ARCHIVUM MATHEMATICUM (BRNO)
Tomus 34 (1998), 353 – 359

PROPERTIES OF A NEW CLASS OF RECURSIVELY DEFINED BASKAKOV-TYPE OPERATORS

Octavian Agratini

Dedicated to Prof. D.D. Stancu on his 70th anniversary

Abstract. By starting from a recent paper by Campiti and Metafune [7], we consider a generalization of the Baskakov operators, which is introduced by replacing the binomial coefficients with other coefficients defined recursively by means of two fixed sequences of real numbers. In this paper, we indicate some of their properties, including a decomposition into an expression which depends linearly on the fixed sequences and an estimation of the corresponding order of approximation, in terms of the modulus of continuity.

1. Introduction

The connections between approximation processes and evolution problems through semigroup theory were deepened in the last years by Altomare [2] and Campiti [6]. At the same time, the class of evolution equations whose solutions can be approximated by constructive approximation processes has been enlarged. In this way, new types of operators have been introduced and studied, see [3], [4].

In their paper [7], published in 1996, Campiti and Metafune considered a modification of the classical Bernstein operators. It was obtained by replacing the binomial coefficients with general ones satisfying a suitable recursive relation. They have obtained for these new operators A_n some quantitative estimations and regulary properties. In their subsequent paper [8] they explored the connections with the semigroup theory and showed that these operators can be used to approximate the solutions of some degenerate second order parabolic problems.

In the present paper we introduce a similar modification of the Baskakov operators and examine the main properties of this new approximation process. If the operators studied in the papers [7], [8] were in connection with functions defined on a bounded interval, now these operators are related to functions defined on an unbounded interval. We obtain a decomposition of the Baskakov operator as a sum of elementary operators and our operators are expressed by a linear combination of these last ones. Also, we prove that the sequences of linear operators obtained

1991 *Mathematics Subject Classification*: 41A35, 41A25, 26D15.
Key words and phrases: Baskakov-type operators, order of approximation, modulus of continuity.
Received March 24, 1997.

here, converge generally not towards the identity operator but towards an operator multiplied by an analytic function.

2. CONSTRUCTION OF THE BASKAKOV-TYPE OPERATORS

We consider two fixed sequences of real numbers: $a = (a_n)_{n\geq 1},\ b = (b_k)_{k\geq 0}$ with $a_1 = b_0$ and the numbers $c_{n,k}$ $(n \geq 1,\ k \geq 0)$ which satisfy the following recursive formulae:

$$c_{n,0} = a_n \ \ n \geq 1; \quad c_{1,k} = b_k,\ \ k \geq 0;$$

$$c_{n+1,k} = c_{n,k} + c_{n+1,k-1}, \quad n \geq 1 \text{ and } k \geq 1. \tag{1}$$

It is clear that these coefficients $c_{n,k}$ are determined uniquely by a and b. According to these sequences, we define the operators $L_n^{\langle a,b\rangle}$ as follows

$$(L_n^{\langle a,b\rangle} f)(x) = \sum_{k=0}^{\infty} c_{n,k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right), \tag{2}$$

where f belongs to $C_B[0,\infty)$, the space of real functions continuous and bounded on $[0,\infty)$.

We observe that $L_n^{\langle a,b\rangle}$ are linear operators. If the sequences a, b are non-negative, then these operators preserve the positivity of the function f. It is easy to check that if $a_n = \lambda,\ n \geq 1$, and $b_k = \lambda,\ k \geq 0$, then

$$c_{n,k} = \lambda \binom{n+k-1}{k}$$

and

$$(L_n^{\langle \lambda,\lambda\rangle} f)(x) = \lambda (V_n f)(x), \tag{3}$$

where V_n represent the Baskakov operators [5].

We recall some useful relations which are fulfilled by V_n:

$$(V_n e_0)(x) = 1, \quad (V_n e_1)(x) = x, \quad (V_n e_2)(x) = x^2 + \frac{x(1+x)}{n}, \tag{4}$$

where $e_j(x) = x^j,\ j = 0, 1, 2$ and $x \geq 0$.

