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Dynamical Systems with Several Equilibria and Natural Liapunov Functions

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Abstract. Dynamical systems with several equilibria occur in various fields of science and engineering: electrical machines, chemical reactions, economics, biology, neural networks. As pointed out by many researchers, good results on qualitative behaviour of such systems may be obtained if a Liapunov function is available. Fortunately for almost all systems cited above the Liapunov function is associated in a natural way as an energy of a certain kind and it is at least nonincreasing along systems solutions.

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1 Introduction

Dynamical systems with several equilibria occur in various fields of science and engineering: electrical machines, chemical reactions, economics, biology, neural networks. These systems are models of either natural or man-made physical systems. In both cases stability properties are required for various reasons but in fact stability means always some "good behaviour" with respect to short-term disturbances. In man-made systems technological operation is connected with stability of the "operating points" i.e. of some constant solutions of the dynamical model.

Technological operation is closely connected with oriented changes from one operating point to another i.e. with transients. With respect to the new operating point (constant solution) the old operating point is a perturbed initial condidtion generating a transient motion (dynamics trajectory) that should end in the new operating point. This is clearly a stability-like property.

Stability is a property of a single solution (equilibrum) and a local one. Linear systems and systems with an almost linear behaviour have a single equilibrum that is globally asymptotically stable.

For systems with several equilibria the usual local concepts of stability are not sufficient for an adequate description. The so-called "global phase portrait" may contain both stable and nonstable equilibria. Of course each of them may be characterized separately since stability is a local concept. Nevertheless global concepts are also required for a better system description.

We consider here a single example: the case of the neural networks. The neural networks are interconnections of simple computing elements whose computational capability is increased by interconnection (emergent collective capacities). This is due to the nonlinear characteristics leading to the existence of several stable equilibria. The network achieves its computing goal if no self-sustained oscillations are present and it always achieves some steady-state (equilibrium) among a finite (while large) number of such states.

This behaviour is met in other systems also. For instance chemical systems or biological communities display several equilibria, according to the external conditions (environment). The models in macroeconomics need several equilibria since in practice this is indeed the case and economic policies (good or bad) are nothing else but "manoeuvres" that take economic systems from one stable equilibrium to another - in the same way as mechanical manoeuvres take engineering systems from one operating point to another.

2 Basic concepts and tools

The basic concepts in the field of the systems with several equilibria come from the papers of Kalman [7] from 1957 and Moser [10] from 1967. Especially the second paper relies on the following remark:

Consider the system

$$\dot{x} = -f(x), x \in \mathbb{R}^n, \tag{1}$$

where $f(x) = grad\, G(x)$ and $G : \mathbb{R}^n \to \mathbb{R}$ has the following properties:

i) $lim_{|x|\to\infty} G(x) = \infty$ and
ii) the number of the critical points is finite.

In this case any solution of (1) approaches asymptotically one of the equilibrum points (which is also a critical point of G — where the gradient i.e. f vanishes). It is only natural to call this behaviour gradient-like but there are other properties that are also important while weaker. With respect to this we shall need some basic notions. Our object will be here the system of ordinary differential equations

$$\dot{x} = f(x, t). \tag{2}$$

Definition 1. a) Any constant solution of (2) is called *equilibrium*. The set of equilibria $\mathcal{E}$ is called *stationary set*.

b) A solution of (2) is called *convergent* if it approaches asymptotically some equilibrium:

$$lim_{t\to\infty}x(t) = c \in \mathcal{E}$$

A solution is called *quasi-convergent* if it approaches asymptotically the stationary set:

$$lim_{t\to\infty}d(x(t), \mathcal{E}) = 0$$

Definition 2. System (2) is called *monostable* if every bounded solution is convergent; it is called *quasi-monostable* if every bounded solution is quasi-convergent.

Definition 3. System (2) is called *gradient-like* if every solution is convergent; it is called *quasi-gradient-like* if every solution is quasi-convergent.

Since there exist also other terms for these notions some comments are necessary. The notion of convergence still defines a solution property and was introduced by Hirsch [5,6]. Monostability has been introduced by Kalman [7] in 1957; sometimes it is called *strict mutability* (Popov [11]) while quasi-mono-stability is called by the same author *mutability* and by other *dichotomy* (Gelig, Leonov, and Yakubovich [3]). In fact for monostable (quasi-monostable) systems some kind of dichotomy occurs: their solutions are either unbounded or tend to an equilibrium (or to the stationary set); in any case self-sustained periodic or almost periodic oscillations are excluded. The quasi-gradient-like property is called sometimes *global asymptotics*.

