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CLASSICAL DIFFERENTIAL GEOMETRY WITH CHRISTOFFEL SYMBOLS OF EHRESMANN ε–CONNECTIONS

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Abstract. We give a method based on an idea of O. Veblen which gives explicit formulas for the covariant derivatives of natural objects in terms of the Christoffel symbols of a symmetric Ehresmann ε-connection.

Introduction

It is well known that tensor calculus originated from the works of Ricci, Levi-Civita and Christoffel about 1900. In 1922 O. Veblen proposed a generalization of covariant differentiation (CD) in [17], [18]. The basic idea was to differentiate the components of a natural object (NO) in affine normal coordinates and also to differentiate the transformation rule of its components from affine normal coordinates to arbitrary coordinates. By successive differentiations, Veblen introduced the concepts of extensions of tensors, affine extensions and affine normal tensors in [17], [18]. Extensions of the metric tensor are independently discovered and used by G. D. Birhoff in relation to physics in [1]. However, in order to differentiate higher order NO's covariantly, one needs higher order Christoffel symbols (CS) or an equivalent concept which unfortunately did not exist at the time of these works. Consequently, this extension procedure was applied to tensors and classical CS and bound by tensor calculus, produced further tensors. It seems that this approach, overhelmed by local formulas and also lacking a conceptual framework compared to the formalism introduced by E. Cartan the same year in [2], did not attract much attention. Later, through the work of C. Ehresmann about 1950, the concept of linear connection on the principle frame bundle has been accepted as the modern substitute for CD. Ehresmann defined and studied also linear connections on higher order frame bundles, generalizing CD to higher order NO's. Since then, CD of NO's has been studied by several geometers (see, for instance, [3], [14], [7]) and different definitions have been proposed which are shown to be equivalent in [7]. Consequently, the foundations of the theory of CD of NO's based on differential forms on frame bundles is now well known and we

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refer the reader to the basic reference [8] for a modern and detailed treatment of this subject.

In this note we give yet another approach to CD of NO's which is based on the following observation: In the above framework, the main object of study is the Lie algebra valued connection form and CS emerge as its components and play a secondary role. Further, CS are now rather intricate objects in higher orders and it is very difficult to do explicit computations with them, in contrast to the classical situation where such computations have paved the way to important discoveries. Therefore one would like to have a framework of CD which incorporates

1. A set of CS which are more elementary than the CS of linear connections.
2. A method which will enable one to compute the covariant derivative of any NO explicitly as in tensor calculus.

Our purpose in this note is to give such a framework which, we believe, updates [17], [18]. As a remarkable fact, it turns out that the CS which we need are the CS of ε-connections, a concept defined by Ehresmann in 1956 in [4] and studied further in [6], [9], [19], [10]. Naturally, these objects are in one to one correspondence with linear connections. However, their transformation rule uses nothing but the group operation of the jet group, that is, chain rule (see (2) below). We would like to indicate here that the local formulas for prolongations of ε-connections are implicit in the formulas for affine extensions in [17], [18].

Our method, which is algorithmic and can be carried out by a computer, is based on the idea of differentiating the transformation rule of the components of a given NO from geodesic coordinates to arbitrary coordinates. This idea is due to Veblen where affine normal coordinates are used instead of geodesic coordinates ([17], [18]). This method enables one to give explicit formulas in terms of the components of the NO and the CS of a given symmetric ε-connection.

The present framework of CD dwells on the following fundamental idea due to those who pioneered CD: Let ξ be a section of a natural bundle $E(M) \to M$ of order k and let ξ^{α} be its components. Then $\frac{\partial \xi^{\alpha}}{\alpha x^{i}}$ does not transform as a section of $V(E(M)) \otimes T^{*}(M)$ due to the derivatives up to order $k+1$ which arise from differentiation. One therefore searches for a structure π on M of order $k+1$ and a correcting term $\sum_{i}^{\alpha}(\xi(x), \pi(x))$ such that $\frac{\alpha \xi^{\alpha}}{\partial x^{i}} + \sum_{i}^{\alpha}(\xi(x), \pi(x))$ will transform properly. Consequently, the present framework seems to be the direct generalization of CD of tensor fields to higher order NO's. However, it seems to have some rather peculiar consequences which we will mention at the end of this note. The relation of the present framework to the formalism of linear connections remains to be clarified.

