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# SECOND ORDER MULTIVALUED BOUNDARY VALUE PROBLEMS

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Abstract. In this paper we use the method of upper and lower solutions to study multivalued Sturm–Liouville and periodic boundary value problems, with Caratheodory orientor field. We prove two existence theorems. One when the orientor field $F(t,x,y)$ is convex–valued and the other when $F(t,x,y)$ is nonconvex valued. Finally we show that the "convex" problem has extremal solutions in the order interval determined by an upper and a lower solution.

## 1. Introduction

The method of upper and lower solutions has been successfully applied to study the existence of multiple solutions for initial and boundary value problems of the first and second order. This method generates solutions of the problem, located in an order interval with the upper and lower solutions serving as bounds. Moreover, this method coupled with some monotonicity type hypotheses, leads to monotone iterative techniques which generate in a constructive way (amenable to numerical treatment) the extremal solutions within the order interval determined by the upper and lower solutions.

This method has been used only in the context of single–valued differential equations and the majority of the works assume that the vector field is continuous in all variables and so they look for solutions in the Banach spaces $C^2(0,b)$. We refer to the books of Bernfeld–Lakshmikantham [1] and Gaines–Mawhin [9], to the papers of Leela [16], Omari [22], Omari–Trombetta [23] and to the references therein. Extensions to Caratheodory vector fields were obtained by Cabada–Nieto [4], Gao–Wang [10], Nieto [19], Nieto–Cabada [20], Nkashama [21] and Papageorgiou–Papalini [27].

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So far the method has not been used in the context of multivalued problems (differential inclusions). The papers on multivalued boundary value problems, use other approaches which require stronger growth conditions on the multivalued vector field. For a sample of these methods we refer to the works of Erbe–Krawcewicz [8], Kandilakis–Papageorgiou [13], Kravvaritis–Papageorgiou [15] and Marano [17]. In this paper we employ the method of upper and lower solutions to study a general class of second order differential inclusions with Sturm–Liouville and periodic boundary conditions. We prove two existence theorems under Caratheodory–type conditions on the multivalued vector field (also known as orientor field). The first existence result assumes that the orientor field has convex values ("convex" existence theorem). The second existence result does not require convexity in the values of the orientor field ("nonconvex" existence theorem). Both theorems establish the existence of solutions in the order interval determined by an upper and a lower solution. Moreover, for the "convex" problem, we prove the existence of extremal solutions for the problem (i.e. we show that the problem has a greatest and a least solution in the order interval). The same set of results is also proved for the periodic problem.

The multivalued problems studied in this paper, arise in many applied situations of interest:

(a) "Closed loop systems": $-x^{''}(t) = f(t, x(t), x^{'}(t), u(t))$ a.e. on $T$, with $u(t) \in U(t, x(t))$ a.e. on $T$. In this case $F(t, x, y) = f(t, x, y, U(t, x))$.

(b) "Implicit differential equations": $f(t, x(t), x^{'}(t), x^{''}(t)) = 0$ a.e. on $T$. In this case $F(t, x, y) = \{-v \in \mathbb{R} : f(t, x, y, v) = 0\}$.

(c) "Deterministic systems with uncertainties": $-x^{''}(t) \in f(t, x(t), x^{'}(t)) + [-r(t, x(t)), r(t, x(t))]$ a.e. on $T$. In this case $F(t, x, y) = f(t, x, y) + [-r(t, x), r(t, x)]$.

(d) "Problems with discontinuous right hand side": $-x^{''}(t) = f(t, x(t), x^{'}(t))$ a.e. on $T$. Since this problem need not have a solution, in order to develop an existence theory, we replace it by a multivalued one which is obtained by, roughly speaking, filling in the gaps at the discontinuity points (see Chang [5]). So we introduce $f_1(t, x, y) = \underline{\lim}_{\substack{x'\to x\\ y'\to y}} f(t, x^{'}, y^{'})$ and $f_2(t, x, y) = \overline{\lim}_{\substack{x'\to x\\ y'\to y}} f(t, x^{'}, y^{'})$. Assuming that these functions are superpositionally measurable ($N$–measurable in the terminology of Chang [5]), we introduce $F(t, x, y) = [f_1(t, x, y), f_2(t, x, y)]$ and consider the problem $-x^{''}(t) \in F(t, x(t), x^{'}(t))$ a.e. on $T$. Note that for every $t \in T$, $f_1(t, \cdot, \cdot)$ is lower semicontinuous and $f_2(t, \cdot, \cdot)$ is upper semicontinuous (see Chang [5]), hence $F(t, \cdot, \cdot)$ is usc (see Klein–Thompson [14]).

To our knowledge the most effective (and often the only) way to analyze such problems, is through the theory of differential inclusions.

## 2. Preliminaries

Let $T = [0, b]$. We will be studying the following Sturm–Liouville and periodic multivalued boundary value problems:

$$\left\{ \begin{array}{ll} -x''(t) \in F(t, x(t), x'(t)) & \text{a.e. on } T \\ (B_0 x)(0) = k_0, \quad (B_1 x)(b) = k_1 & \end{array} \right\} \tag{1}$$

and

$$\left\{ \begin{array}{ll} -x''(t) \in F(t, x(t), x'(t)) & \text{a.e. on } T \\ x(0) = x(b), \quad x'(0) = x'(b) & \end{array} \right\} \tag{2}$$

In problem (1), $(B_0 x)(0) = a_0 x(0) - c_0 x'(0)$ and $(B_1 x)(b) = a_1 x(b) + c_1 x'(b)$, with $a_0, a_1, c_0, c_1 \geq 0$ and $a_0(a_1 b + c_1) + c_0 a_1 \neq 0$. This last condition guarantees that the homogeneous problem $x''(t) = 0$ a.e. on $T$, $(B_0 x)(0) = 0$, $(B_1 x)(b) = 0$, has only the trivial solution. Then for any $h \in L^1(T)$, the solution of $x''(t) = h(t)$ a.e. on $T$, $(B_0 x)(0) = k_0$, $(B_1 x)(b) = k_1$, is given by

$$x(t) = u(t) + \int_0^b G(t, s) h(s)\, ds, \quad t \in T$$

where $u \in C^2(0, b)$ is the solution of $u''(t) = 0$, $t \in T$, $(B_0 u)(0) = k_0$, $(B_1 u)(b) = k_1$ and $G \in C(T \times T)$ is the Green's function of our problem with homogeneous boundary conditions (see for example Bernfeld–Lakshmikantham [1]).

In sections 3 and 4, we deal with the Sturm–Liouville problem and in section 5, we show that the same results with minor modifications in the proofs are also valid for the periodic problem.

So let us start by defining what we mean by a solution of problem (1).

**Definition.** A function $x \in W^{2,1}(T)$ is said to be a "solution of (1)", if there exists $v \in L^1(T)$ such that $v(t) \in F(t, x(t), x'(t))$ a.e. on $T$, $-x''(t) = v(t)$ a.e. on $T$ and $(B_0 x)(0) = k_0$, $(B_1 x)(b) = k_1$.

Now we introduce the notions of upper and lower solutions for problem (1), which are the basic tools in the approach that we follow in this paper.

**Definition.** A function $\phi \in W^{2,1}(T)$ is said to be an "upper solution" for problem (1), if there exists a function $v_1 \in L^1(T)$ such that $v_1(t) \in F(t, \phi(t), \phi'(t))$ a.e. on $T$, $-\phi''(t) \geq v_1(t)$ a.e. on $T$ and $(B_0 \phi)(0) \geq k_0$, $(B_1 \phi)(b) \geq k_1$. Similarly a function $\psi \in W^{2,1}(T)$ is said to be a "lower solution", if there exists $v_0 \in L^1(T)$ such that $v_0(t) \in F(t, \psi(t), \psi'(t))$ a.e. on $T$, $-\psi''(t) \leq v_0(t)$ a.e. on $T$ and $(B_0 \psi)(0) \leq k_0$, $(B_1 \psi)(b) \leq k_1$.