It is obvious that while convergence is associated to solutions, monostability and gradient-like property are associated to systems. At this point we add some properties related to the stationary set (Gelig, Leonov and Yakubovith [3])

Definition 4. The stationary set $\mathcal{E}$ is *uniformly stable* if for any $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that for any t_0 if $d(x(t_0), \mathcal{E}) < \delta$ then $d(x(t), \mathcal{E}) < \varepsilon$ for all $t \geq t_0$.

The stationary set $\mathcal{E}$ is *uniformly globally stable* if it is uniformly Liapunov stable and the system is quasi-gradient-like (has global asymptotics).

The stationary set is *pointwise globally stable* if it is uniformly Liapunov stable and the system is gradient-like.

For autonomus (time-invariant systems) the following Liapunov-type results are available (Gelig, Leonov and Yakubovitch [3]; Leonov, Reitmann and Smirnova [9]).

Lemma 5. *Consider the nonlinear system*

$$\dot{x} = f(x) \tag{3}$$

and assume existence of a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ that is nonincreasing along any solution of (3). *If, additionaly, a bounded on $\mathbb{R}^+$ solution $x(t)$ for which there exists some $\tau > 0$ such that $V(x(\tau)) = V(x(0))$ is an equilibrium then the system is quasi-monostable.*

Lemma 6. *If the assumptions of Lemma* 5 *hold and, additionaly,* $V(x) \to \infty$ *for* $|x| \to \infty$ *then system* (3) *is quasi-gradient-like.*

Lemma 7. *If the assumptions of Lemma* 6 *hold and the set* $\mathcal{E}$ *is discrete (i.e. it consists of isolated equilibria only) then system* (3) *is gradient-like.*

3 Applications from chemical kinetics

3.1. We shall consider first a model from the book of Frank-Kamenetskii [2], studied in the diffusion context by Kružkov and Peregudov [8]; here the diffusion phenomenon will be left aside. The model reads like (3) but under the following assumptions:

i) $f : Q \to \mathbb{R}^n$, $Q = \{x \in \mathbb{R}^n, \, x^i \geq 0, \, i = \overline{1,n}\}$;
ii) $f(0) = 0$;
iii) $\frac{\partial f^i}{\partial x^j} \geq 0$, $\forall x \in Q$, $i, j = \overline{1,n}$;
iv) $\sum_1^n f^i(x) \equiv 0$.

Then the following properties of the system are valid (Halanay and Răsvan [4]):

a) Q is an invariant set of the system;
b) all solutions in Q are bounded;
c) $\sum_1^n |x^i(t) - y^i(t)| \leq \sum_1^n |x^i(\tau) - y^i(\tau)|$ for all $t \geq \tau$, $x(t)$ and $y(t)$ being two solutions of (3) from Q;
d) the function $V(x) = \sum_1^n |f^i(x)|$ is nonincreasing along the solution of (3) i.e. *it is a Liapunov function*; moreover this Liapunov function cannot be constant if it is not identically zero.

We may state now

Theorem 8. *For every* $M > 0$ *there exist equilibria* $\hat{x}$ *such that* $\sum_1^n \hat{x}^i = M$ *and the system is gradient-like on the sets* $\sum_1^n x^i = M$.

Outline of proof: Let $x(t)$ be a solution with $\sum_1^n x^i(0) = M$. From iv) we deduce that $\sum_1^n x^i(0) \equiv M$, the solution is bounded and the ω-limit set is not empty. On the ω-limit set $V(x(t))$ is constant hence it is identically zero; from here we obtain that the ω-limit set consists only of equilibria. Since all solutions are bounded the system is quasi-gradient-like. Using c) we obtain that the system is even gradient-like.

3.2. Consider now the case of a closed chemical system subject to mass-action law and constant temperature—the formal kynetics system ([4]):

$$\begin{aligned} &\dot{c}_i = \textstyle\sum_{j=1}^n (\beta_{ij} - \alpha_{ij})(w_j^+(c) - w_j^-(c)), \; i = \overline{1,m} \\ &w_j^+(c) = k_j^+ \textstyle\prod_1^m (c_i)^{\alpha_{ij}}, \; w_j^-(c) = k_j^- \textstyle\prod_1^m (c_i)^{\beta_{ij}}, \end{aligned} \tag{4}$$

where the nonnegative integers α_{ij}, β_{ij} (stoichiometric coefficients) satisfy the following assumption $(\forall j)(\exists i : \alpha_{ij}+\beta_{ij} \neq 0)$, that is each substance has to participate at least to one reaction either as reactant or as product. Using this assumption we can prove *positivity of the concentrations: if the above assumption holds then* $c_i(0) \geq 0,\ i = \overline{1,m}$ *implies that for any* $i = \overline{1,m}$ *either* $c_i(t) > 0$ *or* $c_i(t) \equiv 0$ *on the entire existence interval of the solution.* Any point c with $c_i > 0,\ i = \overline{1,m}$ is called *admissible*; the *set of the admissible points is called admissible set.* Another property of the system is existence of a *set of conservation laws* that define an *invariant hyperplane.* By writing (4) as follows

