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BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. This paper is concerned with the existence of solutions for some class of functional integrodifferential equations via Leray-Schauder Alternative. These equations arised in the study of second order boundary value problems for functional differential equations with nonlinear boundary conditions.

1. Introduction

Let R^n be the n-dimensional Euclidean space and $|\cdot|$ be any convenient norm in R^n. For a fixed $r \geq 0$, we define C_r to be the Banach space of all continuous functions $\phi : [-r, 0] \to R^n$ endowed with the sup-norm

$$||\phi||_{[-r,0]} = \sup\{|\phi(s)| : -r \leq s \leq 0\}.$$

For any continuous function x defined on the interval $[-r, T], T > 0$ and any $t \in [0, T]$, we denote by x_t the element of C_r defined by

$$x_t(\theta) = x(t + \theta), -r \leq \theta \leq 0.$$

This paper is concerned with the following initial value problem (IVP)

$$\text{(E)} \qquad x'(t) = L(t, x_\tau) + \int_0^T \ell(t, s) F(s, x_s, x'(s)) ds, \quad t \in [0, T],$$

$$\text{(IC)} \qquad x_0 = \phi,$$

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where $L: [0,T] \times C_r \to R^n$, $F: [0,T] \times C_r \times R^n \to R^n$, $\phi: [-r,0] \to R^n$ are continuous functions, $\ell: [0,T] \times [0,T] \to R^n$, is a bounded function and $\tau \in [0,T]$ is a given point.

Integrodifferential equations of the form of (E) arised in the study of boundary value problems (BVP) for functional differential equations with nonlinear boundary conditions. For example we consider the next BVP with nonlinear boundary conditions

$$x''(t) = f(t, x_t, x'(t)), \quad t \in [0,T], \tag{e}$$

$$x_0 = \phi, \quad x(T) = g_1(x_\tau), \tag{BC_1}$$

where $f: [0,T] \times C_r \times R^n \to R^n$, $g_1: C_r \to R^n$ are continuous functions, $\phi \in C([-r,0], R^n)$ is a given function and also, $\tau \in [0,T]$ is a given point. It is clear that the BVP $(e) - (BC_1)$ is equivalent to the following IVP

$$x'(t) = \frac{1}{T} g_1(x_\tau) - \frac{1}{T}\phi(0) + \int_0^T (\frac{\partial}{\partial t} G_1(t,s)) f(s, x_s, x'(s)) ds, \quad t \in [0,T], \tag{1}$$

$$x_0 = \phi, \tag{IC}$$

where

$$G_1(t,s) = \frac{1}{T} \begin{cases} (t-T)s & if \quad 0 \le s \le t \le T \\ t(s-T) & if \quad 0 \le t \le s \le T, \end{cases}$$

is the well known Green's function for the corresponding homogeneous BVP to $(e) - (BC_1)$.

Also, we consider the following BVP with nonlinear boundary conditions

$$x''(t) = f(t, x_t, x'(t)), \quad t \in [0,T], \tag{e}$$

$$x_0 = \phi, \quad x'(T) = g_2(x_\tau), \tag{BC_2}$$

where f, ϕ and τ are as in the previous BVP $(e) - (BC_1)$ and $g_2: C_r \to R^n$ is a continuous function. It is also clear that the BVP $(e) - (BC_2)$ is equivalent to the following IVP

$$x'(t) = g_2(x_\tau) + \int_0^T (\frac{\partial}{\partial t} G_2(t,s)) f(s, x_s, x'(s)) ds, \quad t \in [0,T], \tag{2}$$

$$x_0 = \phi, \tag{IC}$$

where

$$G_2(t,s) = \begin{cases} s & if \quad 0 \le s \le t \le T \\ t & if \quad 0 \le t \le s \le T, \end{cases}$$

is the Green's function for the corresponding homogeneous BVP to $(e)-(BC_2)$.

Obviously, equations (1) and (2) are special forms of the equation (E).

The aim in this paper is to prove existence results for the IVP (E)-(IC) and, consequently, to specify these results to the BVP $(e)-(BC_i)$, $i=1,2$ and some other related BVP conserning functional differential equations with nonlinear boundary conditions.

