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CONJUGACY CRITERIA FOR SECOND ORDER LINEAR DIFFERENCE EQUATIONS

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Abstract. We establish conditions which guarantee that the second order difference equation

$$\Delta^2 x_k + p_k x_{k+1} = 0$$

possesses a nontrivial solution with at least two generalized zero points in a given discrete interval

I. INTRODUCTION.

Consider the second order difference equation

$$\Delta^2 x_k + p_k x_{k+1} = 0, \tag{1.1}$$

where $\Delta^2 x_k = \Delta(\Delta x_k)$, $\Delta x_k = x_{k+1} - x_k$ is the usual forward difference and p_k, $k \in \mathbb{Z}$, is a real-valued sequence. The aim of this paper is to investigate oscillatory properties of (1.1), in particular, we look for conditions which guarantee that (1.1) possesses a nontrivial solution having at least two generalized zero points in a given interval (for the definition of the generalized zero point of a solution of (1.1) and related concepts, see the next section).

In the last years, a considerable effort has been made to establish oscillation theory of equations (1.1) similar to that for *differential* equations

$$y'' + p(t)y = 0, \tag{1.2}$$

see the recent monographs [2,3,10] and the references given therein. It has been shown that in some aspects these theories are quite analogical and, on the other hand, that to treat some problems concerning (1.1), one has to use somewhat different methods than for differential equations, see [4,5,8,9].

Oscillation theory of (1.2) has a long history and was started by the famous paper of Sturm [16]. Since that time, many papers (some of them already in the

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past century) have been devoted to the investigation of conditions which guarantee that (1.2) is oscillatory, i. e., any solution of this equation has infinitely many zero points, see e. g. [15] and the survey paper [18].

On the other hand, the investigation of conjugacy of (1.2) in a given interval (i. e., the existence of a nontrivial solution with at least two zero points in this interval) was started only relatively recently in the paper of Tipler [17]. He proved, among others, that (1.2) is conjugate in $\mathbb{R} = (-\infty, \infty)$ provided $\int_{-\infty}^{\infty} p(t)\, dt > 0$. This result was improved in subsequent papers [6,12,13] and finally in [7] it was proved that (1.2) is conjugate in an arbitrary interval $I = (a, b)$ provided there exist $c \in (a, b)$ and $\varepsilon_1, \varepsilon_2 > 0$ such that

$$(1.3_1) \qquad \varepsilon_1 \int_c^b \exp\left\{2\int_c^x \left[\int_c^t p(s) - \varepsilon_1\right] dt\right\} dx > \frac{\pi}{2},$$

$$(1.3_2) \qquad \varepsilon_2 \int_a^c \exp\left\{2\int_c^x \left[\int_c^t p(s) + \varepsilon_2\right] dt\right\} dx > \frac{\pi}{2}.$$

This conjugacy criterion is based on the combination of the Riccati transformation and the fact that (1.2) is conjugate in (a, b) if and only if there exist linearly independent solutions x, y of this equation such that

$$(1.4) \qquad \int_a^b \frac{|\omega|}{x^2(t) + y^2(t)}\, dt > \pi,$$

where $\omega = xy' - x'y$ is the Wronskian of x, y. The function

$$\alpha(t) = \int_c^t \frac{\omega}{x^2(t) + y^2(t)}\, dt$$

is sometimes called *zero counting function* (in the Borůvka's transformation theory it is called *phase function*) since each crossing of its graph with a multiple of π gives a zero point of any solution y satisfying $y(c) = 0$.

As it is pointed out in [3, p. 422] and explained in more detail in [8], no full discrete analogy of this function is known till now. However, using a suitable modification of the "continuous" method, we will be able to prove a discrete version of (1.3).

The paper is organized as follows. In the next section we recall basic properties of difference equation (1.1). The main results of the paper, conjugacy criteria for (1.1), are given in Section III. In addition to the above mentioned discrete version of (1.3) we "discretize" also one of conjugacy criterion of Tipler [17]. The last section is devoted to remarks and comments concerning possible extensions of the results presented in this paper.