We will briefly recall some basic definitions and facts from multivalued analysis that we will use in the sequel. So let $X$ be a separable Banach space. In what follows we use the following notation:

$$P_{f(c)}(X) = \{A \subseteq X : A \text{ is nonempty, closed (and convex)}\}.$$

If $T = [0, b]$, then a multifunction $F : T \to P_f(X)$ is said to be "measurable", if for every $x \in X$, the function $t \to d(x, F(t)) = \inf[\|x - y\| : y \in F(t)]$ is measurable. In fact if on $T$ we consider the Lebesgue $\sigma$–field $\mathcal{L}$ (the completion of the Borel $\sigma$–field $B(T)$ with respect to the Lebesgue measure), then the above definition of measurability is equivalent to saying that $GrF = \{(t, y) \in T \times X : y \in F(t)\} \in \mathcal{L} \times B(X)$, with $B(X)$ being the Borel $\sigma$–field of $X$ (graph measurability). Finally a third equivalent definition of measurability of $F(\cdot)$, is to say that there exists a sequence of Lebesgue measurable functions $f_n : T \to X$, $n \geq 1$, such that $F(t) = \overline{\{f_n(t)\}}_{n\geq 1}$ for all $t \in T$. For details and additional results, we refer to the survey paper of Wagner [29].

Given a multifunction $F : T \to P_f(X)$, by $S^1_F$ we denote the set of selectors of $F(\cdot)$ which belong in the Lebesgue–Bochner space $L^1(T, X)$; i.e. $S^1_F = \{f \in L^1(T, X) : f(t) \in F(t) \text{ a.e. on } T\}$. In general there is no guarantee that this set is nonempty. Through an easy application of the "Yankov–von Neumann–Aumann selection theorem" (see for example Wagner [29], theorem 5.10), we can check that for a measurable $F(\cdot)$, $S^1_F \neq \varnothing$ if and only if $\inf\{\|y\| : y \in F(t)\} \in L^1(T)$. Also for a measurable $F(\cdot)$, $S^1_F$ is convex (and closed) if and only if for almost all $t \in T$, $F(t)$ is convex (and closed). Also $S^1_F$ is bounded in $L^1(T)$ if and only if $t \to \sup[\|y\| : y \in F(t)]$ belongs in $L^1(T)$. An important property that the set $S^1_F$ has, is that it is "decomposable"; i.e. for all $(f_1, f_2, A) \in S^1_{F_1} \times S^1_{F_2} \times \mathcal{L}$, we have $\chi_A f_1 + \chi_{A^c} f_2 \in S^1_F$. This property, which formally looks like convexity (note that $\chi_{A^c} = 1 - \chi_A$), gives to the set $S^1_F$ a structure very similar to that of a regular convex set (see Papageorgiou [26]).

Next let $Y, Z$ be Hausdorff topological spaces and $G : Y \to 2^Z \setminus \{\varnothing\}$ a multifunction. We say that $G(\cdot)$ is "upper semicontinuous" (usc for short), if for every open set $V \subseteq Z$, the set $G^+(V) = \{y \in Y : G(y) \subseteq V\}$ is open in $Y$. Evidently this is equivalent to saying that for every $C \subseteq Z$ closed, the set $G^-(C) = \{y \in Y : G(y) \cap C \neq \varnothing\}$ is closed in $Y$. When $G(\cdot)$ is closed valued (i.e. for every $y \in Y$, $G(y) \in P_f(Z)$), and $Z$ is regular, upper semicontinuity implies that the graph $GrG = \{(y, z) \in Y \times Z : z \in G(y)\}$ is closed in $Y \times Z$ with the product topology. The converse is true if $\overline{G(Y)} = \overline{\bigcup_{y\in Y} G(y)}$ is compact in $Z$. Another important notion of continuity of $G(\cdot)$, is that of lower semicontinuity. More specifically, we say that $G : Y \to 2^Z \setminus \{\varnothing\}$ is "lower semicontinuous" (lsc for short), if for every $V \subseteq Z$ open, the set $G^-(V) = \{y \in Y : G(y) \cap V \neq \varnothing\}$ is open in $Y$ (equivalently if for every $C \subseteq Z$ closed, the set $G^+(C) = \{y \in Y : G(y) \subseteq C\}$ is closed in $Y$). When $Y$ and $Z$ are metrizable spaces, then we can give some alternative equivalent characterizations of lower semicontinuity. Namely $G(\cdot)$ is lsc if and only if for every $y_n \to y$ in $Y$ as $n \to \infty$, then $G(y) \subseteq \underline{\lim} G(y_n) = \{z \in Z : \lim d_Z(z, G(y_n)) = 0\}$, with $d_Z(\cdot, \cdot)$ denoting a metric which generates the topology of $Z$. Also $G(\cdot)$ is lsc if and only if for every $z \in Z$, $y \to d_Z(z, G(y))$ is an upper semicontinuous $\mathbb{R}_+$–valued function. Note that when $G(\cdot)$ is single–valued, then both notions of upper and lower semicontinuity, coincide with that of continuity. For further results on these and related notions, we refer to the paper of DeBlasi–Myjak [6] and the book of Klein–Thompson [14].

If $Y, Z$ are locally convex spaces, then a multifunction $G : Y \to 2^Z \setminus \{\varnothing\}$ is said to be "bounded", if it maps bounded sets to bounded sets. Now suppose that $Y, Z$ are Banach spaces. An operator $A : Y \to Z$ (not necessarily linear), is said to be "completely continuous", if it is continuous and maps bounded sets in $Y$, into relatively compacts subsets of $Z$.

## 3. Existence results

In this section, assuming the existence of an upper solution $\phi$ and a lower solution $\psi$ such that $\psi \le \phi$, we prove the existence of at least one solution located in the order interval $K = [\psi, \phi] = \{x \in W^{2,1}(T) : \psi(t) \le x(t) \le \phi(t) \text{ for all } t \in T\}$, for both the convex and nonconvex problems.

We start with convex problem. Then our hypotheses on the data of the problem are the following:

$\mathbf{H(F)_1}$: $F : T \times \mathbb{R} \times \mathbb{R} \to P_{f_c}(\mathbb{R})$ is a multifunction such that

(i) for every $(x, y) \in \mathbb{R} \times \mathbb{R}$, $t \to F(t, x, y)$ is measurable;
(ii) for almost all $t \in T$, $(x, y) \to F(t, x, y)$ has closed graph;
(iii) for every $r > 0$, there exists $\gamma_r \in L^1(T)$ such that for almost all $t \in T$ and all $|x|, |y| \le r$ we have $|F(t, x, y)| = \sup[|v| : v \in F(t, x, y)] \le \gamma_r(t)$.

$\mathbf{H_0}$: There exist an upper solution $\phi \in W^{2,1}(T)$ and a lower solution $\psi \in W^{2,1}(T)$ such that $\psi(t) \le \phi(t)$ for all $t \in T$ and also there exists $h \in C(\mathbb{R}_+, (0, +\infty))$ such that $|F(t, x, y)| \le h(|y|)$ for all $t \in T$, $x \in [\psi(t), \phi(t)]$, $y \in \mathbb{R}$ and $\displaystyle\int_\theta^\infty \frac{r}{h(r)}\,dr > \max_{t\in T}\phi(t) - \min_{t\in T}\psi(t)$, with $\theta = \dfrac{1}{b}\max[|\psi(0) - \phi(b)|, |\psi(b) - \phi(0)|]$.