A framework for covariant differentiation of natural objects

We will start by briefly recalling the definition of ε-connections and their CS. Let $\widetilde{P}^{k}(M) \to M$ be the coframe bundle of M of order k. The elements of $\widetilde{P}^{k}(M)$ are k-jets of local diffeomorphisms with source in M and target at the origin of $\mathbb{R}^{n}$ and $\widetilde{P}^{k}(M)$ is a left principal bundle with group $GL_{1}(n, \mathbb{R})$. A dual ε-connection $\widetilde{\Gamma}$ is a $GL_{1}(n, \mathbb{R})$ invariant section of $\widetilde{P}^{k}(M) \to \widetilde{P}^{1}(M)$, where we regard $GL_{1}(n, \mathbb{R})$

as a subgroup of $GL_k(n,\mathbb{R})$ by the canonical injection $GL_1(n,\mathbb{R}) \to (GL_1(n,\mathbb{R}),0)$. If $x^i, \widetilde{x}^i_{j_1}, \dots, \widetilde{x}^i_{j_1\dots j_k}$ are local coordinates on $\widetilde{P}^k(M)$, then $\widetilde{\Gamma}$ is locally determined by its CS $\widetilde{\Gamma}_{j_1j_2}(x), \dots, \widetilde{\Gamma}^i_{j_1\dots j_k}(x)$, which we will also denote by $\widetilde{\Gamma}^i_\mu(x)$, $2 \le |\mu| \le k$, and by the formulas

$$\widetilde{x}^i_{j_1\dots j_s} = \widetilde{x}^i_a \widetilde{\Gamma}^a_{j_1\dots j_s}(x) \qquad 2 \le s \le k\,. \tag{1}$$

It is easy to show that $\widetilde{\Gamma}^i_\mu(x)$ transform by

$$\begin{aligned} p\left(\frac{\partial y^i}{\partial x^j}\right) \bullet (\delta^i_j, \widetilde{\Gamma}^i_{j_1j_2}(x), \dots, \widetilde{\Gamma}_{j_1\dots j_k}(x)) \bullet \left(\frac{\partial x^i}{\partial y^j}, \dots, \frac{\partial^k x^i}{\partial y^{j_1}\dots\partial y^{j_k}}\right) \\ = (\delta^i_j, \widetilde{\Gamma}^i_{j_1j_2}(y), \dots, \widetilde{\Gamma}^i_{j_1\dots j_k}(y)) \end{aligned} \tag{2}$$

where $\bullet$ denotes the group operation of $GL_k(n,\mathbb{R})$ and p denotes the projection $GL_k(n,\mathbb{R}) \to GL_1(n,\mathbb{R})$ ([10]). Note that $\widetilde{\Gamma}^i_{j_k}$ transform as classical CS. If $\widetilde{\varepsilon}^k(M) \to M$ denotes the associated bundle of $\widetilde{P}^k(M)$ with respect to the right action determined by (2), then a ε-connection becomes a section of this bundle. Similarly we can define $\widehat{\varepsilon}^k(M) \to M$ as an associated bundle of the frame bundle $\widehat{P}^k(M) \to M$ ([9], [19]). Using the same notation for bundles and their sheaves of local sections, we have a map $\widetilde{\varepsilon}^k(M) \to \widehat{\varepsilon}^k(M)$ which is locally given by

$$\left(\delta^i_j, \widehat{\Gamma}^i_{j_1j_2}, \dots, \widehat{\Gamma}^i_{j_1\dots j_k}\right) = \left(\delta^i_j, \widetilde{\Gamma}^i_{j_1j_2}, \dots, \widetilde{\Gamma}^i_{j_1\dots j_k}\right)^{-1} \tag{3}$$

where $\widehat{\Gamma}^i_\mu$ denote the CS of $\widehat{\Gamma} \in \widehat{\varepsilon}^k(M)$ and $(\quad)^{-1}$ denotes the inversion in the jet group. If $\widetilde{\Gamma} \in \widetilde{\varepsilon}^k(M)$ and $\widehat{\Gamma} \in \widehat{\varepsilon}^k(M)$ are related by (3), we will denote them by $(\widetilde{\Gamma}, \widehat{\Gamma})$ and call $(\widetilde{\Gamma}, \widehat{\Gamma})$ an associated pair of symmetric ε-connections. Clearly $\widetilde{\Gamma}$ and $\widehat{\Gamma}$ can be defined also without the assumption of symmetry using the same transformation rules.