II. AUXILIARY STATEMENTS.

In this section we recall basic results of oscillation theory of equations (1.1). These results are essentially the same as for differential equation (1.2) since both these equations may be regarded as particular cases of the integral equation

$$y(t) = y(a) + r(a)y'(a)\int_a^t \frac{1}{r(s)} ds + \int_a^t \frac{1}{r(s)} \int_a^s y(\tau) dm(\tau)$$

with Riemann-Stiltjes integrals, see [1]. However, as pointed out in that paper, in some cases this general theory does not apply to (1.1) and special methods have to be used.

In the next treatment, when speaking about an interval $[m,n]$, where $n,m \in \mathbb{Z}$, we actually mean the discrete set $[m,n]\cap\mathbb{Z}$. An interval $(m,m+1]$, $m \in \mathbb{Z}$, is said to contain a *generalized zero point* of a solution y of (1.1) if $y_m \neq 0$ and $y_m y_{m+1} \leq 0$. If $(m,m+1]$, $(n,n+1]$, $m<n$, contain generalized zero points of a solution y of (1.1), then any other solution of this equation has at least one generalized zero point between m and $n+1$. Equation (1.1) is said to be *disconjugate* in $[m,n]$ if any solution of this equation has at most one generalized zero in $(m,n+1]$ and the solution $\tilde{y}$ satisfying $\tilde{y}_m = 0$ has no generalized zero in $(m,n+1]$, in the opposite case (1.1) is said to be *conjugate* in $[m,n]$, see e. g. [10].

If y is a solution of (1.1) which has no generalized zero in $(m,n+1]$, the sequence $w_k = \frac{\Delta y_k}{y_k}$ solves in $[m,n]$ the discrete Riccati equation

$$\Delta w_k + p_k + \frac{w_k^2}{1+w_k} = 0 \tag{2.1}$$

with $1+w_k>0$. This means that (1.1) is disconjugate in a given (discrete) interval if and only if (2.1) has a solution w which satisfies $1+w>0$ in this interval. The last inequality, which has no "continuous" analogy, reflects the fact that in the discrete case we need not only to eliminate "real" zero points of solutions but also sign changes.

Now let us explain what is the main difficulty in extending conjugacy criteria based on (1.4) to difference equation (1.1). After some computations, see also below given Lemma 1, one can find that the only "discrete candidate" which could play the role of the function α in the discrete case is the sequence

$$\gamma_k = \frac{\omega}{x_k x_{k+1} + y_k y_{k+1}}, \quad \omega = y_{k+1}x_k - x_{k+1}y_k,$$

x, y being linearly independent solutions of (1.1). However, in contrast to the continuous case where $x^2+y^2>0$, the sequence $x_{k+1}x_k + y_{k+1}y_k$ may change its sign or may be even identically zero as shows the simple example of the equation $\Delta^2 y_k + 2y_{k+1} = 0$.

We finish this section with two auxiliary statements playing an important role in the proof of Theorem 1 which is the main result of our paper.

Lemma 1. *Let $x = \{x_k,\ k\in\mathbb{Z}\}, y = \{y_k,\ k\in\mathbb{Z}\}$ be any pair of sequences such that $x_k^2+y_k^2>0$ for $k\in\mathbb{Z}$, denote $\omega_k := x_k y_{k+1} - x_{k+1}y_k$ and let $z_k = (x_k+iy_k)(x_k^2+y_k^2)^{-\frac{1}{2}}$. Then*

$$z_{k+1} = \frac{x_k x_{k+1} + y_k y_{k+1} + i\omega_k}{h_k h_{k+1}} z_k, \tag{2.2}$$

where $h_k = (x_k^2+y_k^2)^{\frac{1}{2}}$.

