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GLOBAL EXISTENCE FOR FUNCTIONAL SEMILINEAR INTEGRODIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study the global existence of solutions for first and second order initial value problems for functional semilinear integrodifferential equations in Banach space, by using the Leray-Schauder Alternative or the Nonlinear Alternative for contractive maps.

1. Introduction

In this paper we study the global existence of solutions for initial value problems (IVP for short) for semilinear functional integrodifferential equations. The paper is divided into two parts. In Section 2 we consider the following IVP

$$x'(t) = Ax(t) + \int_0^t f(s, x_s)ds, \quad t \in [0, b], \tag{1.1}$$

$$x_0 = \phi, \tag{1.2}$$

where A is the infinitesimal generator of a linear semigroup in a Banach space X, and $f: [0, b] \times C \to X$ is a function. Here $C = C([-r, 0], X)$ is the Banach space of all continuous functions $\phi : [-r, 0] \to X$ endowed with the sup-norm

$$||\phi|| = sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}.$$

Also for $x \in C([-r, b], X)$ we have $x_t \in C$ for $t \in [0, b]$, $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$. By using topological degree arguments we prove the global existence of a solution of (1.1)–(1.2)

As a model for the equation (1.1) one can take

$$w_t(x, t) = w_{xx}(x, t) + \int_0^t f(s, w(x, s - r))ds, \quad 0 < x < 1, t > 0$$

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$$w(0,t) = w(1,t) = 0, t > 0, w(x,t) = \phi(x,t), \quad -r \le t \le 0$$

for which the equation (1.1) becomes as its abstract formulation. Equation (1.1) has many physical applications. This arises as a very special model for one dimensional heat flow in material with memory [7], [14].

In Section 3 we study the global existence of solutions for second order initial value problems for semilinear functional integrodifferential equations of the form

$$x''(t) = Ax(t) + \int_0^t f(s, x_s, x'(s))ds, \quad t \in [0,b], \tag{1.3}$$

$$x_0 = \phi, \quad x'(0) = \eta, \tag{1.4}$$

where A is a linear infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in a Banach space X, and $f\colon [0,T] \times C \times X \to X$ is a function.

Recent results on global existence, for ordinary, functional, neutral or partial functional differential equations with the aid of the Topological Transversality method, may be found in the works listed in our references, [5], [6], [8], [9], [10]. Our approach here is essentially an application of the Topological Transversality method to obtain global existence results for functional semilinear integrodifferential equations. For other results on integrodifferential equations see [1], [12], [14] and the references cited therein.

It is well known, see e.g [15] (for a simple case of ordinary differential equation and $A = 0$) that only the continuity of f is not sufficient to assure local existence of solutions, even when X is a Hilbert space. Therefore, one has to restrict either the function f or the semi-group operator. Usually restrictions on f are imposed. The function f was assumed to be locally Lipschitz or monotone or completely continuous. Here we assume that $T(t), C(t)$ (defined below) are compact and the function f satisfies the following Caratheodory-type conditions, which not imply that f is completely continuous:

(C_1) For each $t \in [0,b]$ the function $f(t,.) : C \to X$ (resp. $f(t,.,.) : C \times X \to X$) is continuous, and for each $x \in C$ (resp. $x, y \in C \times X$) the function $f(.,x) : [0,b] \to X$ (resp.$f(.,x,y) : [0,b] \to X$) is strongly measurable.

(C_2) For every positive integer k there exists $g_k \in L^1([0,b])$ such that for a.a. $t \in [0,b]$

$$\sup_{\|x\| \le k} |f(t,x)| \le g_k(t) \quad (\text{resp.} \quad \sup_{\|x\|, |y| \le k} |f(t,x,y)| \le g_k(t)).$$

The considerations of this paper are based on the following fixed point results (cf. [2], [4]).

Lemma 1.1 (Leray-Schauder Alternative). *Let S be a convex subset of a normed linear space E and assume $0 \in S$. Let $F\colon S \to S$ be a completely continuous operator, and let*

$$\mathcal{E}(F) = \{x \in S : x = \lambda F x \quad \text{for some} \quad 0 < \lambda < 1\}.$$

Then either $\mathcal{E}(F)$ is unbounded or F has a fixed point.

Lemma 1.2 (Nonlinear Alternative for contractions). *Let U be a bounded open subset of a Banach space E. Assume $0 \in U$ and $G : \overline{U} \to E$ is a contraction. Then either*

(i) G has a unique fixed point in $\overline{U}$, or

(ii) there exists $\lambda \in (0,1)$ and $u \in \partial U$ such that $u = \lambda G u$.

We recal that a map $H : A \to X$ between metric spaces is *contractive* provided

$$d(Hx, Hy) \leq \alpha d(x, y)$$

for all $x, y \in A$, where $0 \leq \alpha < 1$.

2. Global existence for first order IVP

In what follows we let X be a general Banach space and A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t), t \geq 0$ in X satisfying

$$|T(t)| \leq M e^{\omega t}, \quad t \geq 0$$

for some $M \geq 1$ and $\omega \in \mathbb{R}$.

By a *strong solution* of the IVP (1.1)–(1.2) on the interval $[-r, b]$ we mean a function $x : [-r, b] \to X$ which is absolutely continuous whose first derivative $x'(t)$ exists and equals to $Ax(t) + \int_0^t f(s, x_s)ds$ for a.a. $t \in [0, b]$ and which satisfies the initial condition $x_0 = \phi$.

It is known that if $T(t), t \geq 0$ is a strongly continuous semigroup of bounded linear operators in X with infinitesimal generator A, and $x(t)$ is a solution of the IVP (1.1)–(1.2) then

$$x(t) = T(t)\phi(0) + \int_0^t T(t-s) \int_0^s f(\tau, x_\tau) d\tau ds, \quad t \in [0, b]. \tag{2.1}$$

Equation (2.1) is more general than equation (1.1), and a solution of (2.1) is called a *mild solution* of (1.1)–(1.2).

