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ON SKEW 2-PROJECTABLE ALMOST COMPLEX STRUCTURES ON TM

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Abstract. We deal with a $(1,1)$-tensor field α on the tangent bundle TM preserving vertical vectors and such that $J\alpha = -\alpha J$ is a $(1,1)$-tensor field on M, where J is the canonical almost tangent structure on TM. A connection Γ_α on TM is constructed by α. It is shown that if α is a VB-almost complex structure on TM without torsion then Γ_α is a unique linear symmetric connection such that $\alpha(\Gamma_\alpha) = \Gamma_\alpha$ and $\nabla_{\Gamma_\alpha}(J\alpha) = 0$.

INTRODUCTION

In this paper we assume that all manifolds and maps are infinitely differentiable.

Let F be an almost complex structure on $2m$ dimensional manifold M. Recall that F is a (1,1)-tensor field on M such that $F^2 = -Id$, see [9]. It is known [5], [9], that there is not any connection on M, (a linear connection on TM), which can be constructed by a natural operators from F only (without auxiliary geometrical objects).

Let (x^i) be a chart on M and (x^i, x_1^i) be the induced chart on TM. Let $\alpha = (a_j^i dx^j + b_j^i dx_1^j) \otimes \partial/\partial x^i + (c_j^i dx^j + h_j^i dx_1^j) \otimes \partial/\partial x_1^i$ be a (1,1)-tensor field on TM. If α preserves the vertical bundle VTM of vertical vectors on TM, i.e. if $b_j^i = 0$, then $J\alpha = a_j^i dx^j \otimes \partial/\partial x_1^i$, $\alpha J = h_j^i dx^i \otimes \partial/\partial x_1^i$ (here $J = dx^i \otimes \partial/\partial x_1^i$ is the canonical morphism on TM), are semibasic vertical valued forms on TM. We have shown in [2] that if α is an almost complex structure on TM preserving VTM then there is not a connection on TM which can by constructed by a natural operator of zero order from α only.

The complete lift of an almost complex structure F on M is the almost complex structure F^c on TM, which preserves VTM and $JF^c = F^c J$, see [7]. All natural lifts of F on TM, see [3], have these properties.

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When we have studied, [2], some natural operators of first order from the (1,1)-tensor fields α on TM preserving VTM into connections on TM we met with an interesting class of (1,1)-tensor fields α on TM which is very close to the complete lift F^c of a (1,1)-tensor field F on M and for which there are connections on TM constructed by α only. In this paper we study this class.

SKEW 2-PROJECTABLE (1,1)-TENSOR FIELDS ON TM

A (1,1)-tensor field α on TM preserving VTM will be briefly called vertical.

Let us recall that every (1,1)-tensor field $A = a^i_j(x)dx^j \otimes \partial/\partial x^i$ on M determines a semibasic (1,1)-form $\overline{A} = a^i_j(x)dx^j \otimes \partial/\partial x^i_1$ on TM with values in VTM (v-lift of A) and a morphism $\overline{A} : VTM \to VTM$, $\overline{x}^i = x^i$, $\overline{x}^i_1 = a^i_j x^j_1$.

Definition 1. Let A be a regular (1,1)-tensor field on M. A vertical (1,1)-tensor field α on TM is called skew 2-projectable over A if $J\alpha = \overline{A}$, $\alpha J = -\overline{A}$.

In coordinates, if $A = a^i_j(x)dx^j \otimes \partial/\partial x^i$ then a skew 2-projectable (1,1)-tensor over A is of the form

$$\alpha = a^i_j(x)dx^j \otimes \partial/\partial x^i + [c^i_j(x, x_1)dx^j - a^i_j(x)dx^j_1] \otimes \partial/\partial x^i_1, \quad \det a^i_j \neq 0 \ .$$

Then α is a VB-(1,1)-tensor field on TM, i.e. $\alpha(X)$ is a linear and projectable vector field on TM for any projectable and linear vector field X on TM, (see [1]), iff $c^i_j(x, x^1) = c^i_{jk}(x)x^k_1$.