Many papers investigating and generalizing the operators V_n of Baskakov were published. We mention the extension to factorial powers of V_n given by D.D. Stancu in the papers [9], [10], as well as their generalization included in our recent paper [1].

3. PROPERTIES OF $L_n^{\langle a,b\rangle} f$

Lemma 1. *Let $f \in C_B[0,\infty)$ such that $f \geq 0$ and consider the following sequences:* $a = (a_n)_{n\geq 1}, \alpha = (\alpha_n)_{n\geq 1},\ b = (b_k)_{k\geq 0},\ \beta = (\beta_k)_{k\geq 0}$.

(i) *If $a \le \alpha$ and $b \le \beta$ (i.e. $a_n \le \alpha_n,\ n \ge 1$ and $b_k \le \beta_k,\ k \ge 0$) then*

$$L_n^{\langle a,b\rangle} f \le L_n^{\langle \alpha,\beta\rangle} f. \tag{5}$$

(ii) *If there exists:* $\max\left\{\sup_{n\ge1} a_n,\ \sup_{k\ge0} b_k\right\} = M < \infty$, *then*

$$L_n^{\langle a,b\rangle} f \le M(V_n f). \tag{6}$$

Proof. Under the given assumptions, if we denote $c_{n,k}(a,b)$ those coefficients which are obtained from the sequences a, b, it follows $c_{n,k}(a,b) \le c_{n,k}(\alpha,\beta)$ and this implies (5) for every positive $f \in C_B[0,\infty)$.

Because $a \le M$ and $b \le M$, taking into account (5) and (3), the relation (6) follows immediately.

Further on, we shall use the symbol of Kronecker $\delta_{n,m}$.

Theorem 1. *For $f \in C_B[0,\infty)$ and $n \ge 1$ the following identity*

$$(L_n^{\langle a,b\rangle} f)(x) = \sum_{m=1}^{\infty} a_m (A_{m,n} f)(x) + \sum_{m=0}^{\infty} b_m (B_{m,n} f)(x) \tag{7}$$

holds, where:

$$(A_{1,n} f)(x) + (B_{0,n} f)(x) = \frac{\delta_{1,n}}{1+x} f(0); \tag{8}$$

for any $m \ge 2$,

$$(A_{m,n} f)(x) = \begin{cases} 0, & m \ge n+1; \\ \sum\limits_{k=0}^{\infty} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right), & m = n; \\ \sum\limits_{k=1}^{\infty} \binom{n-m+k-1}{k-1} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right), & m \le n-1; \end{cases} \tag{9}$$

for any $m \ge 1$,

$$(B_{m,n} f)(x) = \frac{x^m}{(1+x)^{n+m}} f\left(\frac{m}{n}\right) + \tag{10}$$

$$+(1-\delta_{1,n}) \sum_{k=m+1}^{\infty} \binom{n+k-m-2}{k-m} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right).$$

We call $A_{m,n}$ and $B_{m,n}$ the m^{th} left respectively right operators associated to the operator $L_n^{\langle a,b\rangle}$, $m \in N$.

Proof. Using (1) and (2) we deduce that $L_n^{\langle a,b\rangle}$ depends linearly on a and b. In order to find the elementary operators $A_{m,n}$, $B_{m,n}$ which are associated to the sequences

a and b, firstly we choose $b=0$ and $a=\delta_m$ $(m\geq 2)$ where $\delta_m=(\delta_{m,n})_{n\geq 1}$. We obtain $A_{m,n}f=L_n^{\langle\delta_m,0\rangle}f$. The coefficients $c_{n,k}=c_{n,k}^{(m,0)}$ have the following form:

$$c_{n,k}^{(m,0)}=\begin{cases} 0, & n\leq m-1 \text{ and } k\geq 0, \\ & \quad\text{or} \\ & n\geq m+1 \text{ and } k\geq 0; \\ 1, & n=m \text{ and } k\geq 1; \\ \binom{n-m+k-1}{k-1}, & n\geq m+1 \text{ and } k\geq 1. \end{cases}$$

By (2) and the above identity we arrive at (9).