Now we present the first global existence result for the IVP (1.1)–(1.2).

Theorem 2.1. *Let $f : [0, b] \times C \to X$ be a function satisfying (C_1) and (C_2). Assume that:*

(Hf) There exists an integrable function $m : [0, b] \to [0, \infty)$ such that

$$|f(t, \phi)| \leq m(t)\Omega(\|\phi\|), \quad 0 \leq t \leq b, \phi \in C,$$

where $\Omega : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

Assume also that $T(t), t > 0$ is compact.

Then if

$$\int_0^b \widehat{m}(s)ds < \int_{M\|\phi\|}^{\infty} \frac{ds}{s+\Omega(s)}, \tag{2.2}$$

where $\widehat{m}(t) = max\left\{\omega, M\int_0^t m(s)ds\right\}$, *the IVP (1.1)–(1.2) has at least one mild solution on* $[-r, b]$.

Proof. Consider the space $C([-r,b],X)$ with norm $\|x\|_1 = sup\{|x(t)| : -r \le t \le b\}$. To prove existence of a mild solution of the IVP (1.1)–(1.2) we apply Lemma 1.1. First we obtain the *a priori bounds* for the mild solutions of the IVP $(1.1)_\lambda$–(1.2), $\lambda \in (0,1)$ where $(1.1)_\lambda$ stands for the equation

$$x'(t) = \lambda Ax(t) + \lambda \int_0^t f(s, x_s)ds, \quad t \in [0, b].$$

Let x be a mild solution of the IVP $(1.1)_\lambda$–(1.2). Then we have, by (2.1)

$$|x(t)| \le Me^{\omega t}\|\phi\| + Me^{\omega t}\int_0^t e^{-\omega s}\int_0^s m(\tau)\Omega(\|x_\tau\|)d\tau ds, \quad t \in [0, b].$$

We consider the function μ given by

$$\mu(t) = sup\{|x(s)| : \ -r \le s \le t\}, \quad 0 \le t \le b.$$

Let $t^\star \in [-r, t]$ be such that $\mu(t) = |x(t^\star)|$. If $t^\star \in [0, b]$, by the previous inequality we have

$$\begin{aligned} e^{-\omega t}\mu(t) &\le M\|\phi\| + M\int_0^{t^\star} e^{-\omega s}\int_0^s m(\tau)\Omega(\|x_\tau\|)d\tau ds \\ &\le M\|\phi\| + M\int_0^t e^{-\omega s}\int_0^s m(\tau)\Omega(\mu(\tau))d\tau ds, \quad t \in [0, b]. \end{aligned}$$

If $t^\star \in [-r, 0]$ then $\mu(t) = \|\phi\|$ and the previous inequality holds since $M \ge 1$.

Denoting by $u(t)$ the right-hand side of the above inequality we have

$$u(0) = M\|\phi\|, \quad \mu(t) \le e^{\omega t}u(t), \quad 0 \le t \le b,$$

and

$$\begin{aligned} u'(t) &= Me^{-\omega t}\int_0^t m(s)\Omega(\mu(s))ds \\ &\le Me^{-\omega t}\int_0^t m(s)\Omega(e^{\omega s}u(s))ds, \quad t \in [0, b]. \end{aligned}$$

We remark that

$$\begin{aligned}(e^{\omega t}u(t))' &= \omega e^{\omega t}u(t) + e^{\omega t}u'(t)\\ &\le \omega e^{\omega t}u(t) + M\int_0^t m(s)\Omega(e^{\omega s}u(s))ds\\ &\le \omega e^{\omega t}u(t) + M\Omega(e^{\omega t}u(t))\int_0^t m(s)ds\\ &\le \widehat{m}(t)[e^{\omega t}u(t) + \Omega(e^{\omega t}u(t))], \quad t\in[0,b].\end{aligned}$$

This implies

$$\int_{u(0)}^{e^{\omega t}u(t)} \frac{ds}{s+\Omega(s)} \le \int_0^b \widehat{m}(s)ds < \int_{u(0)}^{\infty}\frac{ds}{s+\Omega(s)}, \quad 0\le t\le b.$$

This inequality implies that there is a constant K such that $u(t)\le K,\ t\in[0,b]$ and hence $\mu(t)\le K,\ t\in[0,b]$. Since for every $t\in[0,b],\quad \|x_t\|\le\mu(t)$, we have

$$\|x\|_1 = sup\{|x(t)| : -r\le t\le b\}\le K,$$

where K depends only on b and on the functions m and Ω.

In the second step we rewrite the IVP (1.1)–(1.2) as follows. For $\phi\in C$ define $\widetilde{\phi}\in C_b,\ C_b = C([-r,b],X)$ by

$$\widetilde{\phi}(t) = \begin{cases}\phi(t), & -r\le t\le 0,\\ T(t)\phi(0), & 0\le t\le b.\end{cases}$$

If $x(t) = y(t)+\widetilde{\phi}(t),\ t\in[-r,b]$ it is easy to see that y satisfies

$$\begin{aligned}y_0 &= 0\\ y(t) &= \int_0^t T(t-s)\int_0^s f(\tau, y_\tau+\widetilde{\phi}_\tau)d\tau ds, \quad 0\le t\le b\end{aligned}$$

if and only if x satisfies

$$x(t) = T(t)\phi(0) + \int_0^t T(t-s)\int_0^s f(\tau,x_\tau)d\tau ds, \quad 0\le t\le b$$

and $x_0=\phi$.