**Remark.** In the second part of hypothesis $H_0$, we recognize the classical Bernstein–Nagumo growth condition, which permits the derivation of a priori bounds for the first derivative of the solution of $-x''(t) \in F(t, x(t), x'(t))$ a.e. on $T$. More precisely, we can prove the following lemma:

**Lemma 1.**

**If** hypothesis $H_0$ holds and $x \in W^{2,1}(T)$ is a solution of $-x''(t) \in F(t, x(t), x'(t))$ a.e. on $T$ such that for all $t \in T$ $\psi(t) \le x(t) \le \phi(t)$,
**then** there exists $N_1$ depending only on $\phi, \psi, h$ such that $|x'(t)| \le N_1$ for all $t \in T$.

**Proof.** Using hypothesis $H_0$, we can find $N_1 > \theta$ such that

$$\max_{t\in T}\phi(t) - \min_{t\in T}\psi(t) < \int_\theta^{N_1} \frac{r}{h(r)}\,dr.$$

We will show that for all $t \in T$, $|x^{'}| \leq N_1$. Proceeding by contradiction, suppose that this is not the case and that there exists $\widehat{t} \in T$ such that $|x^{'}(\widehat{t})| > N_1$. By the mean value theorem, we can find $t_0 \in (0,b)$ such that $x^{'}(t_0) = \frac{x(b)-x(0)}{b} \Rightarrow |x^{'}(t_0)| = \frac{1}{b}|x(b)-x(0)| \leq \frac{1}{b}\max[|\psi(0)-\phi(b)|, |\psi(b)-\phi(0)|]$ (note that $\psi(b)-\phi(0) \leq x(b)-x(0) \leq \phi(b)-\psi(0)$). Hence we have

$$\left|x^{'}(t_0)\right| \leq \theta < N_1 \quad \text{and} \quad \left|x^{'}(\widehat{t})\right| > N_1.$$

Since $x^{'}(\cdot)$ is continuous (recall that $W^{2,1}(T)$ is embedded continuously in $C^1(T)$), we can find $[t_1, t_2] \subseteq T$, where one of the following holds:

(a) $x^{'}(t_1) = \theta,\ x^{'}(t_2) = N_1$ and $\theta < x^{'}(t) < N_1$ for all $t \in (t_1, t_2)$;

(b) $x^{'}(t_1) = N_1,\ x^{'}(t_2) = \theta$ and $\theta < x^{'}(t) < N_1$ for all $t \in (t_1, t_2)$;

(c) $x^{'}(t_1) = -\theta,\ x^{'}(t_2) = -N_1$ and $-N_1 < x^{'}(t) < -\theta$ for all $t \in (t_1, t_2)$;

(d) $x^{'}(t_1) = -N_1,\ x^{'}(t_2) = -\theta$ and $-N_1 < x^{'}(t) < -\theta$ for all $t \in (t_1, t_2)$.

We assume that case (a) holds. Cases (b), (c) and (d) can be treated similarly. We have:

$$-x^{''}(t) \in F\left(t, x(t), x^{'}(t)\right) \quad \text{a.e. on } T$$

$$\Rightarrow \quad \left|x^{''}(t)\right| x^{'}(t) \leq \left|F\left(t, x(t), x^{'}(t)\right)\right| x^{'}(t) \leq h\left(\left|x^{'}(t)\right|\right) x^{'}(t) \quad \text{a.e. on } T$$

$$\Rightarrow \quad \frac{\left|x^{''}(t)\right| x^{'}(t)}{h(|x^{'}(t)|)} \leq x^{'}(t) \quad \text{a.e. on } T \quad (\text{since } h(\cdot) \text{ is } \mathbb{R}_+ \setminus \{0\}\text{–valued}).$$

Integrating over $[t_1, t_2]$, we obtain

$$\int_{t_1}^{t_2} \frac{\left|x^{''}(t)\right| x^{'}(t)}{h(|x^{'}(t)|)}\, dt \leq \int_{t_1}^{t_2} x^{'}(t)\, dt = x(t_2) - x(t_1) \leq \max_{t \in T} \phi(t) - \min_{t \in T} \psi(t).$$

If we let $r = x^{'}(t)$, we have $dr = x^{''}(t)\, dt$ and $x^{'}(t_1) = \theta$, $x^{'}(t_2) = N_1$. Therefore we obtain

$$\int_{\theta_1}^{N_1} \frac{r}{h(r)}\, dr \leq \max_{t \in T} \phi(t) - \min_{t \in T} \psi(t),$$

which contradicts the choice of $N_1$. Since the other cases can be treated similarly, we have proved the lemma. □

We introduce the order interval $K = [\psi, \phi] = \{x \in W^{2,1}(T) : \psi(t) \leq x(t) \leq \phi(t)$ for all $t \in\}$. We are looking for solutions of (1) in the order interval $K$. In the next theorem we prove the existence of such solutions for the convex problem.

**Theorem 2.**

**If** *hypotheses* $H(F)_1$ *and* $H_0$ *hold,*

**then** *problem* (1) *has a solution* $x \in W^{2,1}(T)$ *in* $K = [\psi, \phi]$.

**Proof.** From lemma 1, we know that every solution $x \in W^{2,1}(T)$ of (1), which belongs in $K$, satisfies $|x^{'}(t)| \leq N_1$ for all $t \in T$, with $N_1$ depending on $\phi, \psi, h$ only. Set $N_1 = 1 + \max(N, \|\phi^{'}\|_\infty, \|\psi^{'}\|_\infty)$. Also we define the truncation operator $\tau \in W^{1,1}(T) \to W^{1,1}(T)$ by

$$\tau(x)(t) = \begin{cases} \phi(t) & \text{if} \quad \phi(t) \leq x(t) \\ x(t) & \text{if} \quad \psi(t) \leq x(t) \leq \phi(t) \\ \psi(t) & \text{if} \quad x(t) \leq \psi(t). \end{cases}$$

From Gilbarg–Trudinger [11] (p. 145), we have that for $x \in W^{1,1}(T)$, $\tau(x) \in W^{1,1}(T)$ and

$$\tau(x)^{'}(t) = \begin{cases} \phi^{'}(t) & \text{if} \quad \phi(t) < x(t) \\ x^{'}(t) & \text{if} \quad \psi(t) \leq x(t) \leq \phi(t) \\ \psi^{'}(t) & \text{if} \quad x(t) < \psi(t). \end{cases}$$

In addition we introduce the truncation at $N$ function $q_N : L^1(T) \to L^1(T)$, defined by

$$q_N(h)(t) = \begin{cases} N & \text{if} \quad N \leq h(t) \\ h(t) & \text{if} \quad -N \leq h(t) \leq N \\ -N & \text{if} \quad h(t) \leq -N, \end{cases}$$

and the penalty function $u : T \times \mathbb{R} \to \mathbb{R}$ defined by

$$u(t,x) = \begin{cases} x - \phi(t) & \text{if} \quad \phi(t) \leq x \\ 0 & \text{if} \quad \psi(t) \leq x \leq \phi(t) \\ x - \psi(t) & \text{if} \quad x \leq \psi(t). \end{cases}$$

Let $N_F(x) = S^1_{F(\cdot, \tau(x)(\cdot), q_N(\tau(x)')(\cdot))}$ for every $x \in W^{1,1}(T)$ and define $G : W^{1,1}(T) \to 2^{L^1(T)}$ by

$$G(x) = \{v \in N_F(x) : v(t) \geq v_0(t) \ \text{ a.e. on } \ [x < \psi < \phi]$$

$$\text{and} \ \ v(t) \leq v_1(t) \ \text{ a.e. on } [\psi < \phi < x]\}.$$

*Claim #1:* For every $x \in W^{1,1}(T)$, $G(x)$ is nonempty, closed and convex.