Now the equalities

$$a^i_k a^k_j = -\delta^i_j, \quad c^i_k a^k_j - a^i_k c^k_j = 0 \tag{1}$$

are the coordinate conditions for a skew 2-projectable (1,1)-tensor field α over A to be an almost complex structure (ACS) on TM. If α is overmore VB-tensor field then the second condition of (1) is

$$c^i_{ks} a^k_j - a^i_k c^k_{js} = 0 \ . \tag{2}$$

We want to constructe connections from a skew 2-projectable (1,1)-tensor fields. A connection Γ on $p_M : TM \to M$ can be consider as a (1,1)-tensor field h_Γ on TM (horizontal form of Γ) such that $Tp_M \cdot h_\Gamma = Tp_M$, $h_\Gamma(v) = 0$ for any vertical vector $v \in VTM$, where Tf denotes the tangent prolongation of a map f. Then $h_\Gamma(T(TM)) = H\Gamma$ is the so-called horizontal subbundle of Γ. In coordinates $h_\Gamma =$ $= dx^i \otimes \partial/\partial x^i + \Gamma^i_j dx^j \otimes \partial/\partial x^i_1$ and $(x^i, x^i_1, dx^i, dx^i_1) \in H\Gamma$ if and only if $dx^i_1 =$ $\Gamma^i_j dx^j$, where $\Gamma^i_j(x, x_1)$ are the local functions of Γ. A connection Γ is linear if h_Γ is VB-(1,1)-tensor field on TM, i.e. if $\Gamma^i_j = \Gamma^i_{jk}(x)x^k_1$. Reader is refered to [6] in the case of general connections on fibre bundles.

Remember that a semispray S is a vector field on TM such that $J(S) = V$, where $V = x_1^i\partial/\partial x_1^i$ is the Liouville field the flows of which are the homotheties on individual fibres of TM.

Let α be a general skew 2-projectable (1,1)-tensor fields on TM over a (1,1)-tensor field A on M and $S = x_1^i\partial/\partial x^i + \eta^i(x, x_1)\partial/\partial x_1^i$ be a semispray on TM. Calculating the Lie derivative $L_S\alpha$ and using the denotations $\frac{\partial f}{\partial x^j} := f_j$, $\frac{\partial f}{\partial x_1^j} = f_{j_1}$ we get

$$L_S\alpha = [(a^i_{jk}x_1^k - c^i_j)dx^j + 2a^i_j dx_1^j] \otimes \partial/\partial x^i + [(E^i_j dx^j + F^i_j dx_1^j) \otimes \partial/\partial x_1^i] \ .$$

Let $Y = \xi^i\partial/\partial x^i + \gamma^i\partial/\partial x_1^i$, $\xi^i \neq 0$, be an arbitrary not vertical vector field on TM. Then the vector field

$$L_S\alpha(Y) = [(a^i_{jk}x_1^k - c^i_j)\xi^j + 2a^i_j\gamma^j]\partial/\partial x^i + K^i\partial/\partial x_1^i$$

is a vertical field on TM if and only if

$$2a^i_j\gamma^j = (c^i_j - a^i_{jk}x_1^k)\xi^j, \ \text{i.e. iff}$$

$$\gamma^i = \frac{1}{2}\tilde{a}^i_s(c^s_j - a^s_{jk}x_1^k)\xi^j, \quad \tilde{a}^i_k a^k_j = \delta^i_j \ . \tag{3}$$

We have proved

Proposition 1. *If α is a skew 2-projectable (1,1)-tensor fields on TM over a (1,1)-tensor field A then there is a unique connection Γ_α on TM the horizontal subbundle $H\Gamma_\alpha$ of which is spanned on the vectors Y for which $L_S\alpha(Y) \in VTM$, where S is an arbitrary semispray S on TM.*

Remark 1. Let us emphesize that the connection Γ_α is independent of the choice of the semispray S.

According to the formula (3) the functions

$$\Gamma^i_j = \frac{1}{2}\tilde{a}^i_s(c^s_j - a^s_{jk}x_1^k) \tag{3'}$$

are the local functions of Γ_α. If α is a VB-(1,1)-tensor field then

$$\Gamma^i_j = \frac{1}{2}\tilde{a}^i_s(c^s_{jk} - a^s_{jk})x_1^k \ ,$$

i.e. the connection Γ_α is linear.

Recall that every connection Γ determines a unique semispray $S_\Gamma = x_1^i\partial/\partial x^i + \Gamma^i_j x_1^j\partial/\partial x_1^i$ which is Γ-horizontal. It will be called the semispray of Γ.