Proof. By a direct computation we have

$$
\begin{gathered}
z_{k+1}-z_k=\frac{x_{k+1}+iy_{k+1}}{h_{k+1}}-\frac{x_k+iy_k}{h_k}= \\
\frac{x_k+iy_k}{h_k}\left[\frac{(x_{k+1}+iy_{k+1})h_k}{(x_k+iy_k)h_{k+1}}-1\right]=z_k\left[\frac{(x_{k+1}+iy_{k+1})h_k(x_k-iy_k)}{h_k^2h_{k+1}}-1\right]= \\
\left[\frac{(x_kx_{k+1}+y_ky_{k+1})+i(x_ky_{k+1}-x_{k+1}y_k)}{h_{k+1}h_k}-1\right]z_k= \\
\left[\frac{x_kx_{k+1}+y_ky_{k+1}+i\omega_k}{h_{k+1}h_k}-1\right]z_k. \quad \square
\end{gathered}
$$

Lemma 2. *Let x, y, z be the same as in Lemma 1 and suppose that $\omega_k > 0$. Then $0 < \arg z_{k+1} - \arg z_k < \pi$.*

Proof. From (2.2) $\arg z_{k+1} - \arg z_k = \arg(x_kx_{k+1} + y_ky_{k+1} + i\omega_k)$ and since $\omega_k > 0$ we have the statement. $\square$

III. CONJUGACY CRITERIA.

We start this section with a discrete version of the conjugacy criterion (1.3).

Theorem 1. *Suppose that there exist $\varepsilon_1 > 0, \varepsilon_2 > 0$ such that*

$$
\limsup_{n\to+\infty}\sum_{k=0}^{n}\arctan\frac{\varepsilon_1}{2\alpha_k(p,\varepsilon_1)}>\frac{\pi}{4}, \tag{3.1}
$$

$$
\limsup_{n\to-\infty}\sum_{k=n}^{1}\arctan\frac{\varepsilon_2}{2\beta_k(p,\varepsilon_2)}>\frac{\pi}{4}, \tag{3.2}
$$

where

$$
\begin{gathered}
\alpha_0=1+\varepsilon_1,\quad \beta_1=1+\varepsilon_2,\\
\alpha_k=\alpha_k(p,\varepsilon_1)=\left(\varepsilon_1-\sum_{i=0}^{k-1}p_i+1\right)\prod_{j=0}^{k-1}\left(\varepsilon_1-\sum_{i=0}^{j-1}p_i+1\right)^2 \quad \text{for } k\ge 1,\\
\beta_k=\beta_k(p,\varepsilon_2)=\left(\varepsilon_2-\sum_{i=k-1}^{-1}p_i+1\right)\prod_{j=k+1}^{1}\left(\varepsilon_2-\sum_{i=j-1}^{-1}p_i+1\right)^2 \quad \text{for } k\le 0.
\end{gathered}
$$

Then (1.1) is conjugate in $\mathbb{Z}$.

Proof. In the first part of the proof we consider the interval $[0, +\infty)$ and we will show that the solution x of (1.1) given by the initial conditions $x_0 = 1 = x_1$ has a generalized zero in $[2, \infty)$. Let y be another solution of (1.1) given by the initial condition $y_0 = 1,\ y_1 = 1 + \varepsilon_1$ and let z_k be the same as in Lemmas 1 and 2.

Suppose, by contradiction, that x has no positive generalized zero, i.e. $x_k > 0$ for $k \in [2, +\infty)$. Then $y_k > x_k$ for $k \geq 1$. Indeed, if $y_m \leq x_m$ for some $m \in \mathbb{N}$, then the solution $\tilde{x} = y - x$ satisfies $\tilde{x}_0 = 0$, $\tilde{x}_1 = \varepsilon_1 > 0$ and $\tilde{x}_m \leq 0$. But this contradicts to the separation theorem for generalized zeros of solutions of (1.1) since x is positive throughout $\mathbb{N}$.