The closedness and convexity of $G(x)$ are clear. So it remains to show that $G(x)$ is nonempty. Consider the multifunction $t \to F(t, \tau(x)(t), q_N(\tau(x)^{'})(t))$. We claim that this multifunction has a measurable selector. Note that hypotheses $H(F)_1$ (i) and (ii), in general, do not imply that $t \to F(t, \tau(x)(t), q_N(\tau(x)^{'})(t))$ is measurable (see Papageorgiou [25]). So we can not apply directly the Yankov–von Neumann–Aumann selection theorem and obtain a measurable selector of $t \to F(t, \tau(x)(t), q_N(\tau(x)^{'})(t))$. To obtain the desired measurable selector, we proceed as follows: Let $\{s_n\}_{n \geq 1}$, $\{r_n\}_{n \geq 1}$ be simple functions such that $|s_n(t)| \leq |\tau(x)(t)|$, $|r_n(t)| \leq |q_N(\tau(x)^{'})(t)|$, $s_n(t) \to \tau(x)(t)$ and $r_n(t) \to q_N(\tau(x)^{'})(t)$ a.e. on $T$ as $n \to \infty$. Because of hypothesis $H_0$ (i), for every $n \geq 1$ $t \to F(t, s_n(t), r_n(t))$ is measurable and so we can apply the Yankov–von Neumann–Aumann selection theorem and obtain $v_n : T \to \mathbb{R}$ a measurable function such that $v_n(t) \in F(t, s_n(t), r_n(t))$ a.e. on $T$. By hypothesis $H(F)_1$ (iii) $|v_n(t)| \leq \gamma_r(t)$ a.e. on $T$, $n \geq 1$, with $r > \max[\|\psi\|_\infty, \|\phi\|_\infty, N]$ and $\gamma_r \in L^1(T)$. Invoking the Dunford–Pettis theorem, we may assume that $v_n \xrightarrow{w} \widehat{v}$ in $L^1(T)$ as $n \to \infty$. Then theorem 3.1 Papageorgiou [24], implies that $\widehat{v}(t) \in \operatorname{conv} \overline{\lim}\{v_n(t)\}_{n\geq 1} \subseteq \operatorname{conv} \overline{\lim} F(t, s_n(t), r_n(t)) \subseteq F(t, \tau(x)(t), q_N(\tau(x)^{'})(t))$ a.e on $T$, the last inclusion being a consequence of hypothesis $H(F)_1$ (ii). So $\widehat{v} \in N_F(x)$. Set $v = \chi_{A_1}\widehat{v} + \chi_{A_2}v_0 + \chi_{A_3}v_1$, where $A_1 = [\psi \leq x \leq \phi]$, $A_2 = [x < \psi < \phi]$ and $A_3 = [\psi < \phi < x]$. By decomposability $v \in G(x)$ and so we have proved the claim.

*Claim #2:* $G(\cdot)$ is usc from $W^{1,1}(T)$ into $L^1(T)$ furnished with the weak topology.

We need to show that for every $C \subseteq L^1(T)$ weakly closed, $G^-(C) = \{x \in W^{1,1}(T) : G(x) \cap C \neq \varnothing\}$ is closed. So let $\{x_n\}_{n\geq 1} \subseteq G^-(C)$ and assume that $x_n \to x$ in $W^{1,1}(T)$ as $n \to \infty$. Since $W^{1,1}(T)$ is embedded continuously in $C(T)$ (see Brezis [3]), we also have that $x_n \to x$ in $C(T)$ as $n \to \infty$. Let $v_n \in G(x_n) \cap C$, $n \geq 1$. Then for all $n \geq 1$, $|v_n(t)| \leq \gamma_r(t)$ a.e. on $T$, with $r > \max[\|\psi\|_\infty, \|\phi\|_\infty, N]$. Thus invoking the Dunford–Pettis theorem and by passing to a subsequence if necessary, we may assume that $v_n \xrightarrow{w} v$ in $L^1(T)$ as $n \to \infty$. Then using as before theorem 3.1 of [24], we have that $v(t) \in \operatorname{conv} \overline{\lim}\{v_n(t)\}_{n\geq 1}$ for all $t \in T \setminus S_1$, with $\lambda(S_1) = 0$ ($\lambda$ being the Lebesgue measure on $T$). Note that the truncation maps $\tau : W^{1,1}(T) \to W^{1,1}(T)$ and $q_N : L^1(T) \to L^1(T)$ defined in the beginning of the proof, are continuous. Hence by passing to a suitable subsequence, we can say that $\tau(x_n)(t) \to \tau(x)(t)$ and $q_N(\tau(x_n)^{'})(t) \to q_N(\tau(x)^{'})(t)$ a.e. on $T$. Then using as before theorem 3.1 of [24] and hypothesis $H(F)_1$ (ii), we deduce that $v \in N_F(x)$. If $t \in [x < \psi < \phi] \setminus S_1$, then $t \in [x_n < \psi < \phi]$ for all $n \geq n_0$ and so $v_n(t) \geq v_0(t)$, from which it follows that $v(t) \geq v_0(t)$. Similarly, if $t \in [\psi < \phi < x]$, then $t_n \in [\psi < \phi < x]$ for all $n \geq n_1$ and so $v_n(t) \leq v_1(t)$ from which it follows that $v(t) \leq v_1(t)$. So $v \in G(x) \cap C$ and this proves the claim.

Now let $D = \{x \in W^{2,1}(T) : (B_0x)(0) = k_0, \ (B_1x)(b) = k_1\}$ and define $\widehat{L} : D \subseteq L^1(T) \to L^1(T)$ by $\widehat{L}(x) = -x^{''}$. Set $L = I + \widehat{L}$.

*Claim #3:* $L^{-1} : L^1(T) \to D \subseteq W^{1,1}(T)$ exists and is completely continuous.

First we will show that $R(L) = L^1(T)$ (i.e. the operator $L$ is onto). For this purpose we need to show that for every $h \in L^1(T)$, the Sturm–Liouville problem

$$\left\{ \begin{array}{l} -x^{''}(t) + x(t) = h(t) \ \text{ a.e. on } T \\ (B_0x)(0) = k_0, \ (B_1x)(b) = k_1 \end{array} \right\} \tag{3}$$

has a solution $x \in W^{2,1}(T)$. If $h \in C(T)$, then from Mönch [18] we know that problem (3) has a unique solution. For the general case, let $h \in L^1(T)$ and let $\{h_n\}_{n\geq 1} \subseteq C(T)$ be such that $h_n \to h$ in $L^1(T)$ as $n \to \infty$. Denote by $x_n \in W^{2,1}(T)$, $n \geq 1$, the unique solution of (3) when the right hand side is $h_n$. From Mönch [18] we know that $x_n(t) = u(t) + \int_0^b G(t,s)h_n(s)\,ds$, where $u \in W^{2,1}(T)$ is the unique solution of $u^{''} = 0$, $(B_0u)(0) = k_0$, $(B_1u)(b) = k_1$ and

$G \in C(T \times T)$ is the Green's function of the problem (see section 2). Then it follows that $|x_n(t)| \leq |u(t)| + \|G\|_\infty \|h_n\|_1 \leq u(t) + M_1$ for all $n \geq 1$ and all $t \in T$. So $\{x_n\}_{n\geq 1}$ is uniformly integrable. Recalling that $\|x_n\|_1 + \|x_n^{''}\|_1$ is an equivalent norm on $W^{2,1}(T)$ (see Brezis [3], p.132), we deduce that $\{x_n\}_{n\geq 1}$ is bounded in $W^{2,1}(T)$. Since $W^{2,1}(T)$ is embedded compactly in $W^{1,1}(T)$ and $\{x_n^{''}\}_{n\geq 1}$ is uniformly integrable, by passing to a subsequence if necessary, we may assume that $x_n \to x$ in $W^{1,1}(T)$ and $x_n^{''} \xrightarrow{w} v$ in $L^1(T)$ as $n \to \infty$. Evidently $v = x^{''}$ and so $x \in W^{2,1}(T)$. Moreover $-x^{''}(t) + x(t) = h(t)$ a.e. on $T$, $(B_0 x)(0) = k_0$, $(B_1 x)(b) = k_1$. So problem (3) has a solution $x \in W^{2,1}(T)$ for every $h \in L^1(T)$, i.e. $R(L) = L^1(T)$.