BVP for functional differential equations constitute an interesting area in the theory of functional differential equations. Some recent results on this subject are developed in the papers of Ntouyas, Sficas and Tsamatos [9,10] and Tsamatos and Ntouyas [13]. For a more detail treatment we refer also to the recent books of Erbe, Kong and Zhang [2] and Henderson [8] and the references therein. Boundary conditions considered in these BVPs are usually linear. Results concerning BVPs with nonlinear boundary conditions, but only for ordinary differential equations, were appeared early in the litterature. Among others we refer to [1,3,4,5,6,12].

For the proof of our main existence result in the following, we use the well known topological transversality method by a similar manner to that in [9]. Generally, to be able to apply this method we need the existence of a-priori bounds on the solutions of a certain family of IVPs related to the given IVP (E)-(IC). These a-priori bounds are obtained imposing growth restrictions on the functions involved in the equation (E) in the line of [7] and [11]. Also, here we extend the method developed in [9] to a more general problem including many problems considered in several papers.

2. Preliminaries

If I is an interval of the real line R, by $C(I,R^n)$ and $C^1(I,R^n)$ we denote the space of all continuous and continuously differentiable, respectively, on I R^n-valued functions. Moreover, by

$$||x||_I = \sup\{|x(t)|: t \in I\}$$

and

$$\| x \|_I = \max\{||x||_I, ||x'||_I\}$$

we define the norms $||\cdot||_I$ and $\|\cdot\|_I$ in $C(I,R^n)$ and $C^1(I,R^n)$, respectively. These spaces endowed with the respective norms are obviously Banach spaces. Also, we denote by $L^1(I,R)$ the space of real functions whose absolute value is integrable on I, endowed with the usual norm

$$||x||_1 = \int_I |x(s)| ds.$$

Definition. *By a solution of the IVP (E)-(IC) we mean a function* $x \in C([-r,T],R^n) \cap C^1([0,T],R^n)$ *which satisfies the equation (E) and* $x_0 = x|[-r,0] = \phi$.

We state here a lemma which is essential in the sequel.

Lemma 2.1. *Let* $\Omega_1, \Omega_2 \colon [0,\infty) \to [0,\infty)$ *be nondecreasing functions and* $A, B, d_1, d_2, c_1, c_2, e$ *nonnegative constants such that*

$$Ad_1 \limsup_{x\to\infty} \frac{\Omega_1(x)}{x} + Bd_2 \limsup_{x\to\infty} \frac{\Omega_2(x)}{x} < 1. \tag{3}$$

Then the set

$$S = \{x \in R \colon 0 < x \le A\Omega_1(d_1x + c_1) + B\Omega_2(d_2x + c_2) + e\}$$

is bounded.

Proof. If the set S is unbounded, there exists a sequence (x_ν), with $x_\nu \ne 0$, $\lim_{\nu\to\infty} x_\nu = \infty$ and

$$\begin{aligned} 1 &\le A\frac{\Omega_1(d_1x_\nu + c_1)}{x_\nu} + B\frac{\Omega_2(d_2x_\nu + c_2)}{x_\nu} + \frac{e}{x_\nu} \\ &= A\frac{\Omega_1(d_1x_\nu + c_1)}{d_1x_\nu + c_1}\frac{d_1x_\nu + c_1}{x_\nu} + B\frac{\Omega_2(d_2x_\nu + c_2)}{d_2x_\nu + c_2}\frac{d_2x_\nu + c_2}{x_\nu} + \frac{e}{x_\nu}. \end{aligned}$$

Thus

$$1 \le Ad_1 \limsup_{x_\nu\to\infty} \frac{\Omega_1(x_\nu)}{x_\nu} + Bd_2 \limsup_{x_\nu\to\infty} \frac{\Omega_2(x_\nu)}{x_\nu}$$

which contradicts to (3). □

3. Main Results

Theorem 3.1. *Let* $F \colon [0,T] \times C_r \times R^n \to R^n$, $L \colon [0,T] \times C_r \to R^n$, $\phi \colon [0,T] \to R^n$ *be continuous functions and* $\ell \colon [0,T] \times [0,T] \to R^n$ *be a bounded function with* $\widehat{\ell} \colon [0,T] \to R$, $\widehat{\ell}(t) = \int_0^T \ell(t,s)ds$ *a continuous function. Suppose also that:*