For simplicity of notation, now let $\lambda^i_\nu = \dfrac{\partial^s y^i}{\partial x^{j_1}\dots\partial x^{j_s}}$ and $\sigma^i_\nu = \dfrac{\partial^s x^i}{\partial y^{j_1}\dots\partial y^{j_s}}$ where $\nu = (j_1, \dots, j_s)$.

The main result of this note depends on the following simple

Lemma. *Let $(V; y^i)$ be a coordinate system with $p \in V$ and let $(\widehat{\Gamma}, \widetilde{\Gamma})$ be an associated pair of symmetric ε-connections of order m with CS $\widehat{\Gamma}^i_\nu(y)$ and $\widetilde{\Gamma}^i_\nu(y)$ on V. Then there exists a coordinate system $(U; x^i)$ around p such that at the point p we have*

i. $\dfrac{\partial y^i}{\partial x^j} = \delta^i_j$ *and* $\widehat{\Gamma}^i_\nu(x) = \widetilde{\Gamma}^i_\nu(x) = 0\,.$

ii. $\widehat{\Gamma}^i_\nu(y) = \lambda^i_\nu$ *and* $\widetilde{\Gamma}^i_\nu(y) = \sigma^i_\nu\,.$

Proof. Let (x^i) be the standard coordinates in $\mathbb{R}^n$, $\bar{y}^i$ the coordinates of p and q any point in $\mathbb{R}^n$. Take any map $f : \mathbb{R}^n \to M$ such that $f(q) = p$ and

$$\left(\frac{\partial f^i}{\partial x^j}(q), \frac{\partial^2 f^i}{\partial x^{j_1}\partial x^{j_2}}(q), \ldots, \frac{\partial^m f^i}{\partial x^{j_1}\ldots\partial x^{j_m}}(q)\right) = (\delta^i_j, \widetilde{\Gamma}^i_{j_1 j_2}(\bar{y}), \ldots, \widetilde{\Gamma}_{j_1\cdots j_m}(\bar{y})).$$

Then f restricts to a local diffeomorphism near p and defines a coordinate system $(U; x^i)$ around p.
Both statements are now immediate from (2) and (3). □

We will call $(U; x^i)$ in the above Lemma a (p)-geodesic coordinate system induced by $(V; y^i)$.

The above Lemma shows the conceptual simplicity of the present CS: Pointwise (but not necessarily locally) they are derivatives in a suitable coordinate system.

Now let $E(M) \to M$ be a natural bundle of order k. For simplicity, we will assume that the fiber space of $E(M)$ can be covered by a single coordinate system ξ^β, $1 \le \beta \le n$, which we fix once and for all. Let $\xi \in E(M)$ and suppose that its components transform as

$$\bar{\xi}^\alpha = f^\alpha(\xi^\beta, \lambda^i_\nu) \qquad 1 \le |\nu| \le k \tag{4}$$

If $(x^i, \xi^\beta; \mu^i, \phi^\beta)$ denote the local coordinates on the tangent bundle $T(E(M)) \to M$, we have

$$T(E(M)) \quad \begin{cases} \bar{\xi}^\alpha = f^\alpha(\xi^\beta, \lambda^i_\nu) \\ \bar{\mu}^i = \mu^a \frac{\partial y^i}{\partial x^a} \\ \bar{\phi}^\alpha = \frac{\partial f^\alpha(\xi,\lambda)}{\partial \xi^\beta}\phi^\beta + \frac{\partial f^\alpha(\xi,\lambda)}{\partial \lambda^a_\nu}\lambda^a_{b\nu}\mu^b \end{cases} \tag{5}$$

where $b\nu$ denotes $(b, j_1, \ldots j_s)$. If we let $\mu^i = 0$ in (5), we get the transformation rules of the coordinates $(x^i, \xi^\beta; \phi^\beta)$ of the vertical bundle $V(E(M)) \to E(M)$. Therefore, if $(x^i, \xi^\beta, \phi^\beta_j)$ denote the local coordinates on $V(E(M)) \otimes T^*(M)$, where $T^*(M)$ is pulled back over $E(M)$, we obtain

$$V(E(M)) \otimes T^*(M) \quad \begin{cases} \bar{\xi}^\alpha = f^\alpha(\xi^\beta, \lambda^i_\nu) \\ \bar{\phi}^\alpha_j = \dfrac{\partial f^\alpha(\xi,\lambda)}{\partial \xi^\beta}\phi^\beta_a \dfrac{\partial x^a}{\partial y^j} \end{cases} \tag{6}$$