$$\dot{c} = Gw(c), \tag{5}$$

where $rank\, G = r$ we may obtain by reordering (5) the partition

$$\begin{aligned} \dot{c}^r &= G_{11}w^r(c) + G_{12}w^{m-r}(c), \\ \dot{c}^{m-r} &= G_{21}w^r(c) + G_{22}w^{m-r}(c), \end{aligned} \tag{6}$$

where $det\, G_{11} \neq 0$. Then the following linear invariant manifold is obtained

$$\mathcal{L}(c) \equiv c^{m-r} - G_{21}G_{11}^{-1}c^r = c^{m-r}(0) - G_{21}G_{11}^{-1}c^r(0) \tag{7}$$

called "substance balance hyperplane" that is in fact a linear system of conservation laws.

The equilibrium set of (4) may be quite rich but among the equilibria are of interest the *detailed-balance equilibria* defined by

$$w_j^+(c) = w_j^-(c),\ j = \overline{1,n} \tag{8}$$

and mainly those belonging to the admissible set $Q = \{c \in \mathbb{R}^m,\ c_k > 0,\ k = \overline{1,m}\}$ called *admissible detailed balance equilibria.*

The following result of Zeldovič is valid

Proposition 9. *If* (4) *has an admissible detailed balance equilibrium and in the linear manifold* $\mathcal{L}(c) = q$ *there exists an admissible point then in this manifold there exists a unique detailed balance point.*

If (4) is such that *an admissible detailed balance equilibrium exists* then the following *Liapunov function* may be associated to it:

$$V_{\hat{c}} = \sum_1^m c_k(ln(c_k/\hat{c}_k) - 1) \tag{9}$$

and the following is true (Halanay & Răsvan [4])

Theorem 10. *If an admissible detailed balance equilibrium exists, the following properties of the solutions with* $c_i(0) \geq 0,\ i = \overline{1,m}$ *are valid:*

1. *The solutions are bounded.*

2. *There are no periodic nonconstant solutions with nonnegative components.*
3. *Any equilibrium point with nonnegative components is a detailed balance point.*
4. *The ω-limit set of any solution is composed of equilibrium points only; if such a set contains an admissible detailed balance point it coincides with it being a singleton.*
5. *An admissible detailed balance point is stable in the sense of Liapunov and it is an attractor in the invariant hyperplane that contains it.*
6. *A solution such that $lim_{t\to\infty}c(t)$ exists and has all its components positive is Liapunov stable.*

Some comments are necessary. The first four properties show that, with respect to those solutions that are physically significant, *the system is quasi-gradient-like* but if among the equilibria of a given ω-limit set there is one admissible detailed balance point, the ω-limit set coincides with it; this singleton is stable in the sense of Liapunov and even asymptotically stable when the solutions are reduced to the invariant hyperplane containing this point.

The remarkable property of this system would be existence of an admissible point in the ω-limit set of any solution. In this case the ω-limit set would reduce to it and the attraction domain would coincide with the entire hyperplane. The system would be gradient-like with respect to admissible set Q. Unfortunately this is still an open question. We may nevertheless mention that some recent results for the case of two substances exist (Simon & Farkas [12]).

4 Applications from biology

A. Consider first the model of Volterra type for n species that compete for some resource:

$$\frac{dN_i}{dt} = N_i(\varepsilon_i - \sum_{j=1}^{n} \gamma_{ij} N_j),\ i = \overline{1, n} \tag{10}$$

This model has been studied intensely (e.g. Volterra[14]; Svireżev [13]) for the case ofthe so-called dissipative community: $\varepsilon_i > 0$ and there exist $\alpha_i > 0$ such that the quadratic form $\sum_1^n \sum_1^n \alpha_i \gamma_{ij} x_i x_j$ is positive definite. Here we shall consider the general case because of its similarity to mass action chemical kinetics.