Define $C_b^0 = \{y\in C_b : y_0 = 0\}$ and $F\colon C_b^0\to C_b^0$, by

$$(Fy)(t) = \begin{cases}0, & -r\le t\le 0\\ \int_0^t T(t-s)\int_0^s f(\tau,y_\tau+\widetilde{\phi}_\tau)d\tau ds, & 0\le t\le b.\end{cases}$$

It will now be shown that F is a completely continuous operator.

Let $B_k = \{y \in C_b^0 : ||y||_1 \leq k\}$ for some $k \geq 1$. We first show that F maps B_k into an equicontinuous family. Let $y \in B_k$ and $t_1, t_2 \in [0, b]$ and $\epsilon > 0$. Then if $0 < \epsilon < t_1 < t_2 \leq b$,

$$\begin{aligned}
|(Fy)(t_1) - (Fy)(t_2)| &= \left| \int_0^{t_1} T(t_1 - s) \int_0^s f(\tau, y_\tau + \widetilde{\phi_\tau}) d\tau ds \right. \\
&\quad \left. - \int_0^{t_2} T(t_2 - s) \int_0^s f(\tau, y_\tau + \widetilde{\phi_\tau}) d\tau ds \right| \\
&\leq \left| \int_0^{t_1-\epsilon} [T(t_1 - s) - T(t_2 - s)] \int_0^s f(\tau, y_\tau + \widetilde{\phi_\tau}) d\tau ds \right| \\
&\quad + \left| \int_{t_1-\epsilon}^{t_1} [T(t_1 - s) - T(t_2 - s)] \int_0^s f(\tau, y_\tau + \widetilde{\phi_\tau}) d\tau ds \right| \\
&\quad + \left| \int_{t_1}^{t_2} T(t_2 - s) \int_0^s f(\tau, y_\tau + \widetilde{\phi_\tau}) d\tau ds \right| \\
&\leq \int_0^{t_1-\epsilon} |T(t_1 - s) - T(t_2 - s)| \int_0^s g_{k'}(\tau) d\tau ds \\
&\quad + \int_{t_1-\epsilon}^{t_1} |T(t_1 - s) - T(t_2 - s)| \int_0^s g_{k'}(\tau) d\tau ds \\
&\quad + \int_{t_1}^{t_2} |T(t_2 - s)| \int_0^s g_{k'}(\tau) d\tau ds,
\end{aligned}$$

where $k' = k + ||\widetilde{\phi}||$. The right hand side is independent of $y \in B_k$ and tends to zero as $t_2 - t_1 \to 0$ and ϵ sufficiently small, since the compacteness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology.

Thus F maps B_k into an equicontinuous family of functions.

Notice that we considered here only the case $0 < t_1 < t_2$, since the other cases $t_1 < t_2 < 0$ or $t_1 < 0 < t_2$ are very simple.

It is easy to see that the family B_k is uniform bounded. Next, we show $\overline{FB_k}$ is compact. Since we have shown FB_k is an equicontinuous collection, it suffices by Arzela-Ascoli theorem to show that F maps B_k into a precompact set in X.

Let $0 < t \leq b$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $y \in B_k$ we define

$$\begin{aligned}
(F_\epsilon y)(t) &= \int_0^{t-\epsilon} T(t - s) \int_0^s f(\tau, y_\tau + \widetilde{\phi}_\tau) d\tau ds \\
&= T(\epsilon) \int_0^{t-\epsilon} T(t - s - \epsilon) \int_0^s f(\tau, y_\tau + \widetilde{\phi}_\tau) d\tau ds.
\end{aligned}$$

Since $T(t)$ is a compact operator, the set $Y_\epsilon(t) = \{(F_\epsilon y)(t) : y \in B_k\}$ is precompact in X, for every ϵ, $0 < \epsilon < t$. Moreover, for every $y \in B_k$ we have

$$\begin{aligned}|(Fy)(t) - (F_\epsilon y)(t)| &\leq \int_{t-\epsilon}^{t} |T(t-s) \int_0^s f(\tau, y_\tau + \widetilde{\phi}_\tau)| d\tau ds \\ &\leq \int_{t-\epsilon}^{t} |T(t-s)| \int_0^s g_{k'}(\tau) d\tau ds.\end{aligned}$$

Therefore there are precompact sets arbitrary close to the set $\{(Fy)(t) : y \in B_k\}$. Hence the set $\{(Fy)(t) : y \in B_k\}$ is precompact in X.

It remains to show that $F : C_b^0 \to C_b^0$ is continuous. Let $\{u_n\}_0^\infty \subseteq C_b^0$ with $u_n \to u$ in C_b^0. Then there is an integer q such that $|u_n(t)| \leq q$ for all n and $t \in [0, b]$, so $u_n \in B_q$ and $u \in B_q$. By (C_1) $f(t, u_n(t) + \widetilde{\phi}_t) \to f(t, u(t) + \widetilde{\phi}_t)$ for each $t \in [0, b]$ and since $|f(t, u_n(t) + \widetilde{\phi}_t) - f(t, u(t) + \widetilde{\phi}_t)| \leq 2g_{q'}(t), q' = q + \|\widetilde{\phi}\|$ we have by dominated convergence

$$\begin{aligned}\|Fu_n - Fu\| &= \sup_{t\in[0,b]} \left| \int_0^t T(t-s) \int_0^s [f(\tau, u_n(\tau) + \widetilde{\phi}_\tau) - f(\tau, u(\tau) + \widetilde{\phi}_\tau)] d\tau ds \right| \\ &\leq \int_0^b |T(t-s)| \int_0^s |f(\tau, u_n(\tau) + \widetilde{\phi}_\tau) - f(\tau, u(\tau) + \widetilde{\phi}_\tau)| ds \to 0\end{aligned}$$

Thus F is continuous. This completes the proof that F is completely continuous.