Proposition 2. *Let α be a skew 2-projectable (1,1)-tensor field over A. Then the semispray S_{Γ_α} of the connection Γ_α is just the semispray S on TM for which the Lie derivative $[\alpha(S), S]$ is vertical.*

Proof. Let $S = x_1^i \partial/\partial x^i + b^i \partial/\partial x_1^i$ be an arbitrary semispray. Then

$$[\alpha(S), S] = [(c_j^i - a_{jk}^i x_1^k)x_1^j - 2a_j^i b^j]\partial/\partial x^i + B^i \partial/\partial x_1^i$$

is vertical if and only if

$$b^i = \frac{1}{2}\tilde{a}_s^i(c_j^s - a_{jk}^s x_1^k)x_1^j$$

i.e. iff $S = S_{\Gamma_\alpha}$.

The Frölicher-Nijenhuis bracket $[\alpha, J]$ will be called the torsion of α. We say that α is symmetric if is without torsion, i.e. if $[\alpha, J] = 0$.

In the case of a connection Γ, $\tau_\Gamma = [h_\Gamma, J] = \Gamma_{jk_1}^i dx^j \wedge dx^k \otimes \partial/\partial x_1^i$ is the torsion of the connection Γ.

Lemma 1. Let Γ_α be the connection determined by a skew 2-projectable (1,1)-tensor field α on TM over A. Then

$$\tau_{\Gamma_\alpha} = -\frac{1}{2}\overline{A^{-1}}[\alpha, J] \ .$$

Proof. By direct calculation:

$$[h_{\Gamma_\alpha}, J] = \frac{1}{2}\tilde{a}_s^i(c_{jk_1}^s - a_{jk}^s)dx^j \wedge dx^k \otimes \partial/\partial x_1^i \ ,$$

$$[\alpha, J] = (c_{kj_1}^i + a_{jk}^i)dx^j \wedge dx^k \otimes \partial/\partial x_1^i \ . \tag{4}$$

It completes our proof.

Corollary. The connection Γ_α is without torsion if and only if α is without torsion.

Let Γ be an arbitrary connection ont TM with the local functions Γ_j^i. Let $H\Gamma$ be the horizontal subbundle of Γ. Let α be a skew 2-projectable (1,1)-tensor field over A. Then $\alpha(H\Gamma)$ is the horizontal subbundle of the other connection $\alpha(\Gamma)$. We deduce its local equations.

Let $h_\Gamma = dx^i \otimes \partial/\partial x^i + \Gamma_j^i dx^j \otimes \partial/\partial x_1^i$ be the horizontal form of Γ. Then $\alpha h_\Gamma = = a_j^i dx^j \otimes \partial/\partial x^i + (c_j^i - a_k^i \Gamma_j^k)dx^j \otimes \partial/\partial x_1^i$ and so

$$h_{\alpha(\Gamma)} = dx^i \otimes \partial/\partial x^i + (c_s^i - a_s^i \Gamma_s^k)\tilde{a}_j^s dx^j \otimes \partial/\partial x_1^i$$

is the horizontal form of the connection $\alpha(\Gamma)$, i.e. its local functions are $\overline{\Gamma}_j^i = = (c_s^i - a_k^i \Gamma_s^k)\tilde{a}_j^s$. Then a connection Γ is invariant under α, i.e. $\alpha(H\Gamma) = H\Gamma$ if and only if

$$c_j^i = \Gamma_s^i a_j^s + a_s^i \Gamma_j^s \ . \tag{5}$$

Remember that if a skew 2-projectable (1,1)-tensor field α over A is an almost complex structure on TM then A is an ACS on M.

Proposition 3. *If a skew 2-projectable (1,1)-tensor field over A is an almost complex structure on TM then $\alpha(H\Gamma_\alpha) = H\Gamma_\alpha$.*

Proof. The relation (1) imply

$$\tilde{a}^u_j = -a^u_j,\ c^i_u\tilde{a}^u_j = \tilde{a}^i_u c^u_j,\ c^i_{uk_1}\tilde{a}^u_j = \tilde{a}^i_u c^u_{jk_1},\ a^i_{uk}\tilde{a}^u_j = -\tilde{a}^i_u a^u_{jk}\ . \tag{6}$$

Then using (3) and (6) for the local functions of the connection $\overline{\Gamma} = \alpha(\Gamma_\alpha)$ we get

$$\overline{\Gamma}^i_j = [c^i_u - a^i_t\frac{1}{2}\tilde{a}^t_s(c^s_u - a^s_{uk}x^k_1)]\tilde{a}^u_j = \frac{1}{2}(c^i_u + a^i_{uk}x^k_1)\tilde{a}^u_j = \frac{1}{2}\tilde{a}^i_s(c^s_j - a^s_{jk}x^k_1)\ ,$$

i.e. $\alpha(\Gamma_\alpha) = \Gamma_\alpha$.