Denote by $w_k = x_k x_{k+1} + y_k y_{k+1}$. By Lemma 1 we have

$$z_{k+1} = \frac{w_k + i\varepsilon_1}{h_k h_{k+1}} z_k$$

and hence

$$\arg z_{k+1} = \sum_{j=0}^{k} \arg(w_j + i\varepsilon_1) + \arg z_0 = \sum_{j=0}^{k} \arctan \frac{\varepsilon_1}{w_j} + \frac{\pi}{4}.$$

Further, denote $q_k = \frac{\Delta y_k}{y_k}$. Then $1 + q_k > 0$ and q_k satisfies the discrete Riccati equation (2.1). Hence $\Delta q_k \leq -p_k$ and thus $q_k \leq q_0 - \sum_{i=0}^{k-1} p_i = \varepsilon_1 - \sum_{i=0}^{k-1} p_i$. It follows

$$y_k = \prod_{j=0}^{k-1} (q_j + 1) \leq \prod_{j=0}^{k-1} \left(\varepsilon_1 - \sum_{i=0}^{j-1} p_i + 1 \right).$$

Consequently,

$$\frac{1}{w_k} \geq \frac{1}{2 y_k y_{k+1}} \geq \frac{1}{2\alpha_k}.$$

Since (3.1) holds and $\arg z_0 = \frac{\pi}{4} < \frac{\pi}{2}$, there exists $m \in \mathbb{N} \cup \{0\}$ such that $\arg z_m < \frac{\pi}{2}$ and

$$\arg z_{m+1} \geq \frac{\pi}{4} + \sum_{k=0}^{m} \arctan \frac{\varepsilon_1}{2\alpha_k} \geq \frac{\pi}{2}.$$

Since $0 < \arg z_{m+1} - \arg z_m < \pi$, by Lemma 2 and the fact that $\arg z_m > -\frac{\pi}{2}$, we have $\arg z_{m+1} < \frac{3\pi}{2}$, hence $x_m > 0$ and $x_{m+1} \leq 0$ which means that x has a generalized zero in $(m, m+1]$, a contradiction.

In the second part of this proof we consider the interval $(-\infty, 1]$. We define the backward difference operator $\tilde{\Delta}$ by $\tilde{\Delta} y_k = y_{k-1} - y_k$. Since $\tilde{\Delta}^2 y_k = \Delta^2 y_{k-2}$, equation (1.1) takes the form $\tilde{\Delta}^2 y_k + p_{k-2} y_{k-1} = 0$. Now, let $\bar{y}$ be the solution of (1.1) given by the initial conditions $\bar{y}_1 = 1$, $\bar{y}_0 = 1 + \varepsilon_2$ and let x be the same as in the previous part of the proof. Again, we will show that the assumption $x_k > 0$ for all $k < 0$ leads to a contradiction. Denote $\bar{h}_k = (x_k^2 + \bar{y}_k^2)^{\frac{1}{2}}$ and let $\bar{z}_k = (x_k + i\bar{y}_k)\bar{h}_k^{-1}$. Similarly as in Lemma 1 we have

$$\bar{z}_k = \frac{\bar{w}_k - i\varepsilon_2}{\bar{h}_{k-1}\bar{h}_k} \bar{z}_{k-1}, \quad \bar{w}_k = x_{k-1}x_k + \bar{y}_{k-1}\bar{y}_k$$

and hence

$$\begin{aligned} \bar{z}_{k-1} = \frac{\bar{h}_k \bar{h}_{k-1}}{\bar{w}_k - i\varepsilon_2} \bar{z}_k &= \frac{\bar{h}_k \bar{h}_{k-1}(\bar{w}_k + i\varepsilon_2)}{(x_{k-1}x_k + \bar{y}_{k-1}\bar{y}_k)^2 + (x_{k-1}\bar{y}_k - x_k \bar{y}_{k-1})^2} \bar{z}_k = \\ &= \frac{\bar{h}_{k-1}\bar{h}_k(\bar{w}_k + i\varepsilon_2)}{x_k^2(x_{k-1}^2 + \bar{y}_{k-1}^2) + \bar{y}_k^2(x_{k-1}^2 + \bar{y}_{k-1}^2)} \bar{z}_k = \frac{\bar{w}_k + i\varepsilon_2}{\bar{h}_{k-1}\bar{h}_k} \bar{z}_k. \end{aligned}$$