(H_1) For every bounded subset S *of* C_r *there exists a constant* $\Theta_S \ge 0$ *such that*

$$|L(t_1,u) - L(t_2,u)| \le \Theta_S|t_1 - t_2|$$

for all $t_1, t_2 \in [0,T]$ *and* $u \in S$.

and

(H_2) There exists a constant $M \ge 0$ *such*

$$||x||_{[-r,T]} \le M \quad and \quad ||x'||_{[0,T]} \le M$$

for every solution of the IVP $(E_\lambda)-(IC)$, $\lambda \in (0,1)$, *where* E_λ *stands for the equation*

$$x'(t) = \lambda L(t, x_\tau) + \lambda \int_0^T \ell(t,s) F(s, x_s, x'(s)) ds, \quad t \in [0,T]. \tag{(E_λ)}$$

Then for every $\phi \in C_r$ *the IVP (E)-(IC) has at least one solution.*

Proof. Consider first the case $\phi(0) = 0$. Then the set

$$C = \{x \in C^1([0,T], R^n): x(0) = 0\}$$

is a convex subset of the normed linear space $C^1([0,T], R^n)$ and also $0 \in C$.

Now, we define an operator $R: C \to C^1([0,T], R^n)$ by

$$Rx(t) = \int_0^t L(s, x_\tau) ds + \int_0^t \int_0^T \ell(s,\eta) F(\eta, x_\eta, x'(\eta)) d\eta ds, \quad t \in [0,T],$$

where

$$x_\eta(\theta) = \begin{cases} x(\eta+\theta) & if \quad \eta+\theta \geq 0 \\ \phi(\eta+\theta) & if \quad \eta+\theta < 0. \end{cases}$$

Obviously, $R(C) \subseteq C$.

Our purpose is to prove that R has a fixed point $x \in C$. Then it is clear that the function

$$z(t) = \begin{cases} x(t), t \in [0,T] \\ \phi(t), t \in [-r, 0], \end{cases}$$

is a solution of the IVP (E)-(IC).

Following the same arguments as in [9], it suffices to prove that the operator R is completely continuous and the set

$$E(F) = \{x \in S : x = \lambda Rx \quad \text{for some} \quad 0 < \lambda < 1\}$$

is bounded.

We observe first that R is obviously continuous.

Let now a bounded sequence (x_ν) in C. As in [9], we can prove that there exists a compact set D in C_r such that $x_{\nu t} \in D$ for every ν and every $t \in [0,T]$. Thus, if b_1 is a bound of (x_ν), the set

$$X = [0,T] \times D \times \bar{B}(0, b_1)$$

($\bar{B}(0,b_1)$ is the closed ball in R^n with center 0 and radious b_1) is compact in $[0,T] \times C_r \times R^n$. Then it is obvious that

$$||Rx_\nu||_{[0,T]} \leq TK_1 + T^2 K_2 K_3,$$

where $K_1 = \max\{|L(t,u)|: (t,u) \in [0,T] \times D\}$, $K_2 = \max\{|F(t,u,v)|: (t,u,v) \in X\}$ and $K_3 = \max\{|\ell(t,s)|: (t,s) \in [0,T] \times [0,T]\}$.

Also,

$$||(Rx_\nu)'||_{[0,T]} \leq K_1 + TK_2K_3.$$

Moreover, the sequence (Rx_ν) is equicontinuous. Indeed, for every t_1, t_2 in $[0,T]$ we have

$$|Rx_\nu(t_1) - Rx_\nu(t_2)| = \left| \int_{t_1}^{t_2} (Rx_\nu)'(s)ds \right| \leq (K_1 + TK_2K_3)|t_1 - t_2|. \tag{4}$$

Moreover, taking into accound assumption (H_1) we have

$$|(Rx_\nu)'(t_1) - (Rx_\nu)'(t_2)| \leq \Theta_D|t_1 - t_2| + K_2|\widehat{\ell}(t_1) - \widehat{\ell}(t_2)|. \tag{5}$$

Hence, by (4) and (5) and, moreover, since the function $\widehat{\ell}$ is uniformly continuous on $[0,T]$, we have that the sequence (Rx_ν) is equicontinuous.

