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Fixed Point Theory for Closed Multifunctions

Donal O'Regan

Department Of Mathematics, National University of Ireland,
Galway, Ireland
Email: Donal.ORegan@UCG.IE

Abstract. In this paper some new fixed point theorems of Ky Fan, Leray-Schauder and Furi-Pera type are presented for closed multifunctions.

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1 Introduction

This paper establishes some fixed point theorems for multivalued condensing maps with closed graph. In particular we obtain an analogue of (i). Ky Fan's Fixed Point Theorem, (ii). Leray-Schauder Alternative, and (iii). Furi-Pera Fixed Point Theorem, for such maps. The need for new fixed point theory for closed multifunctions arose out of the study of differential and integral inclusions (see [5,9] and their references). If our operator is compact then a well known result (see [1, page 465]) implies that we may use fixed point theory for upper semicontinuous (u.s.c.) maps. However a new theory is needed if our map is condensing and not compact. We initiated the study in [10,11]. This paper continues this study. In addition we simplify some of the proofs in [10].

For the remainder of this section we describe the type of maps which we will consider in section 2. Suppose X and Z are subsets of Hausdorff topological vector spaces E_1 and E_2 respectively and $F : X \to 2^Z$ a multifunction (here 2^Z denotes the family of nonempty subsets of Z). Given two open neighborhoods U and V of the origins in E_1 and E_2 respectively, a (U, V)-approximate continuous selection [2,3] of F is a continuous function $s : X \to Z$ satisfying

$$s(x) \in (F[(x + U) \cap X] + V) \cap Z \qquad \text{for every } x \in X.$$

F is said to be **approximable** [3] if its restriction $F|_K$ to any compact subset K of X admits a (U, V)-approximate continuous selection for every open neighborhoods U and V of the origins in E_1 and E_2 respectively.

Definition 1. We say $F \in APCG(X, Y)$ if $F : X \to Cc(Y)$ is a closed (i.e. has closed graph), approximable map; here $Cc(Y)$ denotes the family of nonempty, closed subsets of Y.

Definition 2. We say $F \in ACG(X, Y)$ if $F : X \to CD(Y)$ is a closed map; here $CD(Y)$ denotes the family of nonempty, closed, acyclic (see [5]) subsets of Y.

Recall F is acyclic *if for every* $x \in X$, $H^m(F(x)) = \delta_{0m}\mathbb{Z}$, where $\{H^m\}$ denotes the Čech cohomology functor with integer coefficients.

We now recall two results from the literature.

Theorem 3 ([2,3]). *Let Q be a convex, compact subset of a locally convex Hausdorff linear topological space E and $F : Q \to C(Q)$ is a u.s.c., approximable map (here $C(Q)$ denotes the family of nonempty, compact subsets of Q). Then F has a fixed point.*

Let X be a Banach space and Ω_X the bounded subsets of X. The *Kuratowski measure of noncompactness* is the map $\alpha : \Omega_X \to [0, \infty]$ defined by

$$\alpha(Z) = \inf\{\epsilon > 0 : \; Z \subseteq \cup_{i=1}^{n} Z_i \; \text{ and } \; diam\,(Z_i) \le \epsilon\}; \quad \text{here } \; Z \in \Omega_X.$$

Let X_1 and X_2 be Banach spaces. A multivalued map $F : Y \subseteq X_1 \to X_2$ is said to be α-Lipschitzian if it maps bounded sets into bounded sets and if there exists a constant $k \ge 0$ with $\alpha(F(Z)) \le k\,\alpha(Z)$ for all bounded sets $Z \subseteq Y$. We call F a **condensing** map if F is α-Lipschitzian with $k = 1$ and $\alpha(F(Z)) < \alpha(Z)$ for all bounded sets $Z \subseteq Y$ with $\alpha(Z) \ne 0$.

Theorem 4 ([5]). *Let Q be a nonempty, closed, convex subset of a Banach space E. Suppose $F : Q \to CK(Q)$ is a u.s.c., condensing map with $F(Q)$ a subset of a bounded set in E (here $CK(Q)$ denotes the family of nonempty, compact, acyclic subsets of Q). Then F has a fixed point.*

Remark 5. All the results in this paper will be stated and proved when E is a Banach space (the extension to the case when E is a Fréchet space is immediate).

2 Fixed point theory

We begin this section by proving fixed point theorems of Ky Fan [12] type for $APCG$ and ACG maps.