In [2], Prop. 9, we have proved the following assertion. If F is a connection on TM and A, H are semibasic (1,1)-forms on TM with values in VTM then there exists a unique vertical (1,1)-tensor field $\alpha(\Gamma, A, H)$ such that $\alpha(H\Gamma) \subset H\Gamma$ and $J\alpha = A,\ \alpha J = H$. In coordinates

$$\alpha(\Gamma, A, H) = a^i_j dx^j \otimes \partial/\partial x^i + [(\Gamma^i_k a^k_j - h^i_k\Gamma^k_j)dx^j + h^i_j dx^j_1] \otimes \partial/\partial x^i_1\ .$$

Moreover if a, h are almost complex structures on VTM then $\alpha(\Gamma, A, H)$ is also an ACS on TM. This assertion can be reread in the skew 2-projectable case as follows.

Proposition 4. *Let A be a regular (1,1)-tensor field on M and Γ be a connection on TM. Then there is a unique skew 2-projectable (1,1)-tensor field $\alpha(\Gamma, A, -A)$ over A such that $\alpha(H\Gamma) = H\Gamma$. Moreover, if A is an ACS on M then $\alpha(\Gamma, A, -A)$ is also an ACS on TM. If Γ is linear then $\alpha(\Gamma, A, -A)$ is a VB-field. If Γ is without torsion then $\alpha(\Gamma, A, -A)$ is also without torsion.*

As a consequence of Proposition 3 and 4 we can write

Proposition 5. *Let α be an ACS on TM skew 2-projectable over an ACS A on M. Then $\alpha(\Gamma_\alpha, A, -A) = \alpha$.*

Remark 2. Let us recall that every connection Γ on TM determines such an almost complex structure α on TM that $\alpha J = h_\Gamma,\ \alpha h_\Gamma = -J$ but α is not vertical.

Consider the (1,1)-tensor field $A = a^i_j(x)dx^j \otimes \partial/\partial x^i$ on M as a vector bundle morphism $A : TM \to TM$. Then the tangent map $TA : T(TM) \to T(TM)$ has the following coordinate form

$$\begin{aligned} \overline{x}^i &= x^i\ , & \overline{x}^i_1 &= a^i_j(x)x^j_1\ , \\ d\overline{x}^i &= dx^i\ , & d\overline{x}^i_1 &= a^i_{kj}x^k_1 dx^j + a^i_j dx^j_1\ . \end{aligned}$$

Let Γ, $dx_1^i = \Gamma_j^i(x, x_1)dx^j$ be a connection on TM. Let $u = (x, u_1) \in T_xM$, $X = (x, dx) \in T_xM$. Then $\Gamma(X) = (x^i, u_1^i, dx^i, \Gamma_j^i(x, u_1)dx^j)$ is the Γ-lift of X at $u \in T_xM$. Then $TA(\Gamma X) = (x^i, a_j^i u_1^j, dx^i, [a_{kj}^i u_1^k + a_u^i \Gamma_j^u(x, u_1)]dx^j)$ and

$$\begin{aligned} &TA(\Gamma X) - h_\Gamma(TA(\Gamma X)) = \\ &\quad = (x^i, a_j^i u_1^j, 0, [a_{kj}^i u_1^k + a_u^i \Gamma_j^u(x, u_1)]dx^j - \Gamma_j^i(x, a_s^t u_1^s)dx^j) \equiv \\ &\qquad \equiv (x^i, [a_{kj}^i u_1^k + a_t^i \Gamma_j^t(x, u_1)]dx^j - \Gamma_j^i(x^t, a_s^t u_1^s)dx^j) \in T_xM \ . \end{aligned}$$

We get a map $\nabla_u A : T_xM \to T_xM$, $X \to TA(\Gamma X) - h_\Gamma(TA(\Gamma X))$ which is the classical covariant derivative in the case of a linear connection Γ,

$$\nabla^\Gamma A = (a_{kj}^i u_1^k + a_t^i \Gamma_{jk}^t u_1^k - \Gamma_{jt}^i a_k^t u_1^k)dx^j \otimes \partial/\partial x^i \ .$$

Then

$$a_{kj}^i + a_t^i \Gamma_{jk}^t - \Gamma_{jt}^i a_k^t = 0 \tag{7}$$

is the coordinate condition for $\nabla^\Gamma A$ to vanish.