Next let $x_1, x_2 \in D$ and set $x = x_1 - x_2$. Define

$$T_+ = \{t \in T : x(t) > 0\} \quad \text{and} \quad T_- = \{t \in T : x(t) < 0\}$$

both open sets. For $\lambda > 0$ we have

$$\begin{aligned}
\int_0^b \left|x(t) - \lambda x^{''}(t)\right| dt &\geq \int_{T_+} \left|x(t) - \lambda x^{''}(t)\right| dt + \int_{T-} \left|x(t) - \lambda x^{''}(t)\right| dt \\
&\geq \int_{T_+} \left(x(t) - \lambda x^{''}(t)\right) dt - \int_{T_-} \left(x(t) - \lambda x^{''}(t)\right) dt \\
&= \int_{T_+} x(t)\, dt - \int_{T_-} x(t)\, dt - \lambda \int_{T_+} x^{''}(t)\, dt + \lambda \int_{T_-} x^{''}(t)\, dt \\
&= \int_0^b |x(t)|\, dt - \lambda \left[\int_{T_+} x^{''}(t)\, dt - \int_{T_-} x^{''}(t)\, dt\right].
\end{aligned}$$

Let $T_k$ be a component of $T_+$. Then $T_k = (a, c)$ for some $0 < a < c < b$ or $T_k = [0, a)$ for some $0 < a < b$ or finally $T_k = (c, b]$ for some $0 < c < b$. If $T_k = (a, c)$ we have $x(a) = x(c) = 0$ and $x(t) > 0$ for $t \in (a, c)$. Thus $x^{'}(a) \geq 0$ and $x^{'}(c) \leq 0$ and from this it follows that $\int_a^c x^{''}(t)\, dt = x^{'}(c) - x^{'}(a) \leq 0$. If $T_k = [0, a)$, we have $x(a) = 0$ and $x^{'}(a) \leq 0$. Then $\int_0^a x^{''}(t)\, dt = x^{'}(a) - x^{'}(0) \leq 0$, since if $c_0 \neq 0$, $x(0) = \frac{a_0}{c_0} x^{'}(0)$ (recall that $(B_0 x)(0) = 0$ since $x = x_1 - x_2$), while if $c_0 = 0$, then $x^{'}(0) = 0$ and so we fall in the previous case. Hence $\int_{T_+} x^{''}(t)\, dt \leq 0$. In a similar fashion, we can also show that $\int_{T_-} x^{''}(t)\, dt \geq 0$. Thus we have

$$\begin{aligned}
&\int_0^b \left|x(t) - \lambda x^{''}(t)\right| dt \geq \int_0^b |x(t)|\, dt \\
\Rightarrow \quad &\left\|x_1 + \lambda \widehat{L} x_1 - \left(x_2 + \lambda \widehat{L} x_2\right)\right\|_1 \geq \|x_1 - x_2\|.
\end{aligned}$$

From this it follows that $(I+\widehat{L})^{-1} = L^{-1} : L^1(T) \to D \subseteq L^1(T)$ is well–defined and nonexpansive (in fact is the resolvent of the $m$–accretive operator $\widehat{L}$; see Vrabie [28], Lemma 1.1.5, p. 20). Let $\theta > 0$ and consider the set

$$E_\theta = \left\{ x \in D : \|x\|_1 + \|x^{''}\|_1 \leq \theta \right\}.$$

Using once again the fact that $\|x\|_1 + \|x^{''}\|_1$ is an equivalent norm on $W^{2,1}(T)$, we have that $E_\theta$ is bounded in $W^{2,1}(T)$. Since the latter space is embedded compactly in $L^1(T)$, we infer that $E_\theta$ is relatively compact in $L^1(T)$. So from proposition 2.2.1, p. 56 of Vrabie [28], we have that $L^{-1} : L^1(T) \to D \subseteq L^1(T)$ is completely continuous. Now let $V \subseteq L^1(T)$ be bounded, $h \in V$ and let $x = L^{-1}(h)$. Then $-x^{''} + x = h$ and from what we proved earlier, we have

$$\begin{aligned} & \|x\|_1 \leq \|-x^{''} + x\|_1 \leq \sup[\|h\|_1 : h \in V] = |V| < \infty \\ \Rightarrow \quad & \|x^{''}\|_1 \leq 2|V|. \end{aligned}$$

Therefore it follows that $L^{-1}(V) = (I+\widehat{L})^{-1}(V)$ is bounded in $W^{2,1}(T)$. Since the latter is embedded compactly in $W^{1,1}(T)$, it follows that $(I+\widehat{L})^{-1}(V)$ is relatively compact in $W^{1,1}(T)$. Moreover if $h_n \to h$ in $L^1(T)$ as $n \to \infty$ and $x_n = (I+\widehat{L})^{-1}(h_n)$, $n \geq 1$, then from what we proved earlier $x_n \to x = (I+\widehat{L})^{-1}(h)$ in $L^1(T)$ as $n \to \infty$ and $\{x_n\}_{n \geq 1}$ is bounded in $W^{2,1}(T)$. Exploiting once more the compact embedding of $W^{2,1}(T)$ into $W^{1,1}(T)$, we have $x_n \to x$ in $W^{1,1}(T)$ as $n \to \infty$, which proves that $L^{-1} : L^1(T) \to D \subseteq W^{1,1}(T)$ is completely continuous as claimed.

Now let $U : L^1(T) \to L^1(T)$ be defined by $U(x)(\cdot) = u(\cdot, x(\cdot))$ and set $G_1(x) = G(x) - U(x) + x$ for every $x \in W^{1,1}(T)$. Evidently $G_1(\cdot)$ has nonempty, closed and convex values in $L^1(T)$ (see claim #1). Moreover, by virtue of claim #2, $G_1(\cdot)$ is usc from $W^{1,1}(T)$ into $L^1(T)$ equipped with the weak topology. Finally from the definition of the penalty function $u(t,x)$, it follows that for every $x \in W^{1,1}(T)$, $|G_1(x)| = \sup[\|g\|_1 : g \in G_1(x)] \leq \eta^*$, where $\eta^* = \|\gamma\|_1 + b \max[\|\psi\|_\infty, \|\phi\|_\infty]$ and $r > \max[\|\psi\|_\infty, \|\phi\|_\infty, N]$. Therefore by virtue of claim #3, $\overline{L^{-1}G_1(W^{1,1}(T))}$ is compact and $L^{-1}G_1 : W^{1,1}(T) \to D \subseteq W^{1,1}(T)$ has nonempty, closed, convex values and is usc. Thus we can apply theorem 2 of Himmelberg [12] and produce $x \in D$ such that $x \in L^{-1}G_1(x)$; i.e. $x \in D$ is a fixed point of the multivalued map $L^{-1}G_1(\cdot)$. Let $S \subseteq D$ be the set of these fixed points.

*Claim #4:* $S \subseteq K = [\psi, \phi]$.

Let $x \in S$. Then by definition we have

$$\left\{ \begin{array}{l} -x^{''}(t) = v(t) - u(t, x(t)) \quad \text{a.e. on } T \\ (B_0 x)(0) = k_0, \ (B_1 x)(b) = k_1 \end{array} \right\} \tag{4}$$

with $v \in G(x)$; hence $v(t) \in F(t, \tau(x)(t), q_N(\tau(x)^{'})(t))$ a.e. on $T$, $v(t) \geq v_0(t)$ a.e. on $[x < \psi < \phi]$ and $v(t) \leq v_1(t)$ a.e. on $[\psi < \phi < x]$.

