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# INVARIANT VECTOR FIELDS OF HAMILTONIANS

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Abstract. A complete classification of natural transformations of Hamiltonians into vector fields on symplectic manifolds is given herein.

The aim of this paper is to give a classification of natural transformations of Hamiltonians into vector fields. By a Hamiltonian we mean an arbitrary smooth function on a symplectic manifold. It is well known that every Hamiltonian $H$ yields in a natural way a vector field $V_H$ on the symplectic manifold, which is called a Hamiltonian vector field. The coordinates of $V_H$ in a canonical chart $(q,p) : U \longrightarrow \mathbf{R}^n \times \mathbf{R}^n$ (a chart $(q,p)$ on a symplectic manifold is canonical if $dq^i \wedge dp_i$ coincides with the symplectic structure) are given by

$$T(q,p) \circ V_H|_U = \left( q, p, -\frac{\partial}{\partial p} H|_U, \frac{\partial}{\partial q} H|_U \right).$$

The formula written above is invariant i.e. it is immaterial which $(q,p)$ we choose as long as $(q,p)$ is canonical. This is the reason why the transformation $H \longrightarrow V_H$ is natural. The theorem proved in this article says that an arbitrary natural transformation of Hamiltonians into vector fields has the form $H \longrightarrow (\lambda \circ H)V_H$, where $\lambda : \mathbf{R} \longrightarrow \mathbf{R}$ is a smooth function.

Note that our definition of the natural transformation of Hamiltonians into vector fields doesn't agree with the one used by Doupovec and Kurek in [3]. In our paper we study the naturality with respect to all canonical embeddings $f : L \longrightarrow M$, where $(L, \psi)$ and $(M, \omega)$ are arbitrary symplectic manifolds (we call the embedding $f$ canonical if $\omega \circ (T \times T) f = \psi$). The article [3] contains a classification of first order natural transformations of functions on the cotangent bundle into vector fields on the cotangent bundle. Of course, the cotangent bundle may be regarded as a symplectic manifold, but in [3] the authors consider the naturality with respect to the family of maps $T^*f : T^*L \longrightarrow T^*M$, where $f : L \longrightarrow M$ is an embedding and $L$, $M$ are $n$-dimensional manifolds. This family is smaller than the family of all canonical embeddings $T^*L \longrightarrow T^*M$. We think that our

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definition can be justified, because Hamilton's canonical equations are preserved by all canonical embeddings. It is worth pointing out the difference between Hamiltonian mechanics and Lagrangian mechanics. Namely, the only embeddings which preserve Lagrange's equations are of the form $Tf : TL \longrightarrow TM$, (see the definitions of the natural transformations of Lagrangians in [1] and [2]).

As a consequence, we obtain a result different from the result obtained in [3]. We also have to use other methods to prove our theorem. The proof is based on generating functions.

Let $(M, \omega)$ be a symplectic manifold. A smooth function $H : M \longrightarrow \mathbf{R}$ is called a *Hamiltonian* on $M$. We denote by $\mathcal{H}(M)$ the set of all Hamiltonians on $M$, and by $\mathcal{V}(M)$ the set of all smooth vector fields on $M$.

Let $n$ be a fixed positive integer.

**Definition.** A family of maps $\Lambda_{(M,\omega)} : \mathcal{H}(M) \longrightarrow \mathcal{V}(M)$, where $(M, \omega)$ is an arbitrary $2n$-dimensional symplectic manifold, is called a *natural transformation of Hamiltonians into vector fields,* if for every embedding $f : L \longrightarrow M$ of a $2n$-dimensional symplectic manifold $(L, \psi)$ into a $2n$-dimensional symplectic manifold $(M, \omega)$ such that $\omega \circ (T \times T)f = \psi$ and for every Hamiltonian $H$ on $M$ we have $Tf \circ \Lambda_{(L,\psi)}(H \circ f) = \Lambda_{(M,\omega)}(H) \circ f$.

A chart $(q, p) : U \longrightarrow \mathbf{R}^n \times \mathbf{R}^n$ on a symplectic manifold $(M, \omega)$ is said to be canonical if $\omega|_{(T\times T)U} = dq^i \wedge dp_i$.

