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Boundary Layer for Chaffee-Infante Type Equation

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Abstract. This article is concerned with the nonlinear singular perturbation problem due to small diffusivity in nonlinear evolution equations of Chaffee-Infante type. The boundary layer appearing at the boundary of the domain is fully described by a corrector which is "explicitly" constructed. This corrector allows us to obtain convergence in Sobolev spaces up to the boundary.

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1 Introduction

In this article we will study the asymptotic behavior of the solutions of certain reaction diffusion equations with small diffusivity. We will focus on the Chaffee-Infante equation:

$$\frac{\partial u^\varepsilon}{\partial t} - \varepsilon \Delta u^\varepsilon + (u^\varepsilon)^3 - u^\varepsilon = f \quad \text{in} \quad \Omega, \tag{1}$$

where Ω is a two dimensional channel

$$\Omega = (0, 2\pi) \times (0, 1), \tag{2}$$

but the methods apply to more general polynomial nonlinearities and to higher space dimensions.

The initial and boundary conditions associated with (1) and (2) are

$$u^\varepsilon = u_0 \quad \text{at} \quad t = 0, \tag{3}$$

and

$$\begin{cases} u^\varepsilon = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad 1, \\ \text{and periodicity } \ (2\pi) \ \text{ for all functions} \\ \text{in the horizontal } \ (x) \ \text{ direction.} \end{cases} \tag{4}$$

The corresponding "inviscid" equation is the reaction equation:

$$\frac{\partial u^0}{\partial t} + (u^0)^3 - u^0 = f \quad \text{in} \quad \Omega, \tag{5}$$

with the initial condition

$$u^0 = u_0 \quad \text{at} \quad t = 0. \tag{6}$$

We will assume that u_0 satisfies the boundary conditions (4) while f need not vanish at the wall. Thus there is a boundary layer near the wall (at $y = 0$ and $y = 1$) which is the main object under investigation in this article.

We will assume enough smoothness on u_0 and f so that all the calculations hereafter are justified. We will also consider the time T fixed and let the diffusivity ε approaches zero. This is the case since the solutions of the reaction equation (5) may develop internal layers as time approaches infinity. This would prevent us from obtaining a simple boundary layer expansion for the reaction-diffusion equation (1). The long time asymptotics will be considered elsewhere.

The difficulty of the problem lies in the disparity of the boundary conditions of (1) and (5) which makes this a singular perturbation problem. The approach that we take are the ones suggested by Lions [8], Vishik and Lyusternik [17] (see also Temam and Wang [14,15,16]), i.e. the construction and utilization of a corrector. The advantage of this approach, in terms of the common matched asymptotic expansion, is that once we have found the right corrector, the outer expansion for the corrector equation would be trivial (zero) and thus no matching is necessary at all. The other tools that we need here are maximum principle, energy estimates and anisotropic Sobolev imbeddings.

Our method can be carried over to more general reaction-diffusion type equations where the reaction term is a polynomial of odd degree and the leading coefficient positive (see for instance Temam [13]). Note however that the geometry that we consider is flat, our objectives and the type of problems we are interested in are not the same as those occuring with curved boundaries in relation in particular with the Ginzburg Landau equation (see e.g. [11], [12] and the bibliography therein).

Our main results are the following:

Theorem 1. *There exist constants K_j depending on T, u_0 and f only such that*

$$\left\| u^\varepsilon(t;x,y) - u^0(t;x,y) - M\left(t,x,\frac{y}{\sqrt{\varepsilon}}\right) - N\left(t;x,\frac{1-y}{\sqrt{\varepsilon}}\right)\right\|_{L^\infty((0,T)\times\Omega)} \leq K_1\varepsilon^{1/2}, \quad (7)$$

$$\left\| u^\varepsilon(t;x,y) - u^0(t;x,y) - M\left(t,x,\frac{y}{\sqrt{\varepsilon}}\right) - N\left(t;x,\frac{1-y}{\sqrt{\varepsilon}}\right)\right\|_{L^\infty(0,T;L^2(\Omega))} \leq K_2\varepsilon_1^{3/4}, \quad (8)$$

$$\left\| u^\varepsilon(t;x,y) - u^0(t;x,y) - M\left(t,x,\frac{y}{\sqrt{\varepsilon}}\right) - N\left(t;x,\frac{1-y}{\sqrt{\varepsilon}}\right)\right\|_{L^\infty(0,T;H^1(\Omega))} \leq K_3\varepsilon^{1/4}, \quad (9)$$