Proposition 6. *Let α be a skew 2-projectable VB-(1,1)-tensor field without torsion over a (1,1)-tensor field A on M. Let $\alpha(\Gamma_\alpha) = \Gamma_\alpha$. Then A is constant with respect to the covariant derivative according to the linear connection Γ_α, i.e. $\nabla A = 0$.*

Proof. According to (4) the field α is without torsion iff

$$c_{kj}^i + a_{jk}^i = c_{jk}^i + a_{kj}^i \ . \tag{8}$$

For the coordinate functions $\Gamma_j^i = \frac{1}{2}\tilde{a}_s^i(c_{jk}^s - a_{jk}^s)x_1^k$ of the connection Γ_α the condition (5) for $\alpha(\Gamma_\alpha) = \Gamma_\alpha$ reads

$$\tilde{a}_t^i(c_{sk}^t - a_{sk}^t)a_j^s = c_{jk}^i + a_{jk}^i \ . \tag{9}$$

Using the equalities (8) and (9) the left side of the condition (7) in the case of the connection Γ_α gives successively $a_{kj}^i + a_t^i \Gamma_{jk}^t - \Gamma_{jt}^i a_k^t = a_{kj}^i + \frac{1}{2}(c_{jk}^i - a_{jk}^i) - \frac{1}{2}\tilde{a}_s^i(c_{jt}^s - -a_{jt}^s)a_k^t = a_{kj}^i + \frac{1}{2}(c_{kj}^i - a_{kj}^i) - \frac{1}{2}\tilde{a}_s^i(c_{tj}^s - a_{tj}^s)a_k^t = \frac{1}{2}(c_{kj}^i + a_{kj}^i) - \frac{1}{2}(c_{kj}^i + a_{kj}^i) = 0$. The proof is finished.

Proposition 7. *Let α be a skew 2-projectable VB-(1,1)-tensor field without torsion over a (1,1)-tensor field A on M. Let Γ be such a symmetric linear connection on TM that $\alpha(\Gamma) = \Gamma$. Then $\nabla^\Gamma(A) = 0$ if and only if $\Gamma = \Gamma_\alpha$.*

Proof. The equality (5) reads

$$c_{jk}^i = \Gamma_{sk}^i a_j^s + a_s^i \Gamma_{jk}^s \ , \quad \text{i.e.} \quad \Gamma_{js}^i a_k^s = c_{kj}^i - a_s^i \Gamma_{jk}^s \quad \text{as} \quad \Gamma_{jk}^i = \Gamma_{kj}^i \ .$$

Putting it in the condition (7) we get

$$a_{kj}^i + a_t^i \Gamma_{jk}^t + a_s^i \Gamma_{jk}^s - c_{kj}^i = 0 \ , \quad \text{i.e.} \quad 2a_s^i \Gamma_{jk}^s = c_{kj}^i - a_{kj}^i \ .$$

Then according to (8) $\Gamma_{jk}^i = \frac{1}{2}\tilde{a}_s^i(c_{jk}^s - a_{jk}^s)$, i.e. $\Gamma = \Gamma_\alpha$. Then Proposition 6 completes our proof.

Remark 3. Proposition 7 can be reread as follows. Let Γ be a symmetric linear connection and A be a regular (1,1)-tensor field on M. Let $\alpha(\Gamma, A, -A)$ be the (1,1)-tensor field in the sence of Proposition 4. Then $\Gamma_\alpha = \Gamma$ iff $\nabla^\Gamma A = 0$.

Proposition 8. *Let α be a VB-almost complex structure skew 2-projectable without torsion over a (1,1)-tensor field A on TM. Then Γ_α is a unique linear symmetric connection such that $\nabla A = 0$.*

Proof. By Proposition 3 $\alpha(\Gamma_\alpha) = \Gamma_\alpha$. As the connection Γ_α is linear and symmetric (Lemma 1) then Proposition 7 completes our proof.