Secondly, we choose $a=0$ and $b=\delta_m$ $(m\geq 1)$ where $\delta_m=(\delta_{mk})_{k\geq 0}$ and obtain $B_{m,n}f=L_n^{\langle 0,\delta_m\rangle}f$. The new coefficients $c_{n,k}=c_{n,k}^{(0,m)}$ are given by

$$c_{n,k}^{(0,m)}=\begin{cases} 0, & k\leq m-1 \text{ and } n\geq 1, \\ & \quad\text{or} \\ & k\geq m+1 \text{ and } n=1; \\ 1, & k=m \text{ and } n\geq 1; \\ \binom{n+k-m-2}{k-m}, & n\geq 2 \text{ and } k\geq m+1; \end{cases}$$

and we get (10).

Finally, we choose a,b such as $a_1=b_0\neq 0$ and all other terms vanish. We can write

$$\sum_{k=0}^{\infty} c_{n,k}\frac{x^k}{(1+x)^{n+k}}f\left(\frac{k}{n}\right)=a_1(A_{1,n}f)(x)+b_0(B_{0,n}f)(x),\quad n\geq 1.$$

But $c_{1,0}=a_1$ and all others vanish. We have

$$\frac{f(0)}{1+x}=(A_{1,1}f)(x)+(B_{0,1}f)(x),$$

and

$$0=(A_{1,n}f)(x)+(B_{0,n}f)(x),\quad \text{for}\quad n\geq 2.$$

We get (8).

Theorem 2. *If $f\in C_B[0,\infty)$ and the m-th left respectively right elementary operators are defined in (9) and (10) then we have*

(i) *for $m\geq 1$,*

$$\lim_{n\to\infty}(A_{m,n}f)(x)=(1-\delta_{1,m})\frac{x}{(1+x)^m}f(x); \tag{11}$$

(ii) *for $m\geq 0$,*

$$\lim_{n\to\infty}(B_{m,n}f)(x)=(1-\delta_{0,m})\frac{x^m}{(1+x)^{m+1}}f(x), \tag{12}$$

uniformly on any compact $[0,\lambda]$.

Proof. We recall the well-known relation

$$\lim_{n\to\infty}(V_nf)(x)=f(x),\ \text{uniformly on any compact } [0,\lambda]. \tag{13}$$

By using (7), for $m=1$, the statement (i) is clear. For $m\geq 2$, if we make the changes $n-m=p$ and $k-1=i$, we can write

$$\begin{aligned}\lim_{n\to\infty}(A_{m,n}f)(x) &= \frac{x}{(1+x)^m}\lim_{p\to\infty}\sum_{i=0}^{\infty}\binom{p+i}{i}\frac{x^i}{(1+x)^{p+1+i}}f\left(\frac{i+1}{p+m}\right)=\\ &= \frac{x}{(1+x)^m}\lim_{p\to\infty}(V_{p+1}^{*}f)(x).\end{aligned}$$

Here, V_{p+1}^{*} is obtained from V_{p+1} changing the knots $\frac{i}{p+i}$ with $\frac{i+1}{p+m}$. Taking into account (13) it follows (11). The proof of (12) is analogous, so we can omit it.

If we take $f=e_0$ in (9) and (10), and according to (4), we can deduce some useful identities:

$$\text{For } m\geq 2,\quad (A_{m,n}e_0)(x)=\begin{cases} 0, & m\geq n+1,\\ (1+x)^{-n+1}, & m=n,\\ x(1+x)^{-m}, & m\leq n-1.\end{cases} \tag{14}$$

$$\text{For } m\geq 1,\quad (B_{m,n}e_0)(x)=\frac{x^m}{(1+x)^{m+1}}. \tag{15}$$

Indeed, by using (10) we can write:

$$\begin{aligned}\sum_{k=m+1}^{\infty}\binom{n+k-m-2}{k-2}\frac{x^k}{(1+x)^{n+k}} &= \frac{x^m}{(1+x)^{m+1}}\sum_{i=1}^{\infty}\binom{n+i-2}{i}\frac{x^i}{(1+x)^{n-1+i}}\\ &= \frac{x^m}{(1+x)^{m+1}}\left((V_ne_0)(x)-\frac{1}{(1+x)^{n-1}}\right),\end{aligned}$$

which lead us to (15). Relation (14) is to be proved in a similar way.