Finally, the set $\mathcal{E}(F) = \{y \in C_b^0 : y = \lambda F y,\ \lambda \in (0, 1)\}$ is bounded, since for every solution y in $\mathcal{E}(F)$ the function $x = y + \widetilde{\phi}$ is a mild solution of IVP $(1.1)_\lambda$–(1.2), for which we have proved that $\|x\|_1 \leq K$ and hence

$$\|y\|_1 \leq K + \|\widetilde{\phi}\|.$$

Consequently, by Lemma 1.1, the operator F has a fixed point in C_b^0. This means that the IVP (1.1)-(1.2) has a mild solution, completing the proof of the theorem.□

A more general IVP than the IVP (1.1)–(1.2) is the following

$$x'(t) = A(t)x(t) + \int_0^t f(s, x_s) ds, \quad t \in I, \tag{2.4}$$

$$x_0 = \phi, \tag{2.5}$$

where $A(t)$ is a linear closed densely defined operator in a Banach space X, and $f : I \times X \to X$ is a given function.

It is well known that the IVP (2.4)–(2.5) can be written as a nonlinear Volterra integral equation

$$x(t) = W(t, 0)x(0) + \int_0^t W(t, s) \int_0^s f(\tau, x_\tau) d\tau ds, \quad t \in [0, b],$$

with $x_0 = \phi$, where $\{W(t,s) : 0 \leq s \leq t \leq b\}$ is a strongly continuous family of evolution operators on X. We shall make the following assumptions on the evolution system $W(t,s)$:

(W_1) $W(t,s) \in L(X)$, the space of bounded linear transformations on X, whenever $0 \leq s \leq t \leq b$ and for each $x \in X$ the mapping $(t,s) \to W(t,s)x$ is continuous.

(W_2) $W(t,s)W(s,r) = W(t,r), 0 \leq r \leq s \leq t \leq b$.

(W_3) $W(t,t) = I$, the identity operator on X.

(W_4) $W(t,s)$ is a compact operator whenever $t - s > 0$.

Sufficient conditions for $(W_1) - (W_4)$ to hold may be found in Friedman [3]. If the conditions in [3] are satisfied, and if for each $t \in [0,b]$ there is a number λ in the resolvent set of $A(t)$ such that the resolvent $R(\lambda; A(t))$ is compact the generated evolution system will satisfy (W_4).

Theorem 2.2. *Let $\{W(t,s) : 0 \leq s \leq t \leq b\}$ satisfy $(W_1) - (W_4)$ and $f\colon [0,b] \times C \to X$ be a function satisfying (C_1), (C_2) and (Hf).*

Then the IVP (2.4)–(2.5) has at least one mild solution on $[-r,b]$ provided

$$N \int_0^b \int_0^t m(s)dsdt < \int_{\widehat{N}\|\phi\|}^{\infty} \frac{ds}{\Omega(s)}, \tag{2.6}$$

where $N = sup\{|W(t,s)| : 0 \leq s \leq t \leq b\}$ and $\widehat{N} = max\{1, N\}$.

Proof. We apply again Lemma 1.1. In order to apply this Lemma we must establish the *a priori bounds* for the solutions of the IVP $(2.4)_\lambda$–(2.5), $\lambda \in (0,1)$, where $(2.4)_\lambda$ stands for the equation

$$x'(t) = \lambda A(t)x(t) + \lambda \int_0^t f(s, x_s)ds, \quad t \in I.$$

Let x be a solution of the IVP $(2.4)_\lambda$–(2.5). Then we have

$$|x(t)| \leq N\|\phi\| + N \int_0^t \int_0^s m(\tau)\Omega(\|x_\tau\|)d\tau ds, \quad 0 \leq t \leq b,$$

or

$$\mu(t) \leq \widehat{N}\|\phi\| + N \int_0^t \int_0^s m(\tau)\Omega(\mu(\tau))d\tau ds, \quad 0 \leq t \leq b,$$

with μ as defined in the propf of Theorem 2.1.

Denoting by $u(t)$ the right-hand side of the above inequality we have

$$u(0) = \widehat{N}\|\phi\|, \quad \mu(t) \leq u(t), \quad 0 \leq t \leq b,$$

and

$$u'(t) = N \int_0^t m(s)\Omega(\mu(s))ds \leq N\Omega(u(t)) \int_0^t m(s)ds, \quad 0 \leq t \leq b.$$

Then

$$\int_{u(0)}^{u(t)} \frac{ds}{\Omega(s)} \leq N \int_0^b \int_0^t m(s)dsdt < \int_{u(0)}^{\infty} \frac{ds}{\Omega(s)}, \quad 0 \leq t \leq b.$$

This inequality implies that there is a constant K such that $u(t) \leq K, \; t \in [0,b]$ and hence $\mu(t) \leq K, \; t \in [0,b]$. Since for every $t \in [0,b], \; \|x_t\| \leq \mu(t)$, we have

$$\|x\|_1 \leq K,$$

where K depends only on b and the functions m and Ω.

We will rewrite the IVP (2.4)–(2.5) as follows. For $\phi \in C$ define $\widetilde{\phi} \in C_b, C_b = C([-r,b],X)$ by

$$\widetilde{\phi}(t) = \begin{cases} \phi(t), & -r \leq t \leq 0, \\ W(t,0)\phi(0), & 0 \leq t \leq b. \end{cases}$$

If $x(t) = y(t) + \widetilde{\phi}(t), \; t \in [-r,b]$ it is easy to see that y satisfies

$$\begin{aligned} y_0 &= 0 \\ y(t) &= \int_0^t W(t,s) \int_0^s f(\tau, y_\tau + \widetilde{\phi}_\tau), d\tau ds, \quad 0 \leq t \leq b. \end{aligned}$$

Define $C_b^0 = \{y \in C_b : y_0 = 0\}$ and $F \colon C_b^0 \to C_b^0$, by

$$(Fy)(t) = \begin{cases} 0, & -r \leq t \leq 0 \\ \int_0^t W(t,s) \int_0^s (\tau, y_\tau + \widetilde{\phi}_\tau) d\tau ds, & 0 \leq t \leq b. \end{cases}$$

We can prove, as in Theorem 2.1 that F is a completely continuous operator.