Similarly as in the first part of the proof, the assumption $x_k > 0$ for $k < 0$ implies $y_k > x_k > 0$ for all negative integers. Let $\tilde{q}_k = \frac{\tilde{\Delta}\bar{y}_k}{\bar{y}_k}$. By a direct computation we have

$$\tilde{\Delta}\tilde{q}_k = -p_{k-2} - \frac{\tilde{q}_k^2}{1+\tilde{q}_k} \le -p_{k-2}$$

and hence

$$\bar{y}_k = \prod_{j=k+1}^{1}(1+\tilde{q}_j)\bar{y}_1 \le \prod_{j=k+1}^{1}\left(1 - \sum_{i=j+1}^{1} p_{i-2} + \varepsilon_2\right)$$

for $k \le 0$. This implies, in the same way as for $k \ge 1$,

$$\arg \bar{z}_{k-1} = \sum_{j=k}^{1} \arg(\bar{w}_j + i\varepsilon_2) + \arg \bar{z}_1 = \sum_{j=k}^{1} \arctan \frac{\varepsilon_2}{\bar{w}_j} + \frac{\pi}{4} \ge$$

$$\ge \sum_{j=k}^{1} \arctan \frac{\varepsilon_2}{2\beta_k(p,\varepsilon_2)} + \frac{\pi}{4}.$$

Now, (3.2) contradicts to $x_k > 0$ for $k < 0$ in the same way as in the first part of the proof. Consequently, the solution x has at least two generalized zeros, i. e., (1.1) is conjugate on $\mathbb{Z}$. $\square$

Remark 1. If we replace assumptions of Theorem 1 by the assumption:

There exist $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and the integers $m = m(\varepsilon_1) \in [0,\infty)$, $n = n(\varepsilon_2) \in (-\infty, 1]$ such that

$$\sum_{k=0}^{m} \arctan \frac{\varepsilon_1}{2\alpha_k} \ge \frac{\pi}{4}, \quad \sum_{k=n}^{1} \arctan \frac{\varepsilon_2}{2\beta_k} \ge \frac{\pi}{4},$$

the statement of Theorem 1 remains valid.

Corollary 1. *Suppose that*

$$\liminf_{n\to+\infty} \frac{1}{n+1} \sum_{j=0}^{n} \sum_{i=0}^{j-1} p_i =: c_1 > 0, \tag{3.3_1}$$

$$\liminf_{n\to-\infty} \frac{1}{-n+2} \sum_{j=n}^{1} \sum_{i=j-1}^{-1} p_i =: c_2 > 0. \tag{3.3_2}$$

Then (1.1) is conjugate in $\mathbb{Z}$.

Proof. Suppose that (3.3_1) holds. There exists $m \in \mathbb{N}$ such that $\sum_{j=0}^{n}\sum_{i=0}^{j-1} p_i > \frac{3}{4}c_1(n+1)$ whenever $n \in (m,\infty)$. Then

$$\sum_{j=0}^{n}\left(\frac{3}{4}c_1 - \sum_{i=0}^{j-1} p_i\right) < 0.$$

Hence

$$0 < \exp \sum_{j=0}^{n} \left(\frac{3}{4} c_1 - \sum_{i=0}^{j-1} p_i \right) < 1$$

and consequently

$$0 < \prod_{j=0}^{n} \exp \left(\frac{3}{4} c_1 - \sum_{i=0}^{j-1} p_i \right) < 1.$$

