last inequalities in the previous proof.

Inequality (13) gives way to hope that the same method may be generalized to other values of n, k. Unfortunately, we did not succeed to analyze similarly the identities

$$(t(t\ldots(tz')'\ldots)')' = t^{n-1}\,\frac{y^{(n)}}{y} - \frac{z(z-1)\ldots(z-n+1)}{t} + \ldots$$

or

$$(t^{n-2}z)^{(n-1)} = t^{n-1}\,\frac{y^{(n)}}{y} - \frac{z(z-1)\ldots(z-n+1)}{t} + \ldots,$$

where the omitted terms all vanish when $z = \text{const}$.

References

[K] Kiguradze, I. T. and Chanturia, T. A., *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Kluwer Publishers, Dordrecht, 1993.

[E] Elias, U., *Oscillation theory of two-term differential equations*, Kluwer Publishers, Dordrecht, 1997.

[S1] Škerlík, A., *An integral condition of oscillation for the equation* $y''' + p(t)y' + q(t)y = 0$ *with nonnegative coefficients*, Arch. Math. (Brno) **31** (1995), 151-161.

[S2] Škerlík, A., *An integral condition of Property A and B for linear differential equation of third order*, Proceedings of the Conference on Ordinary Differential Equations Poprad, Slovac Republic (1994), 81-85.

U. Elias
Department of Mathematics, Technion
Haifa 32000, ISRAEL
E-mail: elias@techunix.technion.ac.il

A. Škerlík
Department of Mathematics, Technical University
Letná 9, 041 54 Košice, SLOVAKIA
E-mail: skerlik@ccsun.tuke.sk