Theorem 6. *Let Q be a nonempty, convex, closed subset of a Banach space E and suppose $F \in APCG(Q, Q)$ is a condensing map with $F(Q)$ a subset of a bounded set in Q. Then F has a fixed point in Q.*

Proof. Let $x_0 \in Q$. Then [5, Lemma A] guarantees a closed, convex set X with $x_0 \in X$ and

$$X = \overline{co}\,(F(Q \cap X) \cup \{x_0\}).$$

Since $F(Q) \subseteq Q$ implies $F(Q \cap X) \cup \{x_0\} \subseteq Q$ we have $X \subseteq Q$ and so $Q \cap X = X$. Thus

$$X = \overline{co}\,(F(X) \cup \{x_0\}).$$

Since F is condensing we have (using the properties of measure of noncompactness) that X is compact. Thus $F : X \to 2^X$ with X compact and convex. In addition the values of F are closed and $F|_X$ has closed graph. Now [1, page 465] implies $F|_X$ is u.s.c. Consequently $F|_X : X \to C(X)$ is a u.s.c., approximable map and X is convex and compact. Theorem 3 implies that F has a fixed point in X. □

Similarly we have the following result for ACG maps.

Theorem 7 ([11]). *Let Q be a nonempty, convex, closed subset of a Banach space E and suppose $F \in ACG(Q, Q)$ is a condensing map with $F(Q)$ a subset of a bounded set in Q. Then F has a fixed point in Q.*

Proof. Let $x_0 \in Q$ and construct a convex, compact set $X \subseteq Q$ (as in Theorem 6) with $F : X \to 2^X$. In addition the values of F are closed and acyclic and $F|_X$ has closed graph. Now [1] implies $F|_X$ is u.s.c. Consequently $F|_X : X \to CK(X)$ is a u.s.c. map and X is convex and compact. Theorem 4 (or indeed Ky Fan's Fixed Point Theorem [12]) implies that F has a fixed point in X. □

Remark 8. Note Theorem 6 and Theorem 7 can easily be extended to the Fréchet space setting.

We now prove a nonlinear alternative of Leray-Schauder type for ACG and $APCG$ maps. We proved such an alternative in [10]; however here we provide a simpler proof.

Theorem 9. *Let E be a Banach space with U an open, convex subset of E and $x_0 \in U$. Suppose $F \in ACG(\overline{U}, E)$ is a condensing map with $F(\overline{U})$ a subset of a bounded set in E. Then either*

(A1) *F has a fixed point in $\overline{U}$; or*

(A2) *there exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda\, F(u) + (1-\lambda)\{x_0\}$.*

Proof. Without loss of generality assume $x_0 = 0$. Suppose $(A2)$ does not occur and F has no fixed points in ∂U. Let

$$H = \left\{ x \in \overline{U} :\ x \in \lambda\, F(x) \quad \text{for some} \quad \lambda \in [0,1] \right\}.$$

Notice that $H \neq \emptyset$ is closed. To see this let (x_n) be a sequence in H (i.e. $x_n \in \lambda_n\, F(x_n)$ for some $\lambda_n \in [0,1]$) with $x_n \to x_0 \in \overline{U}$. Without loss of generality assume $\lambda_n \to \lambda_0 \in (0,1]$. Since $x_n \in H$ there exists $y_n \in F(x_n)$ with $x_n = \lambda_n\, y_n$. Now $x_n \to x_0$ and $y_n \to \frac{1}{\lambda_0}\, x_0$. The closedness of F implies $\frac{1}{\lambda_0}\, x_0 \in F(x_0)$ so $x_0 \in H$. Thus H is closed. In fact H is compact. To see this notice $H \subseteq \overline{co}\,(F(H) \cup \{0\})$ so if $\alpha(H) \neq 0$, we have

$$\alpha(H) \leq \alpha(F(H)) < \alpha(H),$$

a contradiction. Now since $H \cap \partial U = \emptyset$ there is a continuous function $\mu : \overline{U} \to [0,1]$ with $\mu(H) = 1$ and $\mu(\partial U) = 0$. Define the map J by

$$J(x) = \begin{cases} \mu(x)\, F(x), & x \in \overline{U} \\ \{0\}, & x \in E \backslash \overline{U}. \end{cases}$$

Now it is easy to check that $J : E \to CD(E)$ has closed graph. In addition $J : E \to CD(E)$ is condensing with $J(E)$ a subset of a bounded set in E. To see this note

$$J(A) \subseteq co\, \left(F(\overline{U} \cap A) \cup \{0\} \right)$$

for any subset A of E. Now Theorem 7 implies that there exists $x \in E$ with $x \in J(x)$. Also $x \in U$ since $0 \in U$. Thus $x \in \mu(x)\, F(x) = \lambda\, F(x)$ where $0 \leq \lambda = \mu(x) \leq 1$. Consequently $x \in H$, which implies $\mu(x) = 1$ and so $x \in F(x)$. $\square$

Similarly we have the following nonlinear alternative of Leray-Schauder type for $APCG$ maps.