Now we observe that by assumption (H_2) the set

$$E(F) = \{x \in S : x = \lambda Rx \quad \text{for some} \quad 0 < \lambda < 1\}$$

is bounded. Therefore the operator R has a fixed point in C.

For the proof in the general case, when $\phi(0) \neq 0$, we observe that the transformation

$$y = x - \phi(0),$$

reduces the IVP (E)-(IC) into the following

$$y'(t) = \widehat{L}(t, y_\tau) + \int_0^T \ell(t,s)\widehat{F}(s, y_s, y'(s))ds, \quad t \in [0,T],$$

$$y_0 = \widehat{\phi},$$

where, $\widehat{L}(t,u) = L(t, u + \phi(0))$, $\widehat{F}(t,u,v) = F(t, u + \phi(0), v)$ and $\widehat{\phi} = \phi - \phi(0)$. For the function $\widehat{\phi}$ we have $\widehat{\phi}(0) = 0$. Hence, since the functions $\widehat{L}$,$\widehat{F}$ satisfy the assumptions (H_1), (H_2), the proof of the theorem is complete. □

The applicability of the previous theorem depends upon the existence of an a-priori bound for the solutions of the IVP (E)-(IC). Conditions on L and F which imply the desired a-priori bounds are given by the following theorem.

Theorem 3.2. *Let $F: [0,T] \times C_r \times R^n \to R^n$, $L: [0,T] \times C_r \to R^n$, $\phi: [0,T] \to R^n$ be continuous functions and $\ell: [0,T] \times [0,T] \to R^n$ be a bounded function with $\widehat{\ell}: [0,T] \to R$, $\widehat{\ell}(t) = \int_0^T \ell(t,s)ds$ a continuous function. Suppose also that (H_1) holds and:*

(H_3) There exists a nondecreasing function $\Omega_1\colon[0,\infty)\to[0,\infty)$ *and two real valued functions* p,q *bounded on* $[0,T]$ *and such that*

$$|L(t,u)|\le p(t)\Omega_1(||u||_{[-r,0]})+q(t)$$

for every $(t,u)\in[0,T]\times C_r$
and
(H_4) There exists a nondecreasing function $\Omega_2\colon[0,\infty)\to[0,\infty)$ *and two functions* m,n *in* $L^1([0,T],R)$ *such that*

$$|F(t,u,v)|\le m(t)\Omega_2(\max\{||u||_{[-r,0]},|v|\})+n(t)$$

for every $(t,u,v)\in[0,T]\times C_r\times R^n$.
Then the IVP (E)-(IC) has at least one solution provided that

$$||p||_{[0,T]}T\limsup_{x\to\infty}\frac{\Omega_1(x)}{x}+K_3||m||_1\max\{1,T\}\limsup_{x\to\infty}\frac{\Omega_2(x)}{x}<1, \tag{6}$$

where $K_3=\max\{|\ell(t,s)|\colon(t,s)\in[0,T]\times[0,T]\}$.

Proof. Let x be a solution of the IVP $(E_\lambda)-(IC),\lambda\in(0,1)$. Then for every $t\in[0,T]$ we have

$$||x_t||_{[-r,0]}\le||\phi||_{[-r,0]}+||x||_{[0,T]}.$$

Also, $x(t)=\phi(0)+\int_0^t x'(s)ds$, $t\in[0,T]$. Hence

$$||x||_{[0,T]}\le|\phi(0)|+T||x'||_{[0,T]}. \tag{7}$$

Therefore

$$||x_t||_{[-r,0]}\le||\phi||_{[-r,0]}+T||x'||_{[0,T]},\quad t\in[0,T]. \tag{8}$$