Corollary. *In the case of a VB-almost complex structure on TM skew 2-projectable without torsion over a (1,1)-tensor field A on M there is a unique linear symmetric connection Γ such that $\alpha(\Gamma) = \Gamma$, $\nabla^\Gamma A = 0$. This connection is just the connection Γ_α. Consequently A is an integrable almost complex structure on M, see [9].*

Remark 4. Let Γ be a connection on TM. There is the vertical prolongation $V\Gamma$ of Γ which is a connection on $VTM \to M$, see [7] in the general case of a fibre bundle. In the induced local chart $(x^i, x_1^i, 0, \eta^i)$ on VTM its horizontal subbundle $HV\Gamma$ is determined by the equations

$$d\eta^i = \Gamma^i_{jk_1}\eta^k dx^j \ , \quad dx_1^i = \Gamma^i_j dx^j \ .$$

Analogously to the Proposition 6 it is easy to show that if α is a skew 2-projectable (1,1)-tensor field on TM without torsion such that $\alpha(\Gamma_\alpha) = \Gamma_\alpha$ then $T\alpha(HV\Gamma_\alpha) \subset HV\Gamma_\alpha$.

By direct calculation in the case of a skew 2-projectable (1,1)-tensor field α we obtain for the Nijenhuis tensor $[\alpha, \alpha]$

$$\begin{aligned}\frac{1}{2}[\alpha,\alpha] = {} & (a^i_{su}a^u_j + a^i_k a^k_{js})dx^j \wedge dx^s \otimes \partial/\partial x^i + \\ & + \{(c^i_{su}a^u_j + c^i_{su_1}c^u_j + c^i_u a^u_{js} - a^i_u a^u_{js})dx^j \wedge dx^s + \\ & + (a^i_u a^u_{js} + a^i_{ju}a^u_s + a^i_u c^u_{sj_1} - c^i_{su_1}a^u_j)dx^j_1 \wedge dx^s\} \otimes \partial/\partial x^i_1 \ .\end{aligned} \tag{10}$$

This formula and the well known condition for A to be an integrable almost complex structure, see for example [9], give

Proposition 9. *The Nijenhuis tensor $[\alpha, \alpha]$ of a skew 2-projectable (1,1)-tensor field α over a (1,1)-tensor field A on M is a vertical tangent valued if and only if $[A, A] = 0$, i.e. in the case when α is moreover an ACS iff A is an integrable ACS.*

Proposition 10. *Let α be an almost complex structure on TM skew 2-projectable and symmetric over an integrable almost complex structure A on M. Then the Nijenhuis tensor $[\alpha, \alpha]$ is a semibasic vertical valued 2-form on TM.*

Proof. By Proposition 9 $[\alpha, \alpha]$ is vertical valued. Using the equalities (1) and (4) where $[\alpha, J] = 0$ we get for (10):

$$B^i_{js} = a^i_u a^u_{js} + a^i_{ju}a^u_s + a^i_u c^u_{sj_1} - c^i_{su_1}a^u_j = a^i_u(c^u_{sj_1} - c^u_{js_1} + a^u_{js} - a^u_{sj}) = 0 \ ,$$

i.e. $[\alpha, \alpha] = H^i_{js}dx^j \wedge dx^s \otimes \partial/\partial x^i_1$ is semibasic and vertical valued.

Corollary. *If α is a symmetric VB-almost complex structure skew 2-projectable over A then $[\alpha, \alpha]$ is a semibasic vertical valued 2-form on TM.*