If we substitute (14) and (15) in (7), after few calculations, we can state:

Lemma 2. *The following identity*

$$(L_n^{\langle a,b\rangle}e_0)(x)-\sum_{m\geq 1}b_m\frac{x^m}{(1+x)^{m+1}}=\begin{cases} a_n/(1+x), & n=1,\quad n=2,\\ a_n(1+x)^{-n+1}+\sum\limits_{m=2}^{n-1}a_m x(1+x)^{-m}, & n\geq 3,\end{cases}$$

holds.

Because of the boundedness of the sequences a and b we can define the functions σ,τ,φ, for and $x>0$, as follows

$$\sigma(x)=\sum_{m=2}^{\infty}\frac{a_m x}{(1+x)^m},\quad \tau(x)=\sum_{m=1}^{\infty}b_m\frac{x^m}{(1+x)^{m+1}}$$

and

$$\varphi=\sigma+\tau. \tag{16}$$

Theorem 3. *If $f\in C_B[0,\infty)$, the sequences a,b are bounded and* $\lambda=\max\left\{\sup\limits_{n\geq 1}|a_n|,\sup\limits_{k\geq 0}|b_k|\right\}$ *then*

$$(17)\qquad \text{(i)}\quad |(L_n^{\langle a,b\rangle}f)(x)-f(x)(L_n^{\langle a,b\rangle}e_0)(x)|\le\lambda(1+x(1+x))\omega\left(f,\frac{1}{\sqrt{n}}\right);$$

(18)

$$\text{(ii)}\quad \lambda^{-1}|(L_n^{\langle a,b\rangle}f)(x)-\varphi(x)f(x)|\le(1+x+x^2)\omega\left(f,\frac{1}{\sqrt{n}}\right)+\frac{2|f(x)|}{(1+x)^{n-1}},$$

where $x>0$ and $n\ge 3$;

$$(19)\qquad \text{(iii)}\quad \lim_{n\to\infty}L_n^{\langle a,b\rangle}f=\varphi f,\ \textit{uniformly on any compact } K\subset(0,\infty).$$

Proof. By using lemma 1, for any $g\in C_B[0,\infty)$ and $x\ge 0$, we can write:

$$(20)\qquad |(L_n^{\langle a,b\rangle}g)(x)|\le\sum_{k=0}^{\infty}|c_{n,k}|\frac{x^k}{(1+x)^{n+k}}\left|g\left(\frac{k}{n}\right)\right|\le\lambda V_n(|g|;x),$$

because $|a|\le\lambda$ and $|b|\le\lambda$.

Next, we need the following property of the modulus of continuity $\omega(f,\cdot)$, $f\in C[0,\infty)$: for every $\delta>0$, $x\ge 0$ and $y\ge 0$

$$|f(y)-f(x)|\le(1+\delta^{-2}(x-y)^2)\omega(f,\delta).$$

By using (20), we can write successively

$$|(L_n^{\langle a,b\rangle}f)(x)-f(x)(L_n^{\langle a,b\rangle}e_0)(x)|\le\sum_{k=0}^{\infty}|c_{n,k}|\frac{x^k}{(1+x)^{n+k}}\left|f\left(\frac{k}{n}\right)-f(x)\right|\le$$

$$\le\sum_{k=0}^{\infty}|c_{n,k}|\frac{x^k}{(1+x)^{n+k}}\left(1+\frac{1}{\delta^2}\left(\frac{k}{n}-x\right)^2\right)\omega(f,\delta)\le\lambda V_n(|g|;x)\omega(f,\delta),$$

where $g(t)=1+\delta^{-2}(t-x)^2$.