where M and N are solutions of

$$\frac{\partial M}{\partial t} - \frac{\partial^2 M}{\partial y^2} + M^3 - M + 3g_0M^2 + 3g_0^2M = 0 \quad \text{in} \quad y > 0, \qquad (10)$$

$$M = 0 \quad \text{at} \quad t = 0, \qquad (11)$$

and

$$M = -g_0 \quad \text{at} \quad y = 0, \quad M \to 0 \quad \text{as} \quad y \to +\infty, \qquad (12)$$

$$\frac{\partial N}{\partial t} - \frac{\partial^2 N}{\partial y^2} + N^3 - N + 3g_1N^2 + 3g_1^2N = 0 \quad \text{in} \quad y > 0, \qquad (13)$$

$$N = 0 \quad \text{at} \quad t = 0, \qquad (14)$$

and

$$N = -g_1 \quad \text{at} \quad y = 0, \quad N \to 0 \quad \text{as} \quad y \to +\infty, \qquad (15)$$

where

$$g_0(t;x) = u^0|_{y=0}, \;\; g_1(t;x) = u^0|_{y=1}. \qquad (16)$$

Here the spaces are defined as

$$H^1_p(\Omega) = \left\{v \in H^1(\Omega), v \text{ is periodic in } x \text{ with period } 2\pi\right\}; \qquad (17)$$

$$H^1_{0p}(\Omega) = \left\{v \in H^1_p(\Omega), v = 0 \text{ at } y = 0 \text{ and } y = 1\right\}. \qquad (18)$$

The rest of the article is organized as follows. In the next section we introduce a preliminary form of the corrector and derive some useful estimates; then, in the last section, we derive the correctors (M and N) and prove the main result.

2 The Preliminary Form of the Corrector

It is obvious that u^ε cannot converge to u^0 as ε approaches zero uniformly in Ω. However it is plausible to think that the convergence is true in the interior of Ω since the diffusive coefficient is small. If this is true, $u^\varepsilon - u^0$ can be approximated by a boundary layer type function θ^ε called corrector (see Lions [8]). Considering (1) and (5) we propose that θ^ε be the solution of the following evolution equation

$$\frac{\partial \theta^\varepsilon}{\partial t} - \varepsilon \Delta \theta^\varepsilon + (\theta^\varepsilon)^3 - \theta^\varepsilon + 3u^0(\theta^\varepsilon)^2 + 3(u^0)^2\theta^\varepsilon = 0 \quad \text{in} \quad \Omega, \tag{19}$$

$$\theta^\varepsilon = 0 \quad \text{at} \quad t = 0, \tag{20}$$

$$\theta^\varepsilon = -u^0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = 1. \tag{21}$$

We are led to estimate $w^\varepsilon = u^\varepsilon - u^0 - \theta^\varepsilon$ which satisfies the equation

$$\frac{\partial w^\varepsilon}{\partial t} - \varepsilon \Delta w^\varepsilon + (w^\varepsilon)^3 - w^\varepsilon + 3u^\varepsilon(u^0 + \theta^\varepsilon)w^\varepsilon = \varepsilon \Delta u^0 \quad \text{in} \quad \Omega, \tag{22}$$

$$w^\varepsilon = 0 \quad \text{at} \quad t = 0, \tag{23}$$

$$w^\varepsilon = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = 1. \tag{24}$$

Denoting K a generic constant which may depend on T, u_0 and f but is independent of ε, and which may change from place to place, we obtain:

$$\|\nabla^k u^0\|_{L^\infty((0,T)\times\Omega)} \le K \quad \text{for} \quad k = 0, 1, \dots \tag{25}$$

and by the usual maximum principle

$$\|u^\varepsilon\|_{L^\infty((0,T)\times\Omega)} \le K, \tag{26}$$

$$\|\theta^\varepsilon\|_{L^\infty((0,T)\times\Omega)} \le K, \tag{27}$$

$$\|M\|_{L^\infty((0,T)\times\{y>0\})} + \|N\|_{L^\infty((0,T)\times\{y>0\})} \le K. \tag{28}$$