Also since $\psi \in W^{2,1}(T)$ is a lower solution for (1), by definition we have

$$\left\{ \begin{array}{l} -\psi''(t) \le v_0(t) \quad \text{a.e. on } T \\ (B_0\psi)(0) \le k_0, \ (B_1\psi)(b) \le k_1 \end{array} \right\} \tag{5}$$

with $v_0 \in L^1(T)$, $v_0(t) \in F\left(t, \psi(t), \psi'(t)\right)$ a.e. on $T$. From (4) and (5) it follows that

$$\psi''(t) - x''(t) \ge v(t) - v_0(t) - u(t, x(t)) \quad \text{a.e. on } T$$

Multiply the above inequality with $(\psi - x)_+(t)$ and then integrate over $T = [0, b]$. We obtain:

$$\int_0^b \left(\psi'' - x''\right)(t)(\psi - x)_+(t)\,dt = \int_0^b (v - v_0)(t)(\psi - x)_+(t)\,dt$$

$$- \int_0^b u(t, x(t))(\psi - x)_+(t)\,dt. \tag{6}$$

From Green's formula, we have

$$\begin{aligned} & \int_0^b \left(\psi'' - x''\right)(t)(\psi - x)_+(t)\,dt \\ = \ & (\psi' - x')(b)(\psi - x)_+(b) - \left(\psi' - x'\right)(0)(\psi - x)_+(0) \\ - \ & \int_0^b \left(\psi' - x'\right)(t)(\psi - x)'_+(t)\,dt. \end{aligned}$$

From the boundary conditions, we know that

$$c_1\psi'(b) \le k_1 - a_1\psi(b) \text{ and } c_1x'(b) = k_1 - a_1x(b).$$

If $c_1 = 0$, then $a_1\psi(b) \le k_1$ and $c_1x(b) = k_1$. So $\psi(b) \le x(b)$, from which we have $(\psi - x)_+(b) = 0$. If $c_1 > 0$, then $\psi'(b) \le \frac{k_1}{c_1} - \frac{a_1}{c_1}\psi(b)$ and $-x'(b) = -\frac{k_1}{c_1} + \frac{a_1}{c_1}x(b)$. Hence $(\psi' - x')(b) \le \frac{a_1}{c_1}(x - \psi)(b)$ and so $(\psi' - x')(b)(\psi - x)_+(b) \le \frac{a_1}{c_1}(x - \psi)(b)(\psi - x)_+(b) \le 0$. Thus we always have $(\psi' - x')(b)(\psi - x)_+(b) \le 0$. Similarly we show that $(\psi' - x')(0)(\psi - x)_+(0) \ge 0$. Therefore

$$\int_0^b \left(\psi'' - x''\right)(t)(\psi - x)_+(t)\,dt \le -\int_0^b \left(\psi' - x'\right)(t)(\psi - x)'_+\,dt$$

$$= -\int_0^b \left[(\psi - x)'_+(t)\right]^2 dt \le 0. \tag{7}$$

Also we have

$$\int_0^b (v - v_0)(t)(\psi - x)_+(t)\,dt = \int_{\{\psi > x\}} (v - v_0)(t)(\psi - x)(t)\,dt \ge 0 \tag{8}$$

since $v \in G(x)$. Using (7) and (8) in (6), we obtain

$$
\begin{aligned}
& -\int_0^b u(t,x(t))(\psi - x)_+(t)\,dt \le 0 \\
\Rightarrow\ & \int_0^b (\psi - x)_+^2(t)\,dt \le 0 \ \ \text{(recall the definition of the penalty function } u), \\
\Rightarrow\ & \psi(t) \le x(t) \ \ \text{for all } t \in T.
\end{aligned}
$$

Similarly we can show that $x(t) \le \phi(t)$ for all $t \in T$. Therefore $S \subseteq K = [\psi, \phi]$. But then for every $x \in S$, we have $\tau(x) = x$ and $q_N(\tau(x)') = q_N(x') = x'$. So $x \in D \subseteq W^{2,1}(T)$ solves problem (1). □

We can also have a nonconvex existence theorem. For this purpose the hypotheses on $F(t,x,y)$ are the following:

**H(F)$_2$**: $F : T \times \mathbb{R} \times \mathbb{R} \to P_f(\mathbb{R})$ is a multifunction such that

(i) $(t,x,y) \to F(t,x,y)$ is graph measurable;
(ii) for almost all $t \in T$, $(x,y) \to F(t,x,y)$ is lsc;
(iii) for every $r > 0$, there exists $\gamma_r \in L^1(T)$ such that for almost all $t \in T$ and all $|x|, |y| \le r$, we have $|F(t,x,y)| = \sup[|v| : v \in F(t,x,y)] \le \gamma_r(t)$.

**Theorem 3.**

**If** *hypotheses $H(F)_2$ and $H_0$ hold,*
**then** *problem* (1) *has a solution $x \in W^{2,1}(T)$ in the order interval $K = [\psi, \phi]$.*

**Proof.** As in the proof of theorem 2, for every $x \in W^{1,1}(T)$ let $N_F(x) = S^1_{F(\cdot,\tau(x)(\cdot),q_N(\tau(x)')(\cdot))}$ and let $G : W^{1,1}(T) \to 2^{L^1(T)}$ be defined by

$$
\begin{aligned}
G(x) = \{v \in L^1(T) : \quad & v(t) = f(t) \ \text{ a.e. on } \ [\psi \le x \le \phi] \cap [\psi < \phi], \\
& v(t) = \max[f(t), v_0(t)] \ \text{ a.e. on } \ [x < \psi < \phi], \\
& v(t) = \min[f(t), v_1(t)] \ \text{ a.e. on } \ [\psi < \phi < x], \\
& v(t) = \psi''(t) \ \text{ a.e. on } \ [\psi = \phi], \\
& f \in N_F(x), \ \ f(t) \ge v_0(t) \ \text{ a.e. on } \ [x = \psi < \phi] \\
& \qquad\qquad\quad f(t) \le v_1(t) \ \text{ a.e. on } \ [\psi < \phi = x]\}.
\end{aligned}
$$

By Stampacchia's theorem (see lemma 7.7, p. 145 of Gilbarg–Trudinger [11]), we know that $x'(t) = \psi'(t)$ a.e. on $[x = \psi]$ and $x'(t) = \phi'(t)$ a.e. on $[x = \phi]$. So because of hypothesis $H(F)_2$ (i), we have that for every $x \in W^{1,1}(T)$, $G(x) \ne \varnothing$ and it is clear that it is decomposable and closed.

*Claim #1: $G : W^{1,1}(T) \to P_f(L^1(T))$ is lsc.*

Let $x_n \to x$ in $W^{1,1}(T)$ as $n \to \infty$. We need to show that $G(x) \subseteq \underline{\lim} G(x_n) = \{v \in L^1(T) : v = \lim v_n$ in $L^1(T)$, $v_n \in G(x_n)$, $n \ge 1\} = \{v \in L^1(T) :$

$\lim d_{L^1(T)}(v, G(x_n)) = 0\}$ (see section 2). To this end let $v \in G(x)$. Then by definition we can find $f \in N_F(x)$ such that $v(t) = f(t)$ a.e. on $[\psi \leq x \leq \phi] \cap [\psi < \phi]$, $f(t) \geq v_0(t)$ a.e. on $[x = \psi < \phi]$ and $f(t) \leq v_1(t)$ a.e. on $[\psi < \phi = x]$. By virtue of theorem 4.1 of [24], we can find $g_n \in N_F(x_n)$, $n \geq 1$, such that $g_n \to f$ in $L^1(T)$ as $n \to \infty$. Define $f_n = \max[g_n, v_0]\chi_{A_1^n} + \min[g_n, v_1]\chi_{A_2^n} + g_n\chi_{A_3^n}$, where $A_1^n = [x_n = \psi < \phi]$, $A_2^n = [\psi < \phi = x_n]$, $A_n^3 = T \setminus (A_1^n \cup A_2^n)$. Evidently $f_n \in N_F(x_n)$ and $f_n \to f$ in $L^1(T)$. By passing to a subsequence if necessary, we can say that $f_n(t) \to f(t)$ for all $t \in T \setminus S_1$, $\lambda(S_1) = 0$. Define

$$v_n(t) = \begin{cases} f_n(t) & \text{on} \quad [\psi \leq x_n \leq \phi] \cap [\psi < \phi] \\ \max[f_n(t), v_0(t)] & \text{on} \quad [x_n < \psi < \phi] \\ \min[f_n(t), v_1(t)] & \text{on} \quad [\psi < \phi < x_n] \\ \psi''(t) & \text{on} \quad [\psi = \phi]. \end{cases}$$

Clearly $v_n \in G(x_n)$, $n \geq 1$.