On the other hand, according to (4)

$$V_n(|g|;x)=1+\frac{1}{\delta^2}V_n((\cdot-x)^2;x)=1+\frac{1}{\delta^2}\frac{x(x+1)}{n}.$$

By taking $\delta=1/\sqrt{n}$, the relation (17) follows.

By using (16) and lemma 2, for $n\ge 3$, we evaluate

$$|(L_n^{\langle a,b\rangle}e_0)(x)-\varphi(x)|=\left|\sum_{m=2}^{n-1}\frac{a_m x}{(1+x)^m}+\frac{a_n}{(1+x)^{n-1}}-\sigma(x)\right|\le$$

$$\le\frac{|a_n|}{(1+x)^{n-1}}+\sum_{m=n}^{\infty}\frac{|a_m|x}{(1+x)^m}\le\frac{2\lambda}{(1+x)^{n-1}}.$$

The above inequality and (17) imply

$$|(L_n^{\langle a,b\rangle}f)(x)-\varphi(x)f(x)|\le|(L_n^{\langle a,b\rangle}f)(x)-f(x)(L_n^{\langle a,b\rangle}e_0)(x)|+$$

$$+|(L_n^{\langle a,b\rangle}e_0(x)-\varphi(x)|\cdot|f(x)|\le\lambda(1+x+x^2)\omega\left(f,\frac{1}{\sqrt{n}}\right)+\frac{2\lambda}{(1+x)^{n-1}}|f(x)|,$$

and this completes the proof of (18).

Obviously, (19) is a consequence of (17) and (18).

Let's take the particular case $L_n^{\langle 1,1\rangle}=V_n$, see (3). By using (16) we obtain for any $x>0$ that $\varphi(x)=1$ and thus come across a familiar result.

References

[1] Agratini, O., *Construction of Baskakov-type operators by wavelets*, Rev. d'Analyse Num. et de Théorie de l'Approx., tome **26**(1997), 3-10.

[2] Altomare, F., *Positive projections approximation processes and degenerate diffusion equations*, Conf. Sem. Mat. Univ. Bari, **241**(1991), 43-68.

[3] Altomare, F., Campiti, M., *Korovkin-type Approximation Theory and its Applications*, de Gruyter Studies in Mathematics, Vol. 17, de Gruyter, Berlin/New-York, 1994.

[4] Altomare, F., Romanelli, S., *On some classes of Lototsky-Schnabl operators*, Note Mat., **12**(1992), 1-13.

[5] Baskakov, V. A., *An example of a sequence of linear positive operators in the space of continuous functions*, Dokl. Akad. Nauk SSSR, **113**(1957), 249-251 (in Russian).

[6] Campiti, M., *Limit semigroups of Stancu-Mühlbach operators associated with positive projections*, Ann. Sc. Norm. Sup. Pisa, Cl. Sci., **19**(1992), 4, 51-67.

[7] Campiti, M., Metafune, G., *Approximation properties of recursively defined Bernstein-type operators*, Journal of Approx. Theory, **87**(1996), 243-269.

[8] Campiti, M., Metafune, G., *Evolution equations associated with recursively defined Bernstein-type operators*, Journal of Approx. Theory, **87**(1996), 270-290.

[9] Stancu, D.D., *Two classes of positive linear operators*, Analele Univ. Timişoara, Ser. St. Matem. **8**(1970), 213-220.

[10] Stancu, D.D. *Approximation of functions by means of some new classes of positive linear operators*, in "Numerische Methoden der Approximations Theorie", Vol. 1 (Proc. Conf. Math. Res. Inst., Oberwolfach, 1971; eds. L. Collatz, G. Meinardus), 187-203, Basel: Birkhäuser, 1972.

Babeş-Bolyai University, Faculty of Mathematics and Informatics
Kogălniceanu 1, Ro-3400 Cluj-Napoca, ROMANIA
E-mail: AGRATINI@MATH.UBBCLUJ.RO