The maximum principle applies to w^ε (equation (22)) as well. Indeed let K_1 be a constant independent of ε and larger than $3\|u^\varepsilon(u^0 + \theta^\varepsilon)\|_{L^\infty((0,T)\times\Omega)}$, and consider

$$\tilde{w}^\varepsilon = e^{-(K_1+2)t} w^\varepsilon;$$

we have

$$\frac{\partial \tilde{w}^\varepsilon}{\partial t} - \varepsilon \Delta \tilde{w}^\varepsilon + e^{2(K_1+2)t}(\tilde{w}^\varepsilon)^3 + (K_1 + 2 + 3u^\varepsilon(u^0 + \theta^\varepsilon))\tilde{w}^\varepsilon = \varepsilon e^{-(K_1+1)t}\Delta u^0,$$

It is now easy to observe that

$$\tilde{w}^\varepsilon(t; x, y) \le \varepsilon \|\Delta u^0\|_{L^\infty((0,T)\times\Omega)} \qquad \text{for } (t; x, y) \in (0, T) \times \Omega.$$

We can derive a corresponding lower bound and thus we conclude that

$$\|w^\varepsilon\|_{L^\infty((0,T)\times\Omega)} \le K\varepsilon. \tag{29}$$

This indicates that θ^ε is a good preliminary corrector. Furthermore, standard energy estimates for (22) yield

$$\|w^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq K\varepsilon, \tag{30}$$

$$\|w^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leq K\varepsilon^{1/2}. \tag{31}$$

This again confirms the choice of θ^ε.

To derive $L^\infty(H^1)$ estimates on w^ε we multiply (22) by $-\Delta w^\varepsilon$ and integrate over Ω. We have, after rewriting $u^0 + \theta^\varepsilon$ as $u^\varepsilon - w^\varepsilon$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla w^\varepsilon|^2_{L^2(\Omega)} + \varepsilon|\Delta w^\varepsilon|^2_{L^2(\Omega)} + \int_\Omega 3(w^\varepsilon)^2|\nabla w^\varepsilon|^2 & \\ + 3\int_\Omega (u^\varepsilon)^2|\nabla w^\varepsilon|^2 + 6\int_\Omega u^\varepsilon w^\varepsilon \nabla u^\varepsilon\cdot\nabla w^\varepsilon & \\ - 6\int_\Omega u^\varepsilon w^\varepsilon|\nabla w^\varepsilon|^2 - 3\int_\Omega (w^\varepsilon)^2\nabla u^\varepsilon\cdot\nabla w^\varepsilon & \\ \leq \frac{\varepsilon}{2}|\Delta w^\varepsilon|^2_{L^2(\Omega)} + K|\nabla w^\varepsilon|^2_{L^2(\Omega)} + K\varepsilon^{3/2}. &\end{aligned} \tag{32}$$

For the right-hand side of (32) we have used the following inequality, with f and u replaced by $\varepsilon\Delta u^0$ and w^ε :

$$-\int_\Omega f\Delta u = \int_\Omega \nabla f\nabla u - \int_{y=1} f\frac{\partial u}{\partial y} + \int_{y=0} f\frac{\partial u}{\partial y},$$

and hence

$$\begin{aligned}\left|\int_\Omega f\Delta u\right| &\leq |\nabla f|_{L^2(\Omega)}|\nabla u|_{L^2(\Omega)} + |f|_{L^2(\Gamma)}\left|\frac{\partial u}{\partial y}\right|_{L^2(\Gamma)} \\ &\leq |\nabla f|_{L^2(\Omega)}|\nabla u|_{L^2(\Omega)} + K|f|_{L^2(\Gamma)}|\nabla u|^{1/2}_{L^2(\Omega)}|\Delta u|^{1/2}_{L^2(\Omega)} \\ &\leq |\nabla f|_{L^2(\Omega)}|\nabla u|_{L^2(\Omega)} + \frac{\varepsilon}{2}|\Delta u|^2_{L^2(\Omega)} + |\nabla u|^2_{L^2(\Omega)} + K\varepsilon^{-1/2}|f|^2_{L^2(\Gamma)}.\end{aligned} \tag{33}$$

The treatment of inequality (32) then necessitates estimates on ∇u^ε which can be derived by multiplying (1) by $-\Delta u^\varepsilon$ integrating over Ω and applying the Uniform Gronwall inequality (see e.g. [13]). We also apply (33) with u replaced by u^ε. We find:

$$\|u^\varepsilon\|_{L^\infty(0,\infty;H^1(\Omega))} \leq K\varepsilon^{-1/4}. \tag{34}$$