Moreover, for every $t\in[0,T]$ we have

$$\begin{aligned}|x'(t)|\le{}&|p(t)|\Omega_1(||x_\tau||_{[-r,0]})+|q(t)|\\&+K_3\int_0^T\big(m(s)\Omega_2(\max\{||x_s||_{[-r,0]},|x'(s)|\})+n(s)\big)\,ds.\end{aligned}$$

By (8) and, since Ω_1,Ω_2 are nodecreasing, last inequality reduces to

$$\begin{aligned}|x'(t)|\le{}&||p||_{[0,T]}\Omega_1(||\phi||_{[-r,0]}+T||x'||_{[0,T]})+||q||_{[0,T]}\\&+K_3||m||_1\Omega_2\left(\max\{||\phi||_{[-r,0]}+T||x'||_{[0,T]},||x'||_{[0,T]}\}\right)+||n||_1,t\in[0,T].\end{aligned}$$

Finally, since Ω_2 is nondecreasing we obtain

$$\begin{aligned}||x'||_{[0,T]}\le{}&||p||_{[0,T]}\Omega_1(||\phi||_{[-r,0]}+T||x'||_{[0,T]})+||q||_{[0,T]}\\&+K_3||m||_1\Omega_2(||\phi||_{[-r,0]}+\max\{1,T\}||x'||_{[0,T]}||)+||n||_1.\end{aligned}$$

Hence, by assumption (6) and Lemma 2.1., there exists a constant M_1 such that

$$||x'||_{[0,T]} \le M_1.$$

Then by (7) we have

$$||x||_{[0,T]} \le |\phi(0)| + TM_1$$

and since $x_0 = \phi$,

$$||x||_{[-r,T]} \le ||\phi|| + TM_1 = M_2$$

Thus we proved that for every solution x of the IVP $(E_\lambda) - (IC), \lambda \in (0,1)$, the assumption ($H_2$) of Theorem 3.1. is satisfied fof $M = \max\{M_1, M_2\}$, a constant indepentent of λ. So, the IVP (E)-(IC) has at least one solution. □

The next corollary illustrates the existence result of the above Theorem 3.2. and concerns some special forms of functions Ω_1 and Ω_2.

Corollary 3.3. *Let $L: [0,T] \times C_r \to R^n$, $F: [0,T] \times C_r \times R^n \to R^n$, $\phi: [0,T] \to R^n$ be continuous functions and $\ell: [0,T] \times [0,T] \to R^n$ be a bounded function with $\widehat{\ell}: [0,T] \to R$, $\widehat{\ell}(t) = \int_0^T \ell(t,s)ds$ a continuous function. Suppose also that (H_1) holds and:*

($\widehat{H}_3$) There exists a constant d, $0 \le d \le 1$ and two real valued functions p, q bounded on $[0,T]$ and such that

$$|L(t,u)| \le p(t)(||u||_{[-r,0]})^d + q(t)$$

for every $(t,u) \in [0,T] \times C_r$

and

($\widehat{H}_4$) There exists a constant r, $0 \le r \le 1$ and two functions m, n in $L^1([0,T], R)$ such that

$$|F(t,u,v)| \le m(t)(\max\{||u||_{[-r,0]}, |v|\})^r + n(t)$$

for every $(t,u,v) \in [0,Y] \times C_r \times R^n$.

Then the IVP (E)-(IC) has at least one solution provided that

$$\alpha(d)||p||_{[0,T]}T + \alpha(r)K_3||m||_1 \max\{1,T\} < 1, \tag{9}$$

where

$$\alpha(k) = \begin{cases} 0, & k \in [0,1) \\ 1, & k = 1, \end{cases}$$

Proof. We set $\Omega_1(z) = z^d$ and $\Omega_2(z) = z^r$. Then we have

$$\limsup_{x\to\infty} \frac{\Omega_1(x)}{x} = \alpha(d) \quad \text{and} \quad \limsup_{x\to\infty} \frac{\Omega_2(x)}{x} = \alpha(r).$$

Hence assumption (6) of Theorem 3.2. is reduced to assumption (9) above and the proof is complete. □

4. Applications

Consider now the BVP $(e)-(BC_i), i=1,2$. Since these problems are equivalent to the IVP (1)-(IC) and (2)-(IC), respectively, we have the next existence result which is an immediate consequence of Theorem 3.1.