*Claim #2:* $v_n(t) \to v(t)$ for all $t \in T \setminus S_1$ as $n \to \infty$.

To see this we consider distinct cases:

(a) $t \in [\psi < x < \phi] \setminus S_1$: Then $t \in [\psi < x_n < \phi] \setminus S_1$ for all $n \geq n_0$ (recall that $x_n \to x$ in $C(T)$ as $n \to \infty$). So $v_n(t) = f_n(t) \to f(t) = v(t)$ as $n \to \infty$.

(b) $t \in [x < \psi < \phi] \setminus S_1$: Then $t \in [x_n < \psi < \phi] \setminus S_1$ for all $n \geq n_1$. Then $v_n(t) = \max[f_n(t), v_0(t)] \to \max[f(t), v_0(t)] = v(t)$ as $n \to \infty$.

(c) $t \in [\psi < \phi < x] \setminus S_1$: Then $t \in [\psi < \phi < x_n] \setminus S_1$ for all $n \geq n_2$. Then $v_n(t) = \min[f_n(t), v_1(t)] \to \min[f(t), v_1(t)] = v(t)$ as $n \to \infty$.

(d) $t \in [x = \psi < \phi] \setminus S_1$: If there is a subsequence $\{m\}$ of $\{n\}$ such that $x_m(t) < x(t)$, then $v_m(t) = \max[f_m(t), v_0(t)] \to \max[f_m(t), v_0(t)] = v(t) = f(t)$. If there is a subsequence $\{k\}$ of $\{n\}$ such that $x(t) < x_k(t) < \phi(t)$, then $v_k(t) = f_k(t) \to f(t) = v(t)$ as $k \to \infty$.

(e) $t \in [\psi < \phi = x] \setminus S_1$: This is analyzed as case (d) above.

(f) $t \in [\psi = \phi] \setminus S_1$: Then $v_n(t) = \psi''(t) \to \psi''(t) = v(t)$ as $n \to \infty$.

From all these considerations and the Lebesgue dominated convergence theorem, it follows that every subsequence of $\{v_n\}_{n \geq 1}$ has a further subsequence converging to $v$ in $L^1(T)$. So $v_n \to v$ in $L^1(T)$ as $n \to \infty$. Since $v_n \in G(x_n)$ $n \geq 1$, we infer that $G(x) \subseteq \underline{\lim} G(x_n)$ and so we conclude that $G(\cdot)$ is lsc as claimed.

As in the proof of theorem 2, let $G_1(x) = G(x) - U(x) + x$, where $U(x)(\cdot) = u(\cdot, x(\cdot))$ (see the proof of theorem 2). Evidently $G_1(\cdot)$ has nonempty, closed, decomposable values and is lsc. Apply theorem 3 of Bressan–Colombo [2], to obtain $g_1 : W^{1,1}(T) \to L^1(T)$ a continuous map such that $g_1(x) \in G_1(x)$ for all $x \in W^{1,1}(T)$. Let $L$ be as in the proof of theorem 2 and define $S_0 = \{x \in D : x = L^{-1}g_1(x)\}$. Arguing as in the proof of theorem 2 and using this time Schauder's fixed point theorem, we infer that $S_0 \neq \varnothing$ and $S_0 \subseteq K = [\psi, \phi]$. Then every $x \in S_0$ is a solution of (1). □

## 4. Extremal solutions

Let $\widehat{S}$ denote the set of solutions of problem (1) in the order interval $K = [\psi, \phi]$. In this section we show that the "convex" problem has extremal solutions in $K$; i.e. there exist solutions $x_*, x^* \in \widehat{S}$ such that for every $x \in \widehat{S}$, we have $x_* \leq x \leq x^*$.

**Theorem 4.**

**If** *hypotheses* $H(F)_1$ *and* $H_0$ *hold,*
**then** *problem* (1) *has extremal solutions in the order interval* $K = [\psi, \phi]$.

**Proof.** From theorem 2, we know that $\widehat{S} \neq \varnothing$.

*Claim #1:* $\widehat{S}$ is directed.

To this end let $x_1, x_2 \in \widehat{S}$ and set $x_3 = \max[x_1, x_2]$. From Gilbarg–Trudinger [11] (p. 145), we know that $x_3 \in W^{1,1}(T)$. Let $\tau_3(x)$ and $u_3(t,x)$ be the truncation and penalty functions respectively which correspond to the pair $\{x_3, \phi\}$; i.e. $\tau_3(x)$ and $u_3(t,x)$ are defined by

$$\tau_3(x)(t) = \begin{cases} \phi(t) & \text{if} \quad \phi(t) \leq x(t) \\ x(t) & \text{if} \quad x_3(t) \leq x(t) \leq \phi(t) \\ x_3(t) & \text{if} \quad x(t) \leq x_3(t) \end{cases}$$

$$\text{and } u_3(t,x) = \begin{cases} x - \phi(t) & \text{if} \quad \phi(t) \leq x \\ 0 & \text{if} \quad x_3(t) \leq x \leq \phi(t) \\ x - x_3(t) & \text{if} \quad x \leq x_3(t). \end{cases}$$

For every $x \in W^{1,1}(T)$, let $N_F(x) = S^1_{F(\cdot, \tau_3(x)(\cdot), q_N(\tau_3(x)')(\cdot))}$ and define $G : W^{1,1}(T) \to 2^{L^1(T)}$ by

$$\begin{aligned} G(x) = \{v \in N_F(x) : \quad & v(t) \geq w_1(t) \ \text{ a.e. on } \ [x < x_1] \cap int[x_1 \geq x_2], \\ & v(t) \geq w_2(t) \ \text{ a.e. on } \ [x < x_2] \cap int[x_2 \geq x_1], \\ & v(t) \leq v_1(t) \ \text{ a.e. on } \ [\phi < x]\}, \end{aligned}$$

where $w_1 \in S^1_{F(\cdot, x_1(\cdot), x_1'(\cdot))}$ and $w_2 \in S^1_{F(\cdot, x_2(\cdot), x_2'(\cdot))}$ are such that $-x_1''(t) = w_1(t)$ a.e. on $T$ and $-x_2''(t) = w_2(t)$ a.e. on $T$. As we did in the proof of theorem 2, we can show that $G(\cdot)$ has nonempty, closed, convex values and is usc from $W^{1,1}(T)$ into $L^1(T)$ furnished with the weak topology (denoted henceforth $L^1(T)_w$). Let $G_1(x) = G(x) - U_3(x) + x$, with $U_3(x)(\cdot) = u_3(\cdot, x(\cdot))$. Then $G_1(\cdot)$ is usc from $W^{1,1}(T)$ into $L^1(T)_w$ with nonempty, closed and convex values. Hence if $L = (I + \widehat{L})$ (see the proof of theorem 2), via theorem 2 of Himmelberg [12], we can show that the set $S_f = \{x \in W^{1,1}(T) : x \in L^{-1}G_1(x)\}$ is nonempty.