Combining (26), (27), (29), (32) and (34) we deduce

$$\frac{d}{dt}|\nabla w^\varepsilon|^2_{L^2(\Omega)} + \varepsilon|\Delta w^\varepsilon|^2_{L^2(\Omega)} \leq K|\nabla w^\varepsilon|^2_{L^2(\Omega)} + K\varepsilon^{3/2} + K\varepsilon^2|\nabla u^\varepsilon|^2_{L^2(\Omega)},$$

which implies

$$\|w^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \leq K\varepsilon^{3/4}, \quad \|w^\varepsilon\|_{L^2(0,T;H^2(\Omega))} \leq K\varepsilon^{1/4}. \tag{35}$$

By differentiating the equations in x and repeating the above procedures, we see that the above estimates remain valid for $\partial^k w^\varepsilon/\partial x^k$. This confirms our intuition that tangential derivatives are small even though the normal ones might be large.

3 The Explicit Corrector and the Proof of the Theorem

Since the tangential derivatives are small we tend to neglect them in equation (19). We also expect that θ^ε be a boundary layer type function, i.e. it decays fast in the interior of the domain, thus in terms of matched asymptotic expansions, the outer expansion should be trivial (which is easy to see) and the inner expansion matches the outer one automatically. This leads us to propose M and N defined by (10)–(16) as the inner expansions at $y = 0$ and $y = 1$ respectively. We will check that these expressions are suitable.

We first prove the decay property of M, N, and θ^ε. It is enough to prove this for θ^ε. Let $\eta \in C_0^\infty([0,1])$ be a cut-off function, $\eta \geq 0$.

Standard energy estimates yield

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \eta(\theta^\varepsilon)^2 + \varepsilon\int_\Omega \eta|\nabla\theta^\varepsilon|^2 + \int_\Omega \left(\eta(\theta^\varepsilon)^4 - \eta(\theta^\varepsilon)^2 + 3u^0\eta(\theta^\varepsilon)^3 + 3(u^0)^2\eta(\theta^\varepsilon)^2\right)$$
$$= -\varepsilon\int_\Omega \eta'\frac{\partial\theta^\varepsilon}{\partial y}\theta^\varepsilon = \frac{\varepsilon}{2}\int_\Omega \eta''(\theta^\varepsilon)^2. \quad (36)$$

Using a function of the form

$$\varphi^\varepsilon(t;x,y) = -g_0(t;x)\rho\left(\frac{y}{\sqrt{\varepsilon}}\right) - g_1(t,x)\rho\left(\frac{1-y}{\sqrt{\varepsilon}}\right), \quad (37)$$

with $\rho \in C^\infty([0,1]), \rho(0) = 1,$ supp $\rho \subset [0, \frac{1}{2}]$, and considering $\theta^\varepsilon - \varphi^\varepsilon$, we deduce

$$\|\theta^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq K\varepsilon^{1/4}, \quad (38)$$
$$\|\theta^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leq K\varepsilon^{-1/4}. \quad (39)$$

This together with (36), implies for $\delta \in (0, \frac{1}{2})$,

$$\|\theta^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\delta))} \leq K_\delta\varepsilon^{3/4},$$
$$\|\theta^\varepsilon\|_{L^2(0,T;H(\Omega_\delta))} \leq K_\delta\varepsilon^{1/4},$$

where

$$\Omega_\delta = (0, 2\pi) \times (\delta, 1-\delta), \quad (40)$$

and K_δ is a constant depending on δ, T, f, u_0, but independent of ε.

By reiteration, we deduce

$$\|\theta^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\delta))} \leq K_\delta\varepsilon^{5/4}, \quad (41)$$
$$\|\theta^\varepsilon\|_{L^2(0,T;H^1(\Omega_\delta))} \leq K_\delta\varepsilon^{3/4}. \quad (42)$$

We could reiterate again but our aim now is to obtain estimates on higher order derivatives of θ^ε. For that purpose we multiply (19) by $-\nabla(\eta(y)\nabla\theta^\varepsilon)$ and integrate over Ω.