**Lemma.** *Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold. If $m \in M$ and $H$ is a Hamiltonian on $M$ such that $d_m H \neq 0$ then there exists a canonical chart $(Q, P) : U \longrightarrow \mathbf{R}^n \times \mathbf{R}^n$ on $M$ such that $(Q, P)(m) = 0$ and $Q^1 = H|_U - H(m)$.*

**Proof.** By the Darboux theorem, there exists a canonical chart $(q, p)$ on $M$ such that $(q, p)(m) = 0$. Put $K = H \circ (q, p)^{-1} - H(m)$. Let $(x, y)$ denote the canonical system of coordinates on $\mathbf{R}^n \times \mathbf{R}^n$. Since $K(0) = 0$ and $d_0 K \neq 0$, we can choose a smooth solution $W : \Omega \longrightarrow \mathbf{R}$ (where $\Omega \subset \mathbf{R}^n \times \mathbf{R}^n$) of the differential equation

$$\frac{\partial W}{\partial y_1} = K \circ \left( x|_\Omega, \frac{\partial W}{\partial x} \right) \tag{1}$$

such that $d_0 W = 0$ and

$$\begin{vmatrix} \dfrac{\partial^2 W}{\partial x^1 \partial y_1}(0) & \dots & \dfrac{\partial^2 W}{\partial x^1 \partial y_n}(0) \\ \vdots & & \vdots \\ \dfrac{\partial^2 W}{\partial x^n \partial y_1}(0) & \dots & \dfrac{\partial^2 W}{\partial x^n \partial y_n}(0) \end{vmatrix} \neq 0. \tag{2}$$

Since $d_0 W = 0$ and (2), we can define $(Q, P) : U \longrightarrow \mathbf{R}^n \times \mathbf{R}^n$ (where $U \subset M$) to be the solution of the equations

$$Q = \frac{\partial W}{\partial y} \circ (q|_U, P), \tag{3}$$

$$p|_U = \frac{\partial W}{\partial x} \circ (q|_U, P), \tag{4}$$

which satisfies the condition $(Q, P)(m) = 0$. From (2) it follows that $(Q, P)$ is a chart in a neighbourhood of $m$, which can be easily checked. We may assume that $(Q, P)$ is a chart in the neighbourhood $U$. It is a canonical one, because (3) and (4) are equivalent to $Q^i dP_i + p_i dq^i|_{TU} = d\big(W \circ (q|_U, P)\big)$. According to (3), (1), (4), we have

$$Q^1 = \frac{\partial W}{\partial y_1} \circ (q|_U, P) = K \circ \Big(q|_U, \frac{\partial W}{\partial x} \circ (q|_U, P)\Big) = K \circ (q,p)|_U = H|_U - H(m).$$

This proves the lemma. $\square$

Let $H$ be an arbitrary Hamiltonian on a symplectic manifold $(M, \omega)$. We will denote by $V_H$ the unique vector field on $M$ such that $\omega \circ (V, V_H) = dH \circ V$ for every $V \in \mathcal{V}(M)$. $V_H$ is called the Hamiltonian vector field corresponding to $H$. Of course, if $(L, \psi)$ is another symplectic manifold, $\dim L = \dim M$, and $f : L \longrightarrow M$ is an embedding such that $\omega \circ (T \times T)f = \psi$, then $Tf \circ V_{H\circ f} = V_H \circ f$.

**Theorem.** *If $\Lambda$ is a natural transformation of Hamiltonians into vector fields, then there is one and only one smooth function $\lambda : \mathbf{R} \longrightarrow \mathbf{R}$ such that $\Lambda_{(M,\omega)}(H) = (\lambda \circ H)V_H$ for every $2n$-dimensional symplectic manifold $(M, \omega)$ and every Hamiltonian $H$ on $M$.*

**Proof.** We first observe that if $(M, \omega)$ is a $2n$-dimensional symplectic manifold and $U$ is an open subset of $M$ then for every $H \in \mathcal{H}(M)$ we have $\Lambda_{(U, \omega|_{(T\times T)U})}(H|_U) = \Lambda_{(M,\omega)}(H)|_U$. To see this, it suffices to replace an embedding in the definition of the natural transformation by the inclusion $U \longrightarrow M$.

From now on, $W$ denotes an arbitrary smooth function $\mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}$ such that $d_0 W = 0$ and (2). For every $W$ we can define $(Q, P) : \Omega \longrightarrow \mathbf{R}^n \times \mathbf{R}^n$ (where $\Omega \subset \mathbf{R}^n \times \mathbf{R}^n$) to be the solution of the equations

$$Q = \frac{\partial W}{\partial y} \circ (x|_\Omega, P),$$

$$y|_\Omega = \frac{\partial W}{\partial x} \circ (x|_\Omega, P),$$

which satisfies the condition $(Q, P)(0) = 0$. We may assume that $(Q, P)$ is an embedding, so $(Q, P)$ is a canonical chart on $(\mathbf{R}^n \times \mathbf{R}^n, dx^i \wedge dy_i)$. A trivial computation shows that