Now let ξ be a section of $E(M) \to M$. We then have

$$\xi^\alpha(y) = f^\alpha(\xi^\beta(x), \lambda^i_\nu) \tag{7}$$

on $(U; x^i) \cap (V; y^i)$. Differentiation of (7) gives

$$\frac{\partial \xi^\alpha(y)}{\partial y^j} = \frac{\partial f^\alpha(\xi(x),\lambda)}{\partial \xi^\beta}\frac{\partial \xi^\beta}{\partial x^a}\frac{\partial x^a}{\partial y^j} + \frac{\partial f^\alpha(\xi(x),\lambda)}{\partial \lambda^a_\nu}\lambda^a_{b\nu}\frac{\partial x^b}{\partial y^j} \tag{8}$$

Exactly as in tensor calculus, we now want to eliminate the second term on the RHS of (8), which involves derivatives up to order $k+1$, by means of some structure on M of order $k+1$. For this purpose, we choose an associated symmetric pair

$(\widetilde{\Gamma},\widehat{\Gamma})$ of order $k+1$, some $p\in U\cap V$ and assume that (y^i) in (8) is an arbitrary coordinate system and (x^i) is a p-geodesic coordinate system induced by (y^i). After evaluating all expressions in (8) at p, we substitute $\xi^\beta(x)=f^\beta(\xi^\gamma(y),\sigma^i_\mu)$ into the second expression on the RHS of (8). In view of the Lemma, we now substitute $\lambda^i_j=\sigma^i_j=\delta^i_j$, $\lambda^i_\mu=\widehat{\Gamma}^i_\mu$, $\sigma^i_\mu=\widetilde{\Gamma}^i_\mu$, $2\le|\mu|\le k+1$ into the resulting expression and rewrite (8) in the form

$$\frac{\partial\xi^\alpha(y)}{\partial y^j}-\Theta^\alpha_j(\xi(y),\widehat{\Gamma}(y),\widetilde{\Gamma}(y))=\frac{\partial f^\alpha(\xi(x),\lambda)}{\partial\xi^\beta}\frac{\partial\xi^\beta(x)}{\partial x^j}\tag{9}$$

where the expression Θ^α_j is explicitly known. Now the form of Θ^α_j shows that $\Theta^\alpha_j(\xi(x),\widehat{\Gamma}(x),\widetilde{\Gamma}(x))=0$. Comparing (9) and (6), we see that the differential expressions

$$\frac{\partial\xi^\alpha}{\partial x^j}-\Theta^\alpha_j(\xi(x),\widehat{\Gamma}(x),\widetilde{\Gamma}(x))\tag{10}$$

transform from the induced geodesic coordinates (x^i) to (y^i) by the transition rules of $V(E(M))\otimes T^*(M)$ at p. Now let (y^i) and (z^i) be two arbitrary coordinate systems around p and (x^i) and (w^i) be (p)-geodesic coordinate systems induced by (x^i) and (y^i) respectively. Writing $(y^i;p)\to(x^i;p)\to(w^i;p)\to(z^i;p)$ with the obvious meaning, the above argument shows that the first and the third arrows are induced by the transition rules of the bundle $V(E(M))\otimes T^*(M)$ and it is easy to check that this is also the case with the middle arrow. However note that $\frac{\partial w^i}{\partial x^j}$ need not be δ^i_j in general. Consequently, the composition $(y^i;p)\to(z^i;p)$ is also induced by the transition rules of $V(E(M))\otimes T^*(M)$. To recapitulate the above arguments, we now state

Proposition. *Let $E(M)\to M$ be a natural bundle of order k given by* (4) *and let $(\widehat{\Gamma},\widetilde{\Gamma})$ be an associated pair of symmetric ε-connections of order $k+1$. Then there exists a first order differential operator*

$$\begin{aligned}E(M)&\xrightarrow{\nabla}V(E(M))\otimes T^*(M)\\ \xi^\alpha&\longrightarrow\frac{\partial\xi^\alpha}{\partial x^j}-\Theta^\alpha_j(\xi(x),\widehat{\Gamma}(x),\widetilde{\Gamma}(x))\end{aligned}\tag{11}$$

The above argument shows that the unwanted terms on the RHS of (8) directly give us the correcting term on the LHS of (9), *if we use ε-connections.* In particular, we see that the present framework is valid only for NO's because CS enter our framework through the derivatives in (4) in view of the Lemma and have no meaning otherwise.