Theorem 10. *Let E be a Banach space with U an open, convex subset of E and $x_0 \in U$. Suppose $F \in APCG(\overline{U}, E)$ is a condensing map with $F(\overline{U})$ a subset of a bounded set in E. Then either*

(A1) *F has a fixed point in $\overline{U}$; or*

(A2) *there exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda\, F(u) + (1-\lambda)\{x_0\}$.*

Proof. Without loss of generality assume $x_0 = 0$. Suppose $(A2)$ does not occur and F has no fixed points in ∂U. Let H, μ, J be as in Theorem 9. Now $J : E \to Cc(E)$ has closed graph and J is condensing with $J(E)$ a subset of a bounded set in E. Also an easy argument (see the ideas in [8]; note for any compact subset K of E we have that $F|_K$ is u.s.c. (see [1, page 465])) implies $J : E \to Cc(E)$ is approximable. Now Theorem 6 implies that there exists $x \in E$ with $x \in J(x)$. Also as in Theorem 9 we have $x \in F(x)$. $\square$

Next we prove a new fixed point theorem of Furi-Pera type for ACG and $APCG$ maps. We discuss the case when E is a Hilbert space and then remark about the general situation.

Theorem 11. *Let Q be a closed, convex subset of a Hilbert space E with $0 \in Q$. In addition suppose $F \in APCG(Q,E)$ is a condensing map with $F(Q)$ a subset of a bounded set in E. Also assume*

$$\left.\begin{array}{l} \text{if } \{(x_j,\lambda_j)\}_1^\infty \text{ is a sequence in } \partial Q \times [0,1] \text{ converging to } (x,\lambda) \\ \text{with } x \in \lambda\, F(x) \text{ and } 0 \le \lambda < 1, \text{ then there exists } j_0 \in \{1,2,....\} \\ \text{with } \{\lambda_j\, F(x_j)\} \subseteq Q \text{ for each } j \ge j_0 \end{array}\right\} \qquad (1)$$

holds. Then F has a fixed point in Q.

Remark 12. If $F(\partial Q) \subseteq Q$ then (1) holds.

Proof. Define $r : E \to Q$ by $r(x) = P_Q(x)$ i.e. r is the nearest point projection on Q. Note r is nonexpansive. Consider

$$B = \{x \in E : \; x \in F\, r\,(x)\,\}.$$

Note $F\,r : E \to Cc(E)$ is a condensing map and $F\,r(E)$ is a subset of a bounded set in E. Also $F\,r : E \to Cc(E)$ has closed graph. To see this let (y_n) be a sequence in E with $y_n \to y_0$ and $v_n \in F\,r\,(y_n)$ is such that v_n converges to v_0. Let $z_n = r(y_n)$ and so $v_n \in F(z_n)$ and $z_n \to z_0 = r(x_0)$. Since F has closed graph $v_0 \in F(z_0)$ i.e. $v_0 \in F\,r\,(y_0)$. Finally notice $F\,r : E \to Cc(E)$ is an approximable map. To see this take any compact subset K of E. Note $r : K \to Q$ and $F : Q \to Cc(E)$. A result of [2, page 468] (follow the reasoning in Proposition 3.3; note $F|_{r(K)}$ is u.s.c. [1, page 465]) implies $F\,r : E \to Cc(E)$ is an approximable map. Theorem 6 implies $F\,r$ has a fixed point so $B \neq \emptyset$. We must show B is closed. To see this let (x_n) be a sequence in B (i.e. $x_n \in F\,r\,(x_n)$) with $x_n \to x_0 \in E$. Now since $F\,r$ has closed graph we have $x_0 \in F\,r\,(x_0)$ i.e. $x_0 \in B$. Thus B is closed. In fact B is compact. To see this notice $B \subseteq F\,r\,(B)$. If $\alpha(r(B)) \neq 0$ then

$$\alpha(B) \le \alpha(F\,r\,(B)) < \alpha(r(B)) \le \alpha(B),$$

a contradiction. Thus $\alpha(r(B)) = 0$ and so $\alpha(B) \le \alpha(F\,r\,(B)) \le \alpha(r(B)) = 0$ so B is compact.