The proof of the theorem is now complete. □

Remark. Although the IVP (1.1)-(1.2) is a special case of the IVP (2.4)-(2.5) when $A(t) \equiv A$, constant, the condition (2.2) is different from the condition (2.6). Generally Theorem 2.1 cannot be derived from Theorem 2.2.

By applying the Nonlinear Alternative for contractive maps we have the following result for the IVP (2.4)–(2.5).

Theorem 2.3. *Let $\{W(t,s) : 0 \leq s \leq t \leq b\}$ satisfy $(W_1) - (W_4)$ and $f \colon [0,b] \times C \to X$ be a function satisfying (C_1), (C_2) and (Hf). Moreover we assume that:*

(ℓf) *For $h > 0$ there exists $l_h \geq 0$ such that*

$$|f(t,u) - f(t,v)| \leq l_h \|u - v\|$$

for $t \in [0,b]$ and $u, v \in C$, satisfying $\|u\|, \|v\| \leq h$.

Then if

$$N\int_0^b\int_0^t m(s)dsdt < \int_{\widehat{N}||\phi||}^{\infty}\frac{ds}{\Omega(s)},$$

where $N = sup\{|W(t,s)| : 0 \le s \le t \le b\}, \widehat{N} = max\{1, N\}$ *the IVP (2.4)–(2.5) has a unique mild solution on* $[-r, b]$.

Proof. By Theorem 2.2 there exists a constant K such that $\|x\|_1 < K$, for all solutions of the family of problems

$$\begin{cases} x'(t) = \lambda A(t)x(t) + \lambda\int_0^t f(s, x_s)ds, & 0 \le t \le b \\ x_0 = \lambda\phi, \end{cases}$$

for $\lambda \in [0, 1]$.

Let L be a constant ($> \frac{1}{N}$). In the space $C([-r, b], X)$ consider the two norms:

$$\|x\|_1 = sup\{|x(t)| : t \in [-r, b]\}$$

$$\|x\|_L = sup\{e^{-NLt}|x(t)| : t \in [-r, b]\}.$$

Since $\|x\|_L \le \|x\|_1 \le e^{NLb}\|x\|_L$, these norms are equivalent. Put

$$\overline{U} = \{x \in C([-r, b], X) : \|x\|_1 \le K,\ \ t \in [-r, b]\}$$

and consider the operator $G : \overline{U} \to C([-r, b], X)$ defined by

$$(Gy)(t) = \begin{cases} \phi(t), & -r \le t \le 0 \\ W(t,0)\phi(0) + \int_0^t W(t-s)\int_0^s f(\tau, y_\tau)d\tau ds, & 0 \le t \le b. \end{cases}$$

We shall prove that the operator G from $(\overline{U}, \|.\|_L)$ into $C([-r, b], X), \|.\|_L)$, is a contraction, with $L = \ell_K, \ell_K$ the constant defined in (ℓf). Indeed we have:

$$\begin{aligned}
\|Gx - Gy\|_L &\le sup\left\{e^{-LNt}LN\int_0^t\int_0^s e^{-LN\tau}e^{LN\tau}\|x_\tau - y_\tau\|d\tau ds\right\} \\
&\le LN\|x-y\|_L\left\{e^{-LNt}\int_0^t\int_0^s e^{LN\tau}d\tau ds\right\} \\
&\le LN\|x-y\|_L\left\{e^{-LNt}\frac{1}{LN}\int_0^t\left(e^{LNs}-1\right)ds\right\} \\
&\le \|x-y\|_L\left\{\frac{1}{LN}\left(1-e^{-LNt}\right) - \frac{1}{LN}te^{-LNt}\right\} \\
&\le \|x-y\|_L\left(1-e^{-LNb}\right) \\
&\le \alpha\|x-y\|_L,\ \ \alpha = 1-e^{-LNb} < 1.
\end{aligned}$$

Hence by Lemma 1.2, this completes the proof, since (ii) of the alternative cannot hold by the choice of U. $\square$

3. Global existence for second order IVP

In this section we study the global existence of solutions for IVP (1.3)–(1.4).

In many cases it is advantageous to treat second order abstract differential equations directly rather than to convert them to first order systems. A useful machinery for the study of abstract second order equations is the theory of strongly continuous cosine families.

Given a Banach space X, we say that the family $\{C(t) : t \in \mathbb{R}\}$ in the space $L(X)$ of bounded linear operators on X is a *strongly continuous cosine family* if

(i) C(0)=I;

(ii) $C(t)x$ is strongly continuous in t on $\mathbb{R}$ for each fixed $x \in X$;

(iii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$.

The *strongly continuous sine family* $\{S(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)x = \int_0^t C(s)x ds, \quad x \in X, t \in \mathbb{R}.$$

The *infinitesimal generator* A of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ax = \frac{d^2}{dt^2} C(t)x|_{t=0}, \quad x \in D(A) = \{x \in X : C(.)x \in C^2(\mathbb{R}, X)\}.$$

Assume now that A is a linear infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators from X into itself. Moreover we assume that the adjoint operator $A^\star$ is densely defined i.e $\overline{D(A^\star)} = X^\star$. See [1].