In (11) we regard $V(E(M))\otimes T^*(M)$ as a bundle over M which may not be a vector bundle, that is, $\frac{\partial f^\alpha}{\partial\xi^\beta}$ in (8) may involve ξ. Note that Proposition forces the form of Θ^α_j to be the same in all coordinates (see [15], Corollary 4.3). If we assume that the action given by (4) is a polynomial in ξ, then Θ^α_j will be a polynomial in ξ

of the same degree. Also, since $\widehat{\Gamma}^i_\nu$ and $\widetilde{\Gamma}^i_\nu$ are related by (3), one can express ∇ in terms of only $\widehat{\Gamma}^i_\nu$ or $\widetilde{\Gamma}^i_\nu$, that is, one may differentiate covariantly only with respect to some $\widehat{\Gamma}$ or $\widetilde{\Gamma}$. However, since inversion in $GL_{k+1}(n,\mathbb{R})$ is rather complicated on computational level for large k, it seems more convenient to leave $\widehat{\Gamma}^i_\nu$, $\widetilde{\Gamma}^i_\nu$ as they are in (9). Further, there seems to be no a priori reason, other than some conventions, to prefer one among $\widetilde{\Gamma}$ and $\widehat{\Gamma}$, in fact, one among λ^i_ν and σ^i_ν in (4). For $k=2$, we have $\widetilde{\Gamma}^i_{pq} = -\widehat{\Gamma}^i_{pq}$ which is immediate from (3).

Not surprisingly, we now have

Corollary. *Let $E(M) = T^p_q(M)$, the (p,q)-tensor bundle of M and $\xi \in E(M)$. If we denote the common value $\widetilde{\Gamma}^i_{jk} = -\widehat{\Gamma}^i_{jk}$ by Γ^i_{jk}, then $\frac{\partial \xi^{i_1\cdots i_p}_{j_1\cdots j_q}}{\partial x^m} - \Theta_m{}^{i_1\cdots i_p}_{j_1\cdots j_q}(\xi(x),\Gamma(x))$ is identical with the classical covariant derivative of ξ.*

We will omit the rather straightforward verification of the Corollary. Clearly, the (p,q)-tensor in the Corollary may have any relative weight.

We will now clarify the geometric meaning of ∇. Let $\xi \in E(M)$, $u \in E(M)$ (as a point), $X_p \in T_p(M)$ and suppose that $\xi(p) = u$. Choosing a coordinate system $(U;x^i)$ around p, we have $X_p = X^i_p \frac{\partial}{\partial x^i}\Big|_p$ and $\xi_*(X_p) = X^a_p \frac{\partial}{\partial x^a}\Big|_u + X^a_p \frac{\partial \xi^\beta}{\partial x^a}\frac{\partial}{\partial \xi^\beta}\Big|_u$. We define a tangent vector X^*_u at u by

$$X^*_u = X^a_p \frac{\partial}{\partial x^a}\Big|_u + X^a_p \Theta^\beta_a(\xi,\widetilde{\Gamma},\widehat{\Gamma})\frac{\partial}{\partial \xi^\beta}\Big|_u \tag{12}$$

which lifts X_p and obtain

$$\xi_*(X_p) - X^*_u = X^a_p \frac{\partial \xi^\beta}{\partial x^a}\Big|_u - X^a_p \Theta^\beta_a(\xi,\widetilde{\Gamma},\widehat{\Gamma})\frac{\partial}{\partial \xi^\beta}\Big|_u \tag{13}$$

(13) together with the Proposition shows that the definition of X^*_u does not depend on local coordinates and we obtain the first order operator

$$\begin{aligned} T_p(M)\times E(M) &\longrightarrow V(E(M)) \\ X_p \times \xi &\longrightarrow \nabla_{X_p}(\xi) \end{aligned} \tag{14}$$

where $\nabla_{X_p}(\xi)(p) = \xi_*(X_p) - X^*_u$.