Now let $\varepsilon_1 = \frac{1}{4} c_1$ and $d \in (m+1, \infty)$. We have

$$\sum_{k=0}^{d} \arctan \frac{\varepsilon_1}{2(\varepsilon_1 - \sum_{i=0}^{k-1} p_i + 1) \prod_{j=0}^{k-1} (\varepsilon_1 - \sum_{i=0}^{j-1} p_i + 1)^2} =$$

$$= \sum_{k=0}^{d} \arctan \frac{\varepsilon_1}{2 \prod_{j=0}^{k} (\varepsilon_1 - \sum_{i=0}^{j-1} p_i + 1) \prod_{j=0}^{k-1} (\varepsilon_1 - \sum_{i=0}^{j-1} p_i + 1)} \geq$$

$$\geq R + \sum_{k=m+1}^{d} \arctan \frac{\varepsilon_1}{2 \prod_{j=0}^{k} \exp(\varepsilon_1 - \sum_{i=0}^{j-1} p_i) \prod_{j=0}^{k-1} \exp(\varepsilon_1 - \sum_{i=0}^{j-1} p_i)} =$$

$$= R + \sum_{k=m+1}^{d} \arctan \frac{\varepsilon_1}{2 \prod_{j=0}^{k} \exp(\frac{3}{4} c_1 - \sum_{i=0}^{j-1} p_i - \frac{c_1}{2}) \prod_{j=0}^{k-1} \exp(\frac{3}{4} c_1 - \sum_{i=0}^{j-1} p_i - \frac{c_1}{2})} >$$

$$> R + \sum_{k=m+1}^{d} \arctan \frac{\varepsilon_1}{2 \prod_{j=0}^{k} \exp \frac{-c_1}{2} \prod_{j=0}^{k} \exp \frac{-c_1}{2}} \longrightarrow +\infty$$

as $d \to +\infty$, where

$$R = \sum_{k=0}^{m} \arctan \frac{\varepsilon_1}{2(\varepsilon_1 - \sum_{i=0}^{k-1} p_i + 1) \prod_{j=0}^{k-1} (\varepsilon_1 - \sum_{i=0}^{j-1} p_i + 1)^2}.$$

Similarly we prove that

$$\sum_{k=n}^{1} \arctan \frac{\varepsilon_2}{2(\varepsilon_2 - \sum_{k-1}^{i=-1} p_i + 1) \prod_{j=k+1}^{1} (\varepsilon_2 - \sum_{j-1}^{i=-1} p_i + 1)^2}$$

tends to infinity if $n \to -\infty$ and $\varepsilon_2 = \frac{c_2}{4}$. □

The next statement is a discrete version of Theorem 6 of [17] and gives a sufficient condition for conjugacy of (1.1) in the half-bounded interval $[n, \infty)$.

Theorem 2. *Suppose that $p_k \geq 0$ for $k \in \mathbb{N}$. A sufficient condition for conjugacy of (1.1) in an interval $[n, +\infty)$, $n \in \mathbb{N}$, is that there exist integers l, m with $n < l < m$ such that*

$$\frac{1}{l-n} < \sum_{k=l}^{m} p_k.$$

Proof. We will show that the solution x of (1.1) given by the initial condition $x_n = 0$, $x_{n+1} = 1$ has a generalized zero point in $(n, +\infty)$. For assume it does not. Then without loss of generality we can assume $x_k > 0$ in $(n, +\infty)$ and $\Delta x_k \geq 0$ in $[n, +\infty)$, since if $\Delta x_k < 0$ at some point in $(n, +\infty)$, we would have a generalized zero in $(n, +\infty)$ by the condition $p_k \geq 0$. From (1.1) we obtain

$$\Delta x_{m+1} = \Delta x_l - \sum_{k=l}^{m} x_{k+1} p_k .$$

Since $p_k \geq 0$, using the discrete mean value theorem we have

$$\frac{x_l}{l-n} = \frac{x_l - x_n}{l-n} \geq \Delta x_\kappa \geq \Delta x_l$$

with some $\kappa \in [n+1, l-1]$. Thus $x_l \geq (l-n)\Delta x_l$. Hence

$$\begin{aligned} \Delta x_l - \sum_{k=l}^{m} x_{k+1} p_k \leq \Delta x_l - \sum_{k=l}^{m} x_l p_k \leq \\ \leq \Delta x_l - (l-n)\Delta x_l \sum_{k=l}^{m} p_k = \Delta x_l \left[1 - (l-n) \sum_{k=l}^{m} p_k \right]. \end{aligned}$$