We will show that $S_f \subseteq [x_3, \phi]$. Indeed let $x \in S_f$. Let $A_1 = int[x_1 \geq x_2]$ and $A_2 = int[x_2 \geq x_1]$.We see that $\overline{A_1 \cup A_2} = T$. Then we have

$$x_1^{''}(t) - x^{''}(t) = v(t) - w_1(t) - u_3(t, x(t)) \text{ a.e. on } T \tag{9}$$

and

$$x_2^{''}(t) - x^{''}(t) = v(t) - w_2(t) - u_3(t, x(t)) \text{ a.e. on } T \tag{10}$$

Multiply equation (9) with $(x_1 - x)_+(t)$ and then integrate over $A_1$.
We obtain

$$\int_{A_1} (x_1^{''} - x^{''})(t)(x_1 - x)_+(t)dt = \int_{A_1} (v - w_1)(t)(x_1 - x)_+ dt$$

$$- \int_{A_1} u_3(t, x(t))(x_1 - x)_+ dt \tag{11}$$

Let $(\alpha, \beta)$ be a component of the open set $A_1 \cap \{x_1 > x\}$. Using Green's formula, we have

$$\int_\alpha^\beta (x_1^{''} - x^{''})(t)(x_1 - x)(t)dt = (x_1^{'} - x^{'})(b)(x_1 - x)(b) - (x_1^{'} - x^{'})(a)(x_1 - x)(a)$$
$$- \int_\alpha^\beta (x_1^{'} - x^{'})^2 dt$$

If $0 < \alpha < \beta < b$, then $(x_1 - x)(\alpha) = (x_1 - x)(\beta) = 0$. So we obtain

$$\int_\alpha^\beta (x_1^{''} - x^{''}(t)(x_1 - x)(t)dt = - \int_\alpha^\beta (x_1 - x)^2 dt \leq 0$$

$$\Rightarrow \int_{A_1} (x_1^{''} - x^{''})(t)(x_1 - x)(t)dt \leq 0 \tag{12}$$

Also from the definition of $G(x)$ and since $v \in G(x)$, we have

$$\int_{A_1} (v - w_1)(t)(x_1 - x)_+(t)dt = \int_{A_1 \cap \{x_1 > x\}} (v - w_1)(t)(x_1 - x)(t)dt \geq 0 \tag{13}$$

Using (12) and (13) in (11) we obtain

$$0 \geq - \int_{A_1} u_3(t, x(t))(x_1 - x)_+(t)dt = \int_{A_1} (x_1 - x)(t)(x_1 - x)_+(t)dt$$

$$\Rightarrow \int_{A_1} (x_1 - x)_+^2(t)dt \leq 0, \text{ hence } x(t) \geq x_1(t) \text{ for all } t \in A_1. \tag{14}$$

Similarly multiplying (10) with $(x_2 - x)_+(t)$ and integrating over $A_2$ we obtain

$$\int_{A_2} (x_2 - x)_+^2(t)dt \leq 0, \text{ hence } x(t) \geq x_2(t) \text{ for all } t \in A_2. \tag{15}$$

From (14) and (15) it follows that

$$x_3(t) \leq x(t) \text{ for all } t \in A_1 \cup A_2$$

$$\Rightarrow x_3(t) \leq x(t) \text{ for all } t \in T \text{ by continuity.}$$

Therefore $x \in [x_3, \phi]$, i.e. $S_f \subseteq [x_3, \phi]$. Since $S_f \subseteq \widehat{S}$, we have proved that $\widehat{S}$ is directed as claimed.

*Claim #2:* Every chain $C$ in $\widehat{S}$ has an upper bound in $\widehat{S}$.

From corollary IV.11.7, p. 336 of Dunford–Schwartz [7], we know that there is a sequence $\{x_n\}_{n\geq 1} \subseteq C$ such that $\sup C = \sup\limits_{n\geq 1} x_n$. Evidently we can have $\{x_n\}_{n\geq 1}$ to be increasing. By definition

$$\left\{ \begin{array}{ll} -x_n^{''}(t) = v_n(t) & \text{a.e. on } T \\ (B_0 x_n)(0) = k_0, & (B_1 x_n)(b) = k_1 \end{array} \right\}$$

with $v_n \in S^1_{F(\cdot, x_n(\cdot), x_n'(\cdot))}$. Since $x_n \in [\psi, \phi]$, $n \geq 1$, we have that $|v_n(t)| \leq \gamma_r(t)$ a.e. on $T$ with $r > \max[\|\psi\|_\infty, \|\phi\|_\infty, N_1]$. Hence $\{x_n^{''}\}_{n\geq 1}$ is uniformly integrable. Recalling that $\|x_n\|_1 + \|x_n^{''}\|_1$ is equivalent to the $W^{2,1}(T)$–norm, we infer that $\{x_n\}_{n\geq 1}$ is bounded in $W^{2,1}(T)$. Using the compact embedding of $W^{2,1}(T)$ into $W^{1,1}(T)$, the Dunford–Pettis theorem and by passing to a subsequence if necessary, we may assume that $x_n^{''} \xrightarrow{w} w$ in $L^1(T)$, $x_n \to x$ in $W^{1,1}(T)$, $x_n(t) \to x(t)$ and $x_n^{'}(t) \to x^{'}(t)$ for all $t \in T$ as $n \to \infty$. It is easy to see that $w = x^{''}$, hence $x_n \xrightarrow{w} x$ in $W^{2,1}(T)$ as $n \to \infty$. Moreover from theorem 3.1 of [24] and hypothesis $H(F)_1$ (ii), we obtain as before that

$$\left\{ \begin{array}{ll} -x^{''}(t) \in F(t, x(t), x^{'}(t)) & \text{a.e. on } T \\ (B_0 x)(0) = k_0, & (B_1 x)(b) = k_1 \end{array} \right\}$$

from which we deduce that $x = \sup C \in \widehat{S}$.

Therefore from Zorn's lemma, we infer that $\widehat{S}$ has a maximal element $x^* \in \widehat{S}$. Since $\widehat{S}$ is directed, $x^*$ is unique and is the greatest solution of (1) in $K = [\psi, \phi]$. Similarly we can produce the least solution $x_*$ of (1) in $K = [\psi, \phi]$. $\square$

## 5. Periodic solutions

A careful reading of the proofs of the previous theorems, reveals that the same methods (with few trivial modifications) can be used to analyze the periodic problem (2).

**Definition.** A function $\phi \in W^{2,1}(T)$ is said to be an "upper solution" of (2) if $-\phi^{''}(t) \geq v_1(t)$ a.e. on $T$, with $v_1 \in L^1(T)$, $v_1(t) \in F(t, \phi(t), \phi^{'}(t))$ a.e. on $T$ and $\phi(0) = \phi(b)$, $\phi^{'}(0) \leq \phi^{'}(b)$. Similarly $\psi \in W^{2,1}(T)$ is said to be a "lower solution" of (2) if $-\psi^{''}(t) \leq v_0(t)$ a.e. on $T$, with $v_0 \in L^1(T)$, $v_0(t) \in F(t, \psi(t), \psi^{'}(t))$ a.e. on $T$ and $\psi(0) = \psi(b)$, $\psi^{'}(0) \geq \psi^{'}(b)$.

In this case $\widehat{L} : D \subseteq L^1(T) \to L^1(T)$ is defined by $\widehat{L}(x) = -x''$ for all $x \in D = \{x \in W^{2,1}(T) : x(0) = x(b),\ x'(0) = x'(b)\}$. Using the same techniques that we employed in section 3 and 4 for the analysis of the Sturm–Liouville problem (1), we can have the following theorems (now in hypothesis $H_0$ the upper solution $\phi$ and the lower solution $\psi$, refer to problem (2)).

**Theorem 5.**

**If** *hypotheses* $H(F)_1$ *and* $H_0$ *hold,*
**then** *problem* (2) *has extremal solutions in the order interval* $K = [\psi, \phi]$.

**Theorem 6.**

**If** *hypotheses* $H(F)_2$ *and* $H_0$ *hold,*
**then** *problem* (2) *has a solution* $x \in W^{2,1}(T)$ *in the order interval* $K = [\psi, \phi]$.

**Remark.** These two theorems extend in various ways the works of Cabada–Nieto [4], Gao–Wang [10], Nieto [19], Nieto–Cabada [20] and Omari–Trombetta [23].

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