This construction goes further within the framework of linear connections: As X_p ranges over $T_p(M)$, the collection X^*_u gives a horizontal space H_u at u, that is, a connection on $E(M) \longrightarrow M$. In particular, if $E(M) \longrightarrow M$ is the (semi-holonomic) (co)frame bundle $P^k(M) \longrightarrow M$, then it is not difficult to show that $\{H_u; u \in P^k(M)\}$ is invariant under the action of the whole group $GL_k(n,\mathbb{R})$. Note that this fact is nontrivial in the present framework because $\widetilde{\Gamma}$ and $\widehat{\Gamma}$ are by definition only $GL_1(n,\mathbb{R})$ invariant. This remarkable fact is due to the semidirect product structure $GL_k(n,\mathbb{R}) = GL_1(n,\mathbb{R}) \ltimes B_k(n,\mathbb{R})$ and $B_k(n,\mathbb{R})$ invariance turns out to be the consequence of $GL_1(n,\mathbb{R})$ invariance, a fact already hinted by the definition of the CS of an ε-connection. Thus the definition of a linear connection on the principle (co)frame bundle turns out to be a theorem in the present elementary framework which also does not dwell on the concepts of associated bundles and associated connections.

The formalism of linear connections goes however further: One now defines the $gl_k(n,\mathbb{R})$ valued connection 1-form ω which annihilates $\{H_u; u \in P^k(M)\}$ and defines the CS $\bar{\Gamma}^i_\eta$ of ω as the components of ω, claiming that they are *the Christoffel symbols*. It turns out that $\bar{\Gamma}^i_{jk} = \widetilde{\Gamma}^i_{jk} = -\widehat{\Gamma}^i_{jk}$, which seems to be a mere sign convention but actually comes from inversion and $\bar{\Gamma}^i_\nu$ are now very intricate objects for $|\nu| \geq 3$. We see therefore that this latter construction is not essential for CD of NO's, even though it is fundamental for other purposes. It is very interesting to observe that torsion and curvature are now redefined, the latter also on an arbitrary principal bundle, using the canonical form Θ and the connection form ω whereas these concepts originate in CD of tensor fields which are first order NO's.

We will now briefly clarify the geometric meaning of the second operator in the three term differential sequence constructed in [11]. Let $\widetilde{\Gamma} \in \widetilde{\varepsilon}^{k+1}(M)$ and $\pi^{k+1}_k : \widetilde{\varepsilon}^{k+1}(M) \longrightarrow \widetilde{\varepsilon}^k(M)$ be the projection. In particular, we can differentiate $\pi^{k+1}_k(\widetilde{\Gamma})$ covariantly with respect to $\widetilde{\Gamma}$ which gives an operator

$$\widetilde{\varepsilon}^{k+1}(M) \xrightarrow{\widetilde{D}} V(\widetilde{\varepsilon}^k(M)) \otimes T^*(M) \tag{15}$$

for $k \geq 2$. The order of jets on the RHS of (15) is one less than the one on LHS, but this problem can be easily remedied by choosing the top order CS arbitrarily and passing to some suitable quotient space. It is shown in [11] that $\widetilde{D}(\widetilde{\Gamma}) = 0$ if and only if $\widetilde{\Gamma}^i_\nu$ vanish identically in some local coordinate system. Consequently, (15) extends one step to the left giving an exact sequence. As an interesting observation, note that (15) involves 1-forms and not 2-forms. The same construction works out also with $\widehat{\Gamma}$.

As a remarkable fact, the local formulas which define $\widehat{D}$ are implicit in the works of Veblen. See, for instance, [16], p. 192, [17], p. 568, [18], p.102. These formulas are derived from the equation of a geodesic by successive differentiations. See, for instance, [16], p.192, [17], p.560 and [5], p.52. Further, inversion is effectively used in these works as can be seen from [17], p.555. However, no distinction is made between $\widetilde{\Gamma}$ and $\widehat{\Gamma}$ in their extended tensorial forms and as a very interesting observation, it is always $\widehat{\Gamma}$ which appears in these formulas whereas the classical CS and the curvature tensor arise from $\widetilde{\Gamma}$.

As already indicated above, the above arguments imply the following rather peculiar consequences.

1. The classical CS seem to be the CS of ε-connections and not the CS of linear connections. It is interesting to compare this statement to some assertions in [12], [13] about the origin of CS. However, it is worth noting here that higher order CS,

be they of ε-connections or linear connections, and the present framework of CD do not exist in [12], [13].

2. Classical CD is related to inversion in the jet group as indicated in [10] and therefore to the pseudogroup of local diffeomorphisms on the base manifold.

3. In order to differentiate NO's covariantly, one does not need $GL_k(n,\mathbb{R})$ invariance of linear connections (at least in the "torsion-free" case), but only $GL_1(n,\mathbb{R})$ invariance of ε-connections.

Finally, we would like to indicate that the above conclusions are by no means intended to be final verdicts, but to point out certain mathematical queeries which beg for further clarification.

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