We assume, as in the case of the chemical kinetics, existence of an equilibrium $\hat{N}_i,\ i = \overline{1, n}$ with all $\hat{N}_i > 0$. Associate to (10) the following function:

$$L_{\hat{N}} = \sum_{1}^{n} \hat{N}_i \left(\frac{N_i}{\hat{N}_i} - 1 - \ln \frac{N_i}{\hat{N}_i} \right) \tag{11}$$

which is of the same type as (9); with the new variables $x_i = \ln(N_i/\hat{N}_i)$ we obtain:

$$\frac{dx_i}{dt} = \varepsilon_i - \sum_{j=1}^{n} \gamma_{ij} \hat{N}_j e^{x_j}, \tag{12}$$

$$L_{\hat{N}} = \sum_{1}^{n} \hat{N}_i \left(e^{x_i} - 1 - x_i\right) \tag{13}$$

and it may be easily seen that (12) can be written as:

$$\frac{dx_i}{dt} = -\sum_{j=1}^{n} \gamma_{ij} \frac{\partial L}{\partial x_i}(x_1, \cdots, x_n) \tag{14}$$

i.e. the system is quasi-gradient-like. We have $L(x_1, x_2, \cdots, x_n) > 0$ and also

$$\frac{d}{dt} L(x_1(t), \cdots, x_n(t)) = -\sum_{1}^{n} \sum_{1}^{n} \gamma_{ij} \frac{\partial L}{\partial x_i} \frac{\partial L}{\partial x_j}$$

and if the matrix (γ_{ij}) is nonnegative definite then L is decreasing (nonincreasing along the solutions of (14). Obviously L is bounded for bounded x_i (see the previous section) hence the system is quasi-monostable. Moreover the critical point of L i.e. $x_1 = x_2 = \cdots = x_n = 0$ is globally assymptotically stable. We have also $L(x) \to \infty$ for $|x| \to \infty$ hence according to Lemma 6 the system is quasi-gradient like. Moreover the equilibria of (14) are given by

$$\sum \gamma_{ij} \frac{\partial L}{\partial x_i} = 0$$

and the structure of L shows that they are isolated. We obtained the following

Theorem 11. *If system* (10) *has an equilibrium with al components positive and the matrix* γ_{ij} *is positive definte then it is gradient-like.*

B. An example taken from a different field of biological sciences is the model of evolutionary selection of macromolecular species of Eigen and Schuster (taken from the paper of Cohen and Grossberg [1]):

$$\dot{x}_i = x_i(m_i x_i^{p-1} - q \sum_{k=1}^{n} m_k x_k^p) \tag{15}$$

Remark that if $p = 1$ a special case of (10) is obtained. In fact, as shown is the cited paper, many of the biological models may be obtained from a general neural network model that will be shown next. For this reason we do not insist here on Eigen-Schuster model.

5 Continous-time neural networks

The neural networks are structures that possess "emergent computational capabilities" that is they are interconnected simple computational elements to which interconnections confer increased computational power.

The general model considered here (Cohen and Grossberg [1]) reads

$$\dot{x}_i = a_i(x_i)[b_i(x_i) - \sum_1^n c_{ij} d_j(x_j)],\ i = \overline{1,n}, \tag{16}$$

where $c_{ij} = c_{ji}$. The following Liapunov function is associated

$$V(x) = \frac{1}{2}\sum_1^n \sum_1^n c_{ij} d_i(x_i) d_j(x_j) - \sum_1^n \int_0^{x_i} b_i(\lambda) d_i'(\lambda) d\lambda \tag{17}$$

that is much alike to the Liapunov function of the absolute stability problem.

It can be seen that (16) may be given the form

$$\dot{x} = -A(x)\, grad\, V(x), \tag{18}$$

where the items of $A(x)$ are

$$A_{ij}(x) = \frac{a_i(x_i)}{d_i'(x_i)}\delta_{ij} \tag{19}$$

Also the derivative of V along the solutions of $V(x)$ reads

$$W(x) = -\sum_1^n a_i(x_i) d_i'(x_i)\left[b_i(x_i) - \sum_1^n c_{ij} d_j(x_j)\right]^2 \leq 0$$

provided $a_i(\lambda) > 0$ and $d_i(\lambda)$ are nondecreasing. If additionally $d_i(\cdot)$ are strictly increasing the set, where $W(x) = 0$ consists only of equilibria. It follows that the system is quasi-gradient-like (Lemma 6).

Usually the property required for neural networks is gradient-like behavoiur. This property requires always specific studies since in the general case of (16) the equilibrium set may contain countably many equilibria.

6 Concluding remarks

We have presented here some models occuring in various fields of science and engineering; nevertheless they have some common features. First of all they belong to the class of so called competitive differential systems [5]. They all have many equilibria and require those qualitative concepts that were introduced for such systems (mutability, dichotomy, gradient behaviour). In obtaining the required properties the milestone is to show that the ω-limit sets of the solutions are composed of isolated equilibria only. Usually this goal is achieved using specific methods of differential topology that take into account the structure of differential equations that are competitive [5].

Existence of a suitable Liapunov function may simplify the task of showing that the ω-limit sets are composed of equilibria only; this was supposed to be the mainstream of the present paper and it illustrates that it is desirable to associate a Liapunov function, in a natural way, to any dynamical model. Of course, “guessing” a Liapunov function remains an art and a challenge.

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