It remains to show $B \cap Q \neq \emptyset$. Suppose this is not true i.e. suppose $B \cap Q = \emptyset$. Then there exists $\delta > 0$ with $dist\,(B,Q) > \delta$. Choose $N \in \{1,2,...\}$ such that $1 < \delta\, N$. Define

$$U_i = \left\{ x \in E : \; d(x,Q) \; < \; \frac{1}{i} \right\} \quad \text{for} \quad i \in \{N, N+1, ...\};$$

here d is the metric induced by the norm. Fix $i \in \{N, N+1, ...\}$. Since there is $dist\,(B,Q) > \delta$ then $B \cap \overline{U_i} = \emptyset$. Now Theorem 10 implies (since $B \cap \overline{U_i} = \emptyset$) that there exists $(y_i, \lambda_i) \in \partial U_i \times (0,1)$ with $y_i \in \lambda_i\, F\,r\,(y_i)$. Consequently for each $j \in \{N, N+1, ...\}$ there exists $(y_j, \lambda_j) \in \partial U_j \times (0,1)$ with $y_j \in \lambda_j\, F\,r\,(y_j)$. In particular since $y_j \in \partial U_j$ we have

$$\{\lambda_j\, F\,r\,(y_i)\} \not\subseteq Q \quad \text{for each} \quad j \in \{N, N+1, ...\}. \qquad (2.2)$$

Next let us look at

$$D = \{x \in E : x \in \lambda\, F\, r\,(x) \ \text{ for some } \ \lambda \in [0,1]\,\}.$$

First notice D is closed. To see this let (x_n) be a sequence in D (i.e. $x_n \in \lambda_n\, F\, r\,(x_n)$ for some $\lambda_n \in [0,1]$) with $x_n \to x_0 \in E$ and without loss of generality assume $\lambda_n \to \lambda_0 \in (0,1]$. The closedness of $F\, r$ (see the argument in Theorem 9) implies $\frac{1}{\lambda_0}\, x_0 \in F\, r\,(x_0)$ so $x_0 \in D$ [Alternatively, it is easy to see that $R : E \times [0,1] \to Cc(E)$, given by $R(x,\lambda) = \lambda\, F\, r\,(x)$, has closed graph so it is immediate that D is closed]. In fact D is compact. To see this notice

$$D \subseteq \overline{co}\,(F\, r\,(D) \cup \{0\})$$

and it is easy to check that $\alpha(D) = 0$ (since F is condensing and r is nonexpansive). Thus D is compact (so sequentially compact). This together with $d(y_j, Q) = \frac{1}{j}$, $|\lambda_j| \leq 1$ (for $j \in \{N, N+1, ...\}$) implies that we may assume without loss of generality that $\lambda_j \to \lambda^\star$ and $y_j \to y^\star \in \partial Q$. Also since $y_j \in \lambda_j\, F\, r\,(y_j)$ we have, since R (defined above) : $\overline{U_N} \times [0,1] \to Cc(E)$ has closed graph, that $y^\star \in \lambda^\star\, F\, r\,(y^\star)$. Now $\lambda^\star \neq 1$ since $B \cap Q = \emptyset$. Thus $0 \leq \lambda^\star < 1$. But in this case (1), with $x_j = r(y_j) \in \partial Q$ and $x = y^\star = r(y^\star)$, implies that there exists $j_0 \in \{N, N+1, ...\}$ with $\{\lambda_j\, F\, r\,(y_j)\} \subseteq Q$ for each $j \geq j_0$. This contradicts (2.2). Thus $B \cap Q \neq \emptyset$ i.e. there exists $x \in Q$ with $x \in F\, r\,(x) = F(x)$. $\square$

Remark 13. Of course the result in Theorem 11 holds for certain convex sets in Banach spaces where there is a nearest point retraction that is nonexpansive (or more generally α-Lipschitzian with $k = 1$).

Remark 14. If the map F in Theorem 11 is compact then the Hilbert space can be replaced by any Banach (or indeed Fréchet) space (this is immediate since all we need consider is any continuous retraction r with $r(z) \in \partial Q$ for $z \in E \backslash Q$; note such an r exists (see [7])).

Remark 15. There is an obvious analogue of Theorem 11 for ACG maps.

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