Notice that

$$\varepsilon\int_\Omega \Delta\theta^\varepsilon\eta'\frac{\partial\theta^\varepsilon}{\partial y} = \frac{\varepsilon}{2}\int_\Omega \eta''|\nabla\theta^\varepsilon|^2 - \varepsilon\int_\Omega \eta''\left(\frac{\partial\theta^\varepsilon}{\partial y}\right)^2,$$

$$\left|\int_\Omega \eta\nabla\theta^\varepsilon\nabla((\theta^\varepsilon)^3 + 3u^0(\theta^\varepsilon)^2 + 3(u^0)^2\theta^\varepsilon)\right| \le K\int_\Omega \eta|\nabla\theta^\varepsilon|^2 + K\varepsilon^{5/2},$$

(Thanks to (27), (41) and (42));

hence we have

$$\|\theta^\varepsilon\|_{L^\infty(0,T;H^k(\Omega_\delta))} \le K_\delta\varepsilon^{5/4}, \tag{43}$$

$$\|\theta^\varepsilon\|_{L^2(0,T;H^{k+1}(\Omega_\delta))} \le k_\delta\varepsilon^{3/4}, \quad \text{for} \quad k = 0, 1. \tag{44}$$

The procedure can be repeated for $k = 2, 3$, and with $\partial^k\theta^\varepsilon/\partial x^k$ replacing θ^ε.

Similar estimates hold for $M^\varepsilon(t, x, y) = M(t, x, \frac{y}{\sqrt{\varepsilon}})$ and also for $N^\varepsilon(t, x, y) = N(t; x, \frac{1-y}{\sqrt{\varepsilon}})$. In particular we will have for

$$C_M^\varepsilon(t; x, y) = -yM\left(t; x, \frac{1}{\sqrt{\varepsilon}}\right), \tag{45}$$

$$\|\nabla^k C_M^\varepsilon\|_{L^\infty((0,T)\times\Omega)} \le K\varepsilon^{5/4}, \quad \text{for} \quad k = 0, 1, 2, \ldots \tag{46}$$

$$\left\|\frac{\partial C_M^\varepsilon}{\partial t}\right\|_{L^\infty((0,T)\times\Omega)} \le K\varepsilon^{5/4}. \tag{47}$$

We then consider the quantity

$$q^\varepsilon = \theta^\varepsilon - M^\varepsilon - N^\varepsilon - C^\varepsilon,$$

where $C^\varepsilon = C_M^\varepsilon + C_N^\varepsilon$, $C_N^\varepsilon = -(1-y)N\Big(t, x, \frac{1}{\sqrt{\varepsilon}}\Big)$.

For the sake of simplicity, we now assume that $f \equiv 0$ on $y = 1$ and hence $g_1 \equiv 0$, which further implies $N \equiv 0$. Hence q^ε reduces to

$$q^\varepsilon = \theta^\varepsilon - M^\varepsilon - C_M^\varepsilon\,. \tag{48}$$

It satisfies the equation

$$\begin{aligned}\frac{\partial q^\varepsilon}{\partial t} &- \varepsilon\Delta q^\varepsilon + (\theta^\varepsilon)^3 + 3u^0(\theta^\varepsilon)^2 + 3(u^0)^2\theta^\varepsilon \\ &- (M^\varepsilon)^3 - 3g_0(M^\varepsilon)^2 - 3g_0^2M^\varepsilon - q^\varepsilon \\ &= -\frac{\partial C_M^\varepsilon}{\partial t} + \varepsilon\Delta C_M^\varepsilon + \varepsilon\frac{\partial^2 M^\varepsilon}{\partial x^2} + C_M^\varepsilon \quad \text{in} \quad \Omega,\end{aligned} \tag{49}$$

with initial and boundary conditions (thanks to $N \equiv 0$):

$$q^\varepsilon = 0 \quad \text{at} \quad t = 0, \tag{50}$$

$$q^\varepsilon = 0 \quad \text{on} \quad y = 0 \quad \text{and} \quad y = 1. \tag{51}$$