$$\frac{\partial(Q,P)}{\partial(x,y)}(0) = \begin{bmatrix} B^* - CB^{-1}A & CB^{-1} \\ -B^{-1}A & B^{-1} \end{bmatrix}, \tag{5}$$

where

$$A = \begin{bmatrix} \frac{\partial^2 W}{\partial x^1 \partial x^1}(0) & \dots & \frac{\partial^2 W}{\partial x^1 \partial x^n}(0) \\ \vdots & & \vdots \\ \frac{\partial^2 W}{\partial x^n \partial x^1}(0) & \dots & \frac{\partial^2 W}{\partial x^n \partial x^n}(0) \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{\partial^2 W}{\partial x^1 \partial y_1}(0) & \dots & \frac{\partial^2 W}{\partial x^1 \partial y_n}(0) \\ \vdots & & \vdots \\ \frac{\partial^2 W}{\partial x^n \partial y_1}(0) & \dots & \frac{\partial^2 W}{\partial x^n \partial y_n}(0) \end{bmatrix},$$

$$C = \begin{bmatrix} \frac{\partial^2 W}{\partial y_1 \partial y_1}(0) & \dots & \frac{\partial^2 W}{\partial y_1 \partial y_n}(0) \\ \vdots & & \vdots \\ \frac{\partial^2 W}{\partial y_n \partial y_1}(0) & \dots & \frac{\partial^2 W}{\partial y_n \partial y_n}(0) \end{bmatrix}.$$

Conversely, if $A$, $B$, $C$ are arbitrary $(n \times n)$-matrices such that $A^* = A$, $C^* = C$, and $B$ is non-singular, then it is evident that there is $W$ which gives $(Q, P)$ such that (5).

Suppose

$$\frac{\partial W}{\partial y_1} = x^1. \tag{6}$$

Then the first column of $B$ is $e_1$ (we write $e_1, \dots, e_n$ for the canonical basis of the vector space $\mathbf{R}^n$), and the first column and first row of $C$ are 0. Conversely, if $B$ and $C$ satisfy these conditions, then it is evident that there is $W$ which satisfies (6) and gives $(Q, P)$ such that (5). In this case $(x^1 + h) \circ (Q, P) = x^1|_\Omega + h$ for every $h \in \mathbf{R}$. Thus

$$\begin{aligned} & T(Q,P)\big(\Lambda_{(\mathbf{R}^n \times \mathbf{R}^n, dx^i \wedge dy_i)}(x^1 + h)(0)\big) \\ &= \big(T(Q,P) \circ \Lambda_{(\Omega, dx^i \wedge dy_i|_{(T\times T)\Omega})}(x^1|_\Omega + h)\big)(0) \\ &= \big(T(Q,P) \circ \Lambda_{(\Omega, dx^i \wedge dy_i|_{(T\times T)\Omega})}((x^1 + h) \circ (Q,P))\big)(0) \\ &= \big(\Lambda_{(\mathbf{R}^n \times \mathbf{R}^n, dx^i \wedge dy_i)}(x^1 + h) \circ (Q,P)\big)(0) \\ &= \Lambda_{(\mathbf{R}^n \times \mathbf{R}^n, dx^i \wedge dy_i)}(x^1 + h)(0). \end{aligned}$$

Treating $\Lambda_{(\mathbf{R}^n \times \mathbf{R}^n, dx^i \wedge dy_i)}(x^1 + h)(0)$ as a matrix with one column and $2n$ rows we have

$$\frac{\partial(Q,P)}{\partial(x,y)}(0) \cdot \Lambda_{(\mathbf{R}^n \times \mathbf{R}^n, dx^i \wedge dy_i)}(x^1 + h)(0) = \Lambda_{(\mathbf{R}^n \times \mathbf{R}^n, dx^i \wedge dy_i)}(x^1 + h)(0). \tag{7}$$

Let

$$\Lambda_{(\mathbf{R}^n \times \mathbf{R}^n, dx^i \wedge dy_i)}(x^1 + h)(0) = \begin{bmatrix} X \\ Y \end{bmatrix},$$

where $X, Y \in \mathbf{R}^n$. For $A = -I$, $B = I$, $C = 0$ the formulas (7) and (5) yield

$$\begin{bmatrix} X \\ X + Y \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}.$$

Hence $X = 0$. For $A = 0$ and $C = 0$ the formulas (7) and (5) yield

$$\begin{bmatrix} 0 \\ B^{-1} \quad Y \end{bmatrix} = \begin{bmatrix} B^* & 0 \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ Y \end{bmatrix}.$$