It is known that if $C(t), t \in \mathbb{R}$ is a strongly continuous cosine family with infinitesimal generator A, and $x(t)$ is a solution of the IVP (1.3)–(1.4), then

$$x(t) = C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s) \int_0^s f(\tau, x_\tau, x'(\tau)) d\tau ds, \quad t \in [0, b], \tag{3.1}$$

with $x_0 = \phi$. Equation (3.1) is easier to work with than (1.3)–(1.4) because of the nice properties of the bounded operators $C(t), t \in \mathbb{R}$ and $S(t), t \in \mathbb{R}$, as opposed to the unbounded operator A in equation (1.3).

The global existence result for the IVP (1.3)–(1.4) is the following:

Theorem 3.1. *Let $f: [0, b] \times C \times X \to X$ be a function satisfies $(C_1), (C_2)$ and $C(t)$ (resp. $S(t)$), $t \in [0, b]$ be a strongly continuous cosine (resp. sine) family on X with the infinitesimal generator A as defined above. Assume that:*

$(Hf-1)$ *There exists an integrable function $m : [0, b] \to [0, \infty)$ such that*

$$|f(t, \phi, v)| \leq m(t)\Omega\left(max(\|\phi\|, |v|)\right), \quad 0 \leq t \leq b, \phi \in C, v \in X,$$

where $\Omega : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

Assume also that $C(t), t > 0$ is compact.

Then if

$$M(b+1)\int_0^b \int_0^t m(\tau)d\tau dt < \int_c^\infty \frac{ds}{\Omega(s)}, \tag{3.2}$$

where $M = sup\{|C(t)| : t \in [0,b]\}$, $M' = sup\{|C'(t)| : t \in [0,b]\}$ and $c = (M+M')\|\phi\| + M(1+b)|\eta|$, the IVP (1.3)–(1.4) has at least one mild solution on $[-r, b]$.

Proof. In the space $B = C([-r,b], X) \cap C^1([0,b], X)$ consider the norm

$$\|x\|^\star = max\{\|x\|_r, \|x\|_1\}$$

where

$$\|x\|_r = sup\{|x(t)| : -r \le t \le b\}, \quad \|x\|_1 = sup\{|x'(t)| : 0 \le t \le b\}.$$

To prove existence of a mild solution of the IVP (1.3)–(1.4) we apply Lemma 1.1. First we obtain the *a priori bounds* for the mild solutions of the IVP $(1.3)_\lambda$–(1.4), $\lambda \in (0,1)$ where $(1.3)_\lambda$ stands for the equation

$$x''(t) = \lambda A x(t) + \lambda \int_0^t f(s, x_s, x'(s))ds, \quad t \in [0,b].$$

Let x be a mild solution of the IVP $(1.3)_\lambda$–(1.4). Then we have

$$\begin{aligned} |x(t)| &\le M\|\phi\| + Mb|\eta| + Mb\int_0^t \int_0^s m(\tau)\Omega(max(\|x_\tau\|, |x'(\tau)|))d\tau ds \\ &\le M\|\phi\| + Mb|\eta| + Mb\int_0^t \int_0^s m(\tau)\Omega(\|x_\tau\| + |x'(\tau)|)d\tau ds, \quad t \in [0,b], \end{aligned}$$

or

$$\mu(t) \le M\|\phi\| + Mb|\eta| + Mb\int_0^t \int_0^s m(\tau)\Omega(\mu(\tau) + |x'(\tau)|)d\tau ds, \quad t \in [0,b],$$

with μ as defined in the proof of the Theorem 2.1.

Denoting by $u(t)$ the right-hand side of the above inequality we have

$$u(0) = M\|\phi\| + Mb|\eta|, \quad \mu(t) \le u(t), \quad 0 \le t \le b,$$

and

$$u'(t) \le Mb\int_0^t m(s)\Omega(u(s) + |x'(s)|)ds, \quad 0 \le t \le b.$$

Therefore, if
$$v(t) = sup\{|x'(s)| : s \in [0,t]\},\ \ t \in [0,b]$$
we obtain
$$u'(t) \le Mb\Omega(u(t) + v(t)) \int_0^t m(s)ds, \quad 0 \le t \le b.$$
But
$$x'(t) = C'(t)\phi(0) + S'(t)\eta + \int_0^t C(t-s) \int_0^s f(\tau, x_\tau, x'(\tau))d\tau ds, \quad 0 \le t \le b$$
and thus
$$v(t) \le M'\|\phi\| + M|\eta| + M \int_0^t \int_0^s m(\tau)\Omega(\mu(\tau) + \epsilon(\tau))d\tau ds, \quad t \in [0,b].$$
Denoting by $r(t)$ the right-hand side in the above inequality we have
$$r(0) = M'\|\phi\| + M|\eta|, \quad v(t) \le r(t),\ \ 0 \le t \le b,$$
and
$$r'(t) = M \int_0^t m(s)\Omega(\mu(s) + v(s))ds \le M\Omega(u(t) + r(t)) \int_0^t m(s)ds, t \in [0,b].$$
Hence, we obtain
$$(u(t) + r(t))' \le M(1+b)\Omega(u(t) + r(t)) \int_0^t m(s)ds, \quad t \in [0,b].$$
This implies
$$\int_{u(0)+r(0)}^{u(t)+r(t)} \frac{ds}{\Omega(s)} \le M(1+b) \int_0^b \int_0^t m(s)dsdt < \int_c^\infty \frac{ds}{\Omega(s)}, \quad 0 \le t \le b.$$
This inequality implies that there is a constant K such that
$$u(t) + r(t) \le K, \quad t \in [0,b].$$
Then
$$|x(t)| \le \mu(t) \le u(t), \quad t \in [0,b]$$
$$|x'(t)| \le v(t) \le r(t), \quad t \in [0,b]$$
and hence
$$\|x\|^\star \le K,$$
where K depends only on b and on the functions m and Ω.