By hypothesis, the factor in brackets is negative. If $\Delta x_l > 0$, then $\Delta x_{m+1} < 0$, implying a generalized zero in $(m+1, \infty)$. If $\Delta x_l = 0$, then $\Delta x_m < 0$ since $\sum_{k=l}^{m} x_{k+1} p_k > 0$ by assumption. In either case, x has a generalized zero point in $(m+1, \infty)$ and so (1.1) is conjugate in $[n, \infty)$. $\square$

IV. REMARKS.

(i) Our main result, the conjugacy criterion given in Theorem 1, is essentially based on the application of the Riccati transformation. In the continuous case, another powerfull tool of oscillation theory is the variational principle consisting in the relation between disconjugacy of the equation

$$(r(t)y')' + p(t)y = 0 \tag{4.1}$$

in the interval $[a, b]$ and positivity of the quadratic functional

$$\mathcal{J}(y) = \int_a^b \left[r(t)y'^2 - p(t)y^2 \right] \, dt$$

in the class of functions satisfying $y(a) = 0 = y(b)$. In the discrete case we have a similar statement, equation (1.1) is disconjugate in the interval $[m, n]$ if and only if

$$\sum_{k=m}^{n} \left[(\Delta y_k)^2 - p_k y_{k+1}^2 \right] > 0$$

for every nontrivial $y = \{y_k\}_{k=m}^{n+1}$ satisfying $y_m = 0 = y_{n+1}$. The application of the continuous variational principle usually gives weaker results than the Riccati technique. In the discrete case we have a rather different situation. Using the variational principle it is not difficult to prove that (1.1) is conjugate in $\mathbb{Z}$ whenever

$$\sum_{k=-\infty}^{\infty} p_k > 0 \tag{4.2}$$

(the proof of this statement is essentially the same as that given in [12] for (4.1)). If $p_k = 0$ except for one index, say $n \in \mathbb{Z}$, where $p_n > 0$, then (4.2) is trivially satisfied hence (1.1) is conjugate in $\mathbb{Z}$. However, an easy reasoning shows that Theorem 1 does not apply in this simple case.

(ii) Theorem 1 gives sufficient conditions for conjugacy of (1.1) in $\mathbb{Z}$. Remark 1 given below Theorem 1 shows that conditions of this theorem may be easily modified to give the conjugacy criterion in any (sufficiently long) subinterval of $\mathbb{Z}$.

(iii) Throughout the paper we investigate oscillation properties of solutions of second order equation of the form (1.1). Instead of this equation consider a more general equation

$$\Delta(r_k \Delta y_k) + p_k y_{k+1} = 0. \tag{4.3}$$

If $r_k > 0$ in the interval under consideration, say $[0, N]$, then the transformation

$$y_k = h_k u_k, \quad \text{with } h_0 = 1,\ h_{k+1} = \frac{1}{h_k r_k}, \tag{4.4}$$

transforms (4.3) into the equation of the form (1.1). Indeed, by a direct computation one can verify the identity

$$h_{k+1}\left[\Delta(r_k \Delta y_k) + p_k y_{k+1}\right] = \Delta(r_k h_{k+1} h_k \Delta u_k) + h_{k+1}[\Delta(r_k \Delta h_k) + p_k h_{k+1}] u_{k+1}$$

which yields the required transformation. If the sequence r_k changes its sign, the transformation (4.4) does not preserve the sign of the transformed sequences and one has to investigate (4.3) directly. However, a closer examination of the proofs of Lemmas 1,2 and of Theorem 1 shows that the sign changes of r_k make difficulties and it remains an open problem whether Theorem 1 extends to the more general equation (4.3). Note also that in contrast to the continuous oscillation theory of the second order differential equations (4.1) which requires the function r to be positive, oscillation theory of (4.3) may be established under a weaker assumption $r_k \neq 0$, see [2].

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