Since $B$ is an arbitrary non-singular matrix such that $Be_1 = e_1$, from $BY = Y$ we deduce that the vectors $e_1$ and $Y$ are not linearly independent. Finally, for every $h \in \mathbf{R}$ there exists $\lambda(h) \in \mathbf{R}$ such that

$$\Lambda_{(\mathbf{R}^n \times \mathbf{R}^n, dx^i \wedge dy_i)}(x^1 + h)(0) = \lambda(h) \frac{\partial}{\partial y_1}(0) = \lambda(h) \cdot V_{x^1 + h}(0).$$

Fix a $2n$-dimensional symplectic manifold $(M, \omega)$, $m \in M$, and $H \in \mathcal{H}(M)$. If $d_m H \neq 0$ then let $(Q, P) : U \longrightarrow \mathbf{R}^n \times \mathbf{R}^n$ be the chart from the lemma. Since $(Q, P)$ is canonical, $dx^i \wedge dy_i \circ (T \times T)(Q, P) = \omega|_{(T \times T)U}$ and we have

$$\begin{aligned}
&\Lambda_{(M,\omega)}(H)(m) \\
&= T(Q,P)^{-1}\big(\Lambda_{((Q,P)(U), dx^i \wedge dy_i|_{(T\times T)(Q,P)(U)})}\big(H \circ (Q,P)^{-1}\big)(0)\big) \\
&= T(Q,P)^{-1}\big(\Lambda_{(\mathbf{R}^n \times \mathbf{R}^n, dx^i \wedge dy_i)}\big(x^1 + H(m)\big)(0)\big) \\
&= T(Q,P)^{-1}\big(\lambda\big(H(m)\big) \cdot V_{x^1 + H(m)}(0)\big) \\
&= \lambda\big(H(m)\big) \cdot T(Q,P)^{-1}\big(V_{H \circ (Q,P)^{-1}}(0)\big) \\
&= \lambda\big(H(m)\big) \cdot V_H(m),
\end{aligned}$$

and so

$$\Lambda_{(M,\omega)}(H)(m) = \lambda\big(H(m)\big) \cdot V_H(m). \tag{8}$$

By (8), for every $t \in \mathbf{R}$

$$\Lambda_{(\mathbf{R}^n \times \mathbf{R}^n, dx^i \wedge dy_i)}(x^1)(te_1, 0) = \lambda(t) \frac{\partial}{\partial y_1}(te_1, 0). \tag{9}$$

Therefore $\lambda$ is a smooth function, since $\Lambda_{(\mathbf{R}^n \times \mathbf{R}^n, dx^i \wedge dy_i)}(x^1)$ is a smooth vector field.

If $d_m H = 0$, then (8) also holds. In order to prove this, we take an open $U \subset M$, $m_i \in M$ $(i \in \mathbf{N},)$ and $H' \in \mathcal{H}(M)$ such that $m \in \overline{U}$, $\lim_{i \longrightarrow \infty} m_i = m$, $H'|_U = H|_U$, and $d_{m_i} H' \neq 0$ for $i \in \mathbf{N}$. We have

$$\Lambda_{(M,\omega)}(H)|_U = \Lambda_{(U, \omega|_{(T\times T)U})}(H|_U) = \Lambda_{(U, \omega|_{(T\times T)U})}(H'|_U) = \Lambda_{(M,\omega)}(H')|_U,$$

which gives $\Lambda_{(M,\omega)}(H)(m) = \Lambda_{(M,\omega)}(H')(m)$ by the continuity of $\Lambda_{(M,\omega)}(H)$ and $\Lambda_{(M,\omega)}(H')$. Moreover,

$$\begin{aligned} &\Lambda_{(M,\omega)}(H')(m) \\ &\quad= \lim_{i\longrightarrow\infty} \Lambda_{(M,\omega)}(H')(m_i) \\ &\quad= \lim_{i\longrightarrow\infty} \lambda\big(H'(m_i)\big)\cdot V_{H'}(m_i) \\ &\quad= \lambda\big(H'(m)\big)\cdot V_{H'}(m). \end{aligned}$$

By the continuity of $H'$, $V_{H'}$, $H$, and $V_H$, $\lambda\big(H'(m)\big)\cdot V_{H'}(m) = \lambda\big(H(m)\big)\cdot V_H(m)$. Combining these we obtain (8).

The uniqueness of $\lambda$ is clear from (9), and the proof has been completed. □

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