Notice that

$$\begin{aligned}
(\theta^\varepsilon)^3 - (M^\varepsilon)^3 &= q^\varepsilon((\theta^\varepsilon)^2 + \theta^\varepsilon M^\varepsilon + (M^\varepsilon)^2) + C_M^\varepsilon((\theta^\varepsilon)^2 + \theta^\varepsilon M^\varepsilon + (M^\varepsilon)^2),\\
3u^0(\theta^\varepsilon)^2 - 3g_0(M^\varepsilon)^2 &= 3u^0(\theta^\varepsilon + M^\varepsilon)q^\varepsilon + 3u^0(\theta^\varepsilon + M^\varepsilon)C_M^\varepsilon + 3(u^0 - g_0)(M^\varepsilon)^2,\\
3(u^0)^2\theta^\varepsilon - 3g_0^2 M^\varepsilon &= 3(u^0)^2 q^\varepsilon + 3(u^0)^2 C_M^\varepsilon + 3(u^0 + g_0)(u^0 - g_0)M^\varepsilon;
\end{aligned}$$

hence we may rewrite (49) as

$$\begin{aligned}
\frac{\partial q^\varepsilon}{\partial t} &- \varepsilon \Delta q^\varepsilon + ((\theta^\varepsilon)^2 + \theta^\varepsilon M^\varepsilon + (M^\varepsilon)^2)q^\varepsilon\\
&+ 3u^0(\theta^\varepsilon + M^\varepsilon)q^\varepsilon + 3(u^0)^2 q^\varepsilon - q^\varepsilon = \tilde{f} \quad \text{in} \quad \Omega,
\end{aligned} \tag{49'}$$

where

$$\begin{aligned}
\tilde{f} = &- \frac{\partial C_M^\varepsilon}{\partial t} + \varepsilon \Delta C_M^\varepsilon + \varepsilon \frac{\partial^2 M^\varepsilon}{\partial x^2} + C_M^\varepsilon\\
&- \big((\theta^\varepsilon)^2 + \theta^\varepsilon M^\varepsilon + (M^\varepsilon)^2 + 3u^0(\theta^\varepsilon - M^\varepsilon) + 3(u^0)^2\big)C_M^\varepsilon\\
&- 3(u^0 - g_0)(M^\varepsilon)^2 - 3(u^0 + g_0)(u^0 - g_0)M^\varepsilon.
\end{aligned} \tag{52}$$

By the choice of g_0, $\frac{u^0 - g_0}{y}$ remains bounded on $(0,T) \times \Omega$. In order to obtain an L^∞ estimate on $\tilde{f}$ (sharp in terms of dependence on ε), we need to obtain an L^∞ bound on yM. Consider $(1+y)M$ which satisfies the equation

$$\begin{aligned}
\frac{\partial((1+y)M)}{\partial t} &- \frac{\partial^2}{\partial y^2}((1+y)M) + \frac{1}{(1+y)^2}((1+y)M)^3 + \frac{3g_0}{1+y}((1+y)M)^2\\
&+ 3g_0^2(1+y)M - (1+y)M = -2\frac{\partial M}{\partial y},
\end{aligned} \tag{53}$$

and

$$\frac{\partial}{\partial t}\left(\frac{\partial M}{\partial y}\right) - \frac{\partial^2}{\partial y^2}\left(\frac{\partial M}{\partial y}\right) + 3M^2\frac{\partial M}{\partial y} + 6g_0 M\frac{\partial M}{\partial y} + 3g_0^2\frac{\partial M}{\partial y} - \frac{\partial M}{\partial y} = 0. \tag{54}$$

We see that $\frac{\partial M}{\partial y}$ satisfies a maximum principle and hence $(1+y)M$ too.

This combined with (27), (28), (46) and (47) yields

$$\|\tilde{f}\|_{L^\infty((0,T)\times\Omega)} \le K\varepsilon^{1/2}. \tag{55}$$

This further implies, via a maximum principle type argument as that for w^ε,

$$\|q^\varepsilon\|_{L^\infty((0,T)\times\Omega)} \le K\varepsilon^{1/2}. \tag{56}$$

It is also easy to check, thanks to (46), (47) and the boundedness of $\frac{u^0-g_0}{y}$, that

$$\|\tilde{f}\|_{L^2(0,T;L^2(\Omega))} \leq K\varepsilon^{3/4}. \tag{57}$$

Thus standard energy estimates yield

$$\|q^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq K\varepsilon^{3/4}, \tag{58}$$

$$\|q^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \leq K\varepsilon^{1/4}. \tag{59}$$

The theorem then follows from (29), (30), (35), (46), (56), (58) and (59).
This completes the proof.

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