Theorem 4.1. *Let $f\colon[0,T]\times C_r\times R^n\to R^n$ and $g_i\colon C_r\to R^n$, $i=1,2$ be continuous functions. Suppose also that (H_4) holds, with f in place of F, and:*

(H_3') There exists a nondecreasing function $\Omega_i'\colon[0,\infty)\to[0,\infty)$, $i=1,2$ such that

$$|g_i(z)|\le\Omega_i'(||z||),\quad i=1,2$$

for every $z\in C_r$. Then for every $\phi\in C_r$ the BVP $(e)-(BC_i), i=1,2$, has at least one solution provided that

$$A_i\max\{1,T\}\limsup_{x\to\infty}\frac{\Omega_i'(x)}{x}+||m||_1\max\{1,T\}\limsup_{x\to\infty}\frac{\Omega_2(x)}{x}<1,$$

where

$$A_i=\begin{cases}\frac{1}{T}, i=1\\ 1, i=2.\end{cases}$$

Proof. The BVP $(e)-(BC_i), i=1,2$ are equivalent to the IVP $(i)-(IC), i=1,2$, respectively. For these IVP the assumption (H_1) is, obviously satisfied. Thus the proof is similar to that of Theorem 3.1. and 3.2. with some obvious modifications. □

Now we consider the following BVP

$$x''(t)=f(t,x_t,x'(t)),\quad t\in[0,T], \tag{e}$$

$$x_0=\phi,\quad \alpha x(T)+\beta x'(T)=g(x_\tau), \tag{BC}$$

where f, ϕ and τ are as in the previous BVP $(e)-(BC_i), i=1,2$, $g\colon C_r\to R^n$ is a continuous function and α,β are real constants such that

$$\alpha T+\beta\ne0.$$

It is clear that the BVP $(e)-(BC)$ is equivalent to the following IVP

$$x'(t)=\frac{g(x_\tau)}{\alpha T+\beta}-\frac{\alpha\phi(0)}{\alpha T+\beta}+\int_0^T(\frac{\partial}{\partial t}G(t,s))f(s,x_s,x'(s))ds,\quad t\in[0,T], \tag{10}$$

$$x_0=\phi, \tag{IC}$$

where

$$G(t,s)=\frac{1}{\alpha T+\beta}\begin{cases}(\alpha t-\alpha T-\beta)s & if\quad 0\le s\le t\le T\\ t(\alpha s-\alpha T-\beta) & if\quad 0\le t\le s\le T,\end{cases}$$

is the Green's function for the corresponding homogeneous BVP to $(e)-(BC)$.

The BVP (e)-(BC) is more general than BVP $(e)-(BC_i), i=1,2$. Hence the next theorem generalizes the result of the previous Theorem 4.1. A closely related BVP is studied in [10,13].

Theorem 4.2. *Let* $f\colon [0,T]\times C_r\times R^n \to R^n$ *and* $g\colon C_r \to R^n$ *be continuous functions. Suppose also that* (H_4) *holds, with* f *in place of* F*, and:*

(H_3') *There exists a nondecreasing function* $\Omega\colon [0,\infty)\to[0,\infty)$ *such that*

$$|g(z)| \le \Omega(||z||)$$

for every $z\in C_r$. *Then for every* $\phi\in C_r$ *The BVP* $(e)-(BC)$ *has at least one solution provided that*

$$\frac{1}{\alpha T+\beta}\max\{1,T\}\limsup_{x\to\infty}\frac{\Omega(x)}{x}+||m||_1\max\{1,T\}\limsup_{x\to\infty}\frac{\Omega_2(x)}{x}<1.$$

Proof. Since the BVP $(e)-(BC)$ is equivalent to the IVP $(10)-(IC)$, the proof is immediate. □

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