In order to apply Lemma 1.1 we must prove that the operator $F : B \to B$ defined by

$$(Fy)(t) = \begin{cases} \phi(t), & -r \le t \le 0 \\ C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)\int_0^s f(\tau, y_\tau, y'(\tau))d\tau ds, & 0 \le t \le b. \end{cases}$$

is a completely continuous operator. We proceed as in Theorem 2.1.

Let $B_k = \{y \in B : \|y\|^\star \le k\}$ for some $k \ge 1$. We first show that F maps B_k into an equicontinuous family. Let $y \in B_k$ and $t_1, t_2 \in [0, b]$ and $\epsilon > 0$. Then if $0 < \epsilon < t_1 < t_2 \le b$,

$$\begin{aligned}|(Fy)(t_1) - (Fy)(t_2)| = &|C(t_1)\phi(0) - C(t_2)\phi(0) + S(t_1)\eta \\ &- S(t_2)\eta + \int_0^{t_1} S(t_1 - s)\int_0^s f(\tau, y_\tau, y'(\tau))d\tau ds \\ &- \int_0^{t_2} S(t_2 - s)\int_0^s f(\tau, y_\tau, y'(\tau))d\tau ds\Big| \\ &\le |C(t_1) - C(t_2)||\phi(0)| + |S(t_1) - S(t_2)||\eta| \\ &+ \int_0^{t_1-\epsilon} |S(t_1 - s) - S(t_2 - s)| \int_0^s g_k(\tau)d\tau ds \\ &+ \int_{t_1-\epsilon}^{t_1} |S(t_1 - s) - S(t_2 - s)| \int_0^s g_k(\tau)d\tau ds \\ &+ \int_{t_1}^{t_2} |S(t_2 - s)| \int_0^s g_k(\tau)d\tau ds\end{aligned}$$

and

$$\begin{aligned}|(Fy)'(t_1) - (Fy)'(t_2)| = &|C'(t_1)\phi(0) - C'(t_2)\phi(0) + S'(t_1)\eta \\ &- S'(t_2)\eta + \int_0^{t_1} C(t_1 - s)\int_0^s f(\tau, y_\tau, y'(\tau))d\tau ds \\ &- \int_0^{t_2} C(t_2 - s)\int_0^s f(\tau, y_\tau, y'(\tau))d\tau ds\Big| \\ &\le |C'(t_1) - C'(t_2)||\phi(0)| + |S'(t_1) - S'(t_2)||\eta| \\ &+ \int_0^{t_1-\epsilon} |C(t_1 - s) - C(t_2 - s)| \int_0^s g_k(\tau)d\tau ds \\ &+ \int_{t_1-\epsilon}^{t_1} |C(t_1 - s) - C(t_2 - s)| \int_0^s g_k(\tau)d\tau ds \\ &+ \int_{t_1}^{t_2} |C(t_2 - s)| \int_0^s g_k(\tau)d\tau ds.\end{aligned}$$

The right hand sides are independent of $y \in B_k$ and tends to zero as $t_2 - t_1 \to 0$ and ϵ sufficiently small, since $C(t), S(t), C'(t), S'(t)$ are uniformly continuous for

$t > 0$ and the compacteness of $C(t), S(t)$ for $t > 0$ implies the continuity in the uniform operator topology. The compactness of $S(t)$ follows from that of $C(t)$ and the Proposition 2.1 of [11] and Lemma 2.5 of [13].

Thus F maps B_k into an equicontinuous family of functions.

The equicontinuity for the cases $t_1 < t_2 \leq 0$ and $t_1 \leq 0 \leq t_2$ follows from the uniform continuity of ϕ on $[-r, 0]$ and from the relation

$$|(Fy)(t_1) - (Fy)(t_2)| \leq |\phi(t_1) - (Fy)(t_2)| \leq |(Fy)(t_2) - (Fy)(0)| + |\phi(0) - \phi(t_1)|$$

respectively.

It is easy to see that the family B_k is uniform bounded.

Next, we show that for each fixed t the set $\{(Fy)(t) : y \in B_k\}$ is precompact in X. Let $0 < t \leq b$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $y \in B_k$ we define

$$(F_\epsilon y)(t) = C(t)\phi(0) + S(t)\eta + \int_0^{t-\epsilon} S(t-s) \int_0^s f(\tau, y_\tau, y'(\tau)) d\tau ds.$$

Since $S(t)$ is a compact operator, the set $Y_\epsilon(t) = \{(F_\epsilon y)(t) : y \in B_k\}$ is precompact in X, for every ϵ, $0 < \epsilon < t$. Moreover, for every $y \in B_k$ we have

$$|(Fy)(t) - (F_\epsilon y)(t)| \leq \int_{t-\epsilon}^t |S(t-s)| \int_0^s g_k(\tau) d\tau ds,$$

$$|(Fy)'(t) - (F_\epsilon y)'(t)| \leq \int_{t-\epsilon}^t |C(t-s)| \int_0^s g_k(\tau) d\tau ds.$$

Hence the set $\{(Fy)(t) : y \in B_k\}$ is precompact in X.