Remark 5. Let us recall the complete lift Γ^c of a connection Γ on TM, see for example [8]. If $h_\Gamma = dx^i \otimes \partial/\partial x^i + \Gamma^i_j(x, x_1) dx^j \otimes \partial/\partial x^i_1$ is the horizontal form of a connection Γ then $Ti_1 \cdot i_2 \cdot Th_\Gamma \cdot i_2 \cdot Ti_1 = dx^i \otimes \partial/\partial x^i + dx^i_1 \otimes \partial/\partial x^i_1 + \Gamma^i_j(x, \xi) dx^j \otimes \partial/\partial \xi^i + +[(\Gamma^i_{jk}(x, \xi) x^k_1 + \Gamma^i_{jk_1}(x, \xi)\eta^k) dx^j + \Gamma^i_j(x, \xi) dx^j_1] \otimes \partial/\partial \eta^i$ is the horizontal form of the connection Γ^c on $p_{TM} : T(TM) \to TM$, where i_1 and i_2 are the canonical involutions on TTM and $TT(TM)$, $i_1(x, x_1, \xi, \eta) = (x, \xi, x_1, \eta)$. If Γ is linear and without torsion then also Γ^c is linear and without torsion. In the case of a symmetric VB-almost complex structure α skew 2-projectable over an ACS A on M the connection Γ_α is linear and symmetric then the complete lift Γ^c_α is also linear and symmetric. This means that there is on TM such an ACS which determines a connection on TM, i.e. linear connection on $p_{TM} : T(TM) \to TM$ without auxiliary geometrical objects. Remember that in the case of an ACS on M such a connection has not to exist. We will comment this situation in detail. Let $F : TM \to TM$ be an ACS on M and $\Gamma \equiv h_\Gamma : TTM \to TTM$ be a linear connection. Let $f : M \to N$ be a local diffeomorphism. Recall that F_M, F_N or Γ_M, Γ_N are f-related if $F_N \cdot Tf = Tf \cdot F_M$ or $\Gamma_N \cdot TTf = TTf \cdot \Gamma_M$. By [4] there is not any linear connection Γ which can be constructed from an ACS F only by a natural operator Φ which means that if F_M, F_N are f-related then also $\Phi(F_M)$, $\Phi(F_N)$ are f-related. Certainly in the case of a symmetric VB-almost complex structure α skew 2-projectable over an ACS A on M the operator $\Phi : \alpha \to \Gamma^c_\alpha$ is "M-natural", i.e. if $\alpha_N \cdot TTf = TTf \cdot \alpha_M$ then also $\Phi(\alpha_N) \cdot TTTf = TTTf \cdot \Phi(\alpha_M)$. But Φ is not "TM-natural" because if $f : TM \to TN$ is an arbitrary local diffeomorphism then $Tf \cdot \alpha_M \cdot Tf^{-1}$ need not be an VB-almost complex structure on TN. Readers are kindly refered to [7] for more detail information on theory of natural operations.

Example. Let $A = a^i_j(x) dx^j \otimes \partial/\partial x^i$ be a regular (1,1)-tensor field on M. Let $\overline{A} = a^i_j dx^j \otimes \partial/\partial x^i_1$ be the semibasic VTM-valued (1,1)-form on TM determined by A. Let $S = x^i_1 \partial/\partial x^i + \eta^i(x, x_1) \partial/\partial x^i_1$ be a semispray. Then the Lie derivative

$$\alpha \equiv L_S \overline{A} = -a^i_j dx^j \otimes \partial/\partial x^i + [(a^i_{jk} x^k_1 - \eta^i_{k_1} a^k_j) dx^j + a^i_j dx^j_1] \otimes \partial/\partial x^i_1$$

is a skew 2-projectable (1,1)-tensor field α on TM over $-A$. If S is a spray, i.e. $L_V S = S$, then $L_S \overline{A}$ is a VB-form.

Recall, see [4], that the Lie derivative $L_S J$ determines the connection Γ_S with the local functions $\Gamma^i_j = \frac{1}{2} \eta^i_{j_1}$.

As $\Gamma^i_j = -\frac{1}{2} \tilde{a}^i_s (2a^s_{jk} x^k_1 - \eta^s_{k_1} a^k_j)$ are the local functions of the connection $\Gamma_{L_S \overline{A}}$ then it is easy to see, that

$$\alpha(\Gamma_{L_S \overline{A}}) = \Gamma_S \ .$$

If $A = Id\Big|_{TM}$ then $\Gamma_{L_S \overline{A}} = \Gamma_S$.

We will discuss the conditions for $L_S\overline{A}$ to be an ACS on TM. In this case the second equation of (1) reads

$$a^i_s \eta^s_{j_1} - \eta^i_{s_1} a^s_j = -2a^i_{jk} x^k_1 \ .$$

The map $y \to Ay - yA$ is singular and so the last equation has not to be solvable. Therefore if A is an ACS on M then such a semispray S that $L_S\overline{A}$ is an ACS on TM has not to exist.

If A is an ACS on M and Γ_S is the connection determined by a semispray S then by the Proposition 4 $\alpha(\Gamma_S, A, -A)$ is an ACS on TM.

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