Next, we show that $F : B \to B$ is continuous. Let $\{u_n\}_0^\infty \subseteq C_b^0$ with $u_n \to u$ in C_b^0. Then there is an integer q such that $|u_n(t)|, |u_n'(t)| \leq q$ for all n and $t \in [0, b]$, so $u_n, u_n' \in B_q$ and $u, u' \in B_q$. By (C_1) $f(t, u_n(t), u_n'(t)) \to f(t, u(t), u'(t))$ for each $t \in [0, b]$ and since $|f(t, u_n(t), u_n'(t)) - f(t, u(t), u'(t))| \leq 2g_{q'}(t)$ we have by dominated convergence

$$\begin{aligned} \|Fu_n - Fu\| &= \sup_{t \in [0,b]} \left| \int_0^t S(t-s) \int_0^s [f(\tau, u_n(\tau), u_n'(\tau)) - f(\tau, u(\tau), u'(\tau))] d\tau ds \right| \\ &\leq \int_0^b |S(t-s)| \int_0^s |f(\tau, u_n(\tau), u_n(\tau)) - f(\tau, u(\tau), u'(\tau))| ds \to 0 \end{aligned}$$

and

$$\begin{aligned} &\|(Fu_n)' - (Fu)'\| \\ &\quad = \sup_{t \in [0,b]} \left| \int_0^t C(t-s) \int_0^s [f(\tau, u_n(\tau), u_n'(\tau)) - f(\tau, u(\tau), u'(\tau))] d\tau ds \right| \\ &\quad \leq \int_0^b |C(t-s)| \int_0^s |f(\tau, u_n(\tau), u_n(\tau)) - f(\tau, u(\tau), u'(\tau))| ds \to 0 \end{aligned}$$

This completes the proof that F is completely continuous.

Finally, the set $\mathcal{E}(F) = \{y \in B : y = \lambda F y, \lambda \in (0,1)\}$ is bounded, as we proved in the first part. Consequently, by Lemma 1.1, the operator F has a fixed point in B. This means that the IVP (1.3)–(1.4) has a mild solution, completing the proof of the theorem. □

Also by applying the Nonlinear Alternative for conrtactive maps we have the following result for the IVP (1.3)–(1.4).

Theorem 3.2. *Let $f: [0,b] \times C \times X \to X$ be a function satisfying (C_1), (C_2) and $(Hf-1)$. Assume that:*

$(\widehat{\ell f})$ *For $h > 0$ there exists $l_h \geq 0$ such that*

$$|f(t,u,v) - f(t,w,z)| \leq l_h(\|u-w\| + \|v-z\|_1)$$

for $t \in [0,b]$ and $u, w \in C$, $v, z \in X$ satisfying $\|u\|, \|v\|_1, \|w\|, \|z\|_1 \leq h$.

Assume also that $C(t), t > 0$ is compact.

Then if

$$M(b+1)\int_0^b \int_0^t m(\tau)d\tau dt < \int_c^\infty \frac{ds}{\Omega(s)},$$

where $M = sup\{|C(t)| : 0 \leq s \leq t \leq b\}$, $M' = sup\{|C'(t)| : 0 \leq s \leq t \leq b\}$and $c = (M + M')\|\phi\| + M(1+b)|\eta|$, the IVP (1.3)–(1.4) has a unique mild solution on $[-r,b]$.

Proof. By Theorem 3.1 there exists a constant K such that $\|x\|^\star < K$, for all solutions of the family of problems

$$\begin{cases} x''(t) = \lambda A x(t) + \lambda \int_0^t f(s, x_s, x'(s))ds, & 0 \leq t \leq b, \\ x_0 = \lambda\phi, \quad x'(0) = \lambda\eta, \end{cases}$$

for $\lambda \in [0,1]$.

Let L be a constant and $Q = M(1+b)$. In the space $B = C([-r,b],X) \cap C^1([0,b],X)$ consider the two norms:

$$\|x\|^\star = max\{\|x\|_r, \|x\|_1\}$$

$$\|x\|_L = max\{\|x\|_{Lr}, \|x\|_{L1}\},$$

where

$$\|x\|_r = sup\{|x(t)| : -r \leq t \leq b\}, \quad \|x\|_1 = sup\{|x'(t)| : 0 \leq t \leq b\},$$

$$\|x\|_{Lr} = sup\{e^{-QLt}|x(t)| : t \in [-r,b]\}, \ \|x\|_{L1} = sup\{e^{-QLt}|x'(t)| : t \in [0,b]\}.$$

Since $||x||_L \leq ||x||^\star \leq e^{QLb}||x||_L$, these norms are equivalent. Put

$$\overline{U} = \{x \in C([-r,b],X) \cap C^1([0,b],X) : ||x||_r, ||x||_1 \leq K\}$$

and consider the operator $G : \overline{U} \to C([-r,b],X) \cap C^1([0,b],X)$ defined by

$$(Gy)(t) = \begin{cases} \phi(t), & -r \leq t \leq 0 \\ C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)\int_0^s f(\tau, y_\tau, y'(\tau))d\tau ds, & 0 \leq t \leq b. \end{cases}$$

We have, if $L = l_K$, that:

$$\begin{aligned} ||Gx - Gy||_L &\leq sup\left\{e^{-LQt}L\int_0^t (|S(t-s)| + |C(t-s)|)\right. \\ &\quad \left.\times \int_0^s (||x_\tau - y_\tau|| + |x'(\tau) - y'(\tau)|)d\tau ds, t \in [0,b]\right\} \\ &\leq sup\left\{e^{-LQt}LQ\int_0^t\int_0^s e^{-LQ\tau}e^{LQ\tau}(||x_\tau - y_\tau||\right. \\ &\quad \left.+|x'(\tau) - y'(\tau)|)d\tau ds\right\} \\ &\leq LQ||x-y||_L\left\{e^{-LQt}\int_0^t\int_0^s e^{LQ\tau}d\tau ds\right\} \\ &\leq LQ||x-y||_L\left(\frac{1}{L^2Q^2} - \frac{e^{-LQb}}{L^2Q^2} - \frac{be^{-LQb}}{LQ}\right) \\ &\leq \frac{1}{LQ}(1 - e^{-LQb})||x-y||_L \\ &\leq (1 - e^{-LQb})||x-y||_L. \end{aligned}$$

This means that G is a contraction. Hence by Lemma 1.2